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Modified Mathematical Models in Biology by the Means of Caputo Derivative of a Function with Respect to Another Exponential Function

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Abstract In this article, the researcher considered some well-known mathematical models of ordinary differential equations applied in biology such as the bacterial growth, the natural FC solution models for vegetables, the biological phospholipids pathway, the glucose absorption by the body and the spread of epidemics. The ordinary differential equations for each model are fractionalized by the means of Caputo derivative of a function with respect to certain exponential function. In each model, we embed the concept fractionalization associated with a chosen exponential function in order to modify the given model. Consequently, various propositions are evoked by hypothetically allowing some modifications in several mathematical models of biology. The results are further visualized by providing the graphs of Mittag-Leffler function on various parameters. The graphs' analysis explored the behavior of the solution for every modified model. In this study, the solutions of the modified models are all of the Mittag–Leffler form while all original models are solved by the means of exponential function. Slight changes of the behavior of the solutions are due to the assumptions and the change of parameters.

Keywords Fractional Calculus, Mittag-Leffler Function, Monod Equation, Caputo-type Derivatives

1. Mathematical Background

Fractional calculus is a branch of Mathematical analysis that explores all sort of powers for a given operator, and such powers can be real or even complex numbers [1,8,10,11,15,20,21]. For the last five years the applications of fractional calculus have gained much attention in the fields of economics, physics, engineering and biology [11,14,18].

By swapping the order of the ordinary derivative with the fractional integral operator [9,16], one can reconstruct the definition of Riemann–Liouville fractional derivative. In this way, the Laplace transformation depends on the initial conditions of this new integer order derivative, in contrast to the initial conditions of fractional order derivative while using the Riemann–Liouville fractional derivative.

In this article, we considered an analogous concept namely the fractional derivative and the integral of a function with respect to another function via the following definitions [1].

Definition 1 [2,3]

Let $\theta > 0$. $n \in N$. I is an interval $-\infty \le a \le b \le \infty$. f is integrable function defined on I and $g \in C^1(I)$ such that g is strictly increasing and $g' \neq 0$. for all $t \in I$. Then the fractional integral of function f with respect to another function g is given by

$$I_{a+}^{\theta.g}f(t) := \frac{1}{\Gamma(\theta)}\int_a^t g'(\tau)[g(t)-g(\tau)]^{\theta-1}f(\tau)d\tau. \quad (1)$$

Note that, if $g(t) = t$ the above assertion reduces to the Riemann-Liouville fractional integral.

By the means of definition 1, we can consider the Caputo's derivative of f with respect to g as follows:

Definition 2 [2,3]

Let $\theta > 0$. $n = [\theta] \in N. I = (a.b)$ is an interval $-\infty \le a \le b \le \infty. f \in C^n(I). g \in C^1(I)$ two functions such that g is strictly increasing and $g' \neq 0$. for all $t \in I$. The right fractional derivatives of f with respect to $g$are respectively given by

$$cD_{a+}^{\theta.g}f(t) := I_{a+}^{n-\theta.g}\left(\frac{1}{g'(t)}\frac{d}{dt}\right)^n f(t). \quad (2)$$

Remark 1

If $f(t) = [g(t)-g(a+)]^{\beta-1}$. $\beta > 1$ then:

$$cD_{a+}^{\theta.g}f(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\theta)}[g(t)-g(a+)]^{\beta-\theta-1} \quad (3)$$

We recall the Mittag-Leffler functions that have been considered by many researchers due to their central role in the studies of fractional differential equations and their applications, one may refer to some recent monographs studied by [12-14].

As Almeida recently mentioned in [2], the compositions of Mittag-Leffler functions with other functions can potentially be useful. The use of Integro-differential equations can provide solid theoretical underpinnings for these investigations. For that we provide the following:

Remark 2

If $f(t) = E_\theta\{\lambda[g(t) - g(a+)]^{\beta-\theta-1}\}. \lambda \in R$

$$cD_{a+}^{\theta.\,g}f(t) = \lambda f(t) \quad (4)$$

Proposition 1

In the current approach we consider the following application. Given any model of the form

$$e^{\beta t}\frac{dM(t)}{dt} = \alpha M(t) \quad (5)$$

First, we perform a fractionalization of the first order differential equation by introducing the fractional derivative of $M(t)$with respect to the strictly increasing function $g(t) = e^{\beta t}$. The new model will then be:

$$cD_{a+}^{\theta.\,g}M_\theta(t) = \alpha M_\theta(t).$$

$$\theta \in (0.1].\ with initial\ condition M_\theta(0) = M_0 \quad (6)$$

$$cD_{a+}^{\theta.\,g}\ M_\theta(t) = \frac{1}{\Gamma(1-\theta)}\int_0^1\left(\frac{e^{-\beta\tau}-e^{-\beta t}}{\beta}\right)^{-\theta}\frac{dM_\theta}{d\tau}\,d\tau. \quad (7)$$

This is a particular case of the Caputo-type fractional derivative of a function with respect to another function as we assumed that $g(t) = e^{\beta t}$ with the restriction of $\theta \in (0.1]$. It is evident that Equation (6) is an Eigenvalue problem for the Caputo-type fractional operator defined in Equation (7), whose solution is given by

$$M_\theta(t) = M_0 E_\theta\left[\frac{\alpha}{\beta^\theta}\left(1-e^{-\beta t}\right)^\theta\right] \quad (8)$$

2. Problem Statement

The emphasis of this work is the models represented by ODEs, for example the following exponential population growth

Where $C(t)$ is the population as a function of time t,

$C_0 = C(0)$: the initial population size when t = 0,

α: growth (decay) constant derived from observations or data collection within a limited time interval.

$$\frac{dC(t)}{dt} = \alpha\, C(t)\ \rightarrow C(t) = C_0\, e^{\alpha t}$$

Certainly, if we investigate the same population growth, we might find different value of α. *That is* α *itself might be a of function of time in another words* $\alpha = \alpha(t)$. *In this case, the solution of the equation above might fail to be accurate especially in long term. Indeed, the exponential growth might not be realistic when time is sufficiently large. For that, we allow some modifications and assumptions to provide more accurate and realistic models.*

1. The Microbial Growth

The Monod equation is a mathematical model that describes microorganism development. It was named for scientist Jack Mono, who proposed utilizing an equation to link the growth rates of microorganisms (microbial) in an aqueous medium with the bone concentrations of nutrients that organisms consume, such as sugar for bacteria. This model is employed in a variety of industries, including microbial sewage treatment, pharmaceutical component fermentation, and food fermentation. In this work, we explored the classical model for microbial growth that involves a single microbial species in a batch mode chemostat or bioreactor which has been mathematically modeled in the equations from (4 – 6).

$$\frac{dS}{dt} = -\alpha M(S)X \quad (9)$$

$$\frac{dX}{dt} = M(S)X - k_d X. \quad (10)$$

satisfying the initial conditions $S(0) = S_0 > 0. X(0) = X_0 > 0$. Where the notation

$X = X(t)$: The biomass,

$S = S(t)$: The concentration of the substrate in the chemostat

k_d: The decay constant

$M(S)$: Function of s which is widely regarded as the célèbre Monod equation:

$$M(S) = \mu_{max}\frac{S}{S+k_s}.\ \ k_s\ a\ positive\ constant \quad (11)$$

Note that by using from (9) and (10), one can readily deduce that

$$\frac{dS}{dX} = \frac{-\alpha M(S)}{M(S)-k_d} = f(S)$$

Proposition 2

"The Caputo fractional derivate of $S(t)$ with respect to$e^{\beta t}$".

We let $M(S) = e^{-\beta t}\frac{S}{X}$. $\beta > 1$ in assertion (9), that is

$$\frac{dS}{dt} = -M(S)\alpha X \rightarrow e^{\beta t}\frac{dS}{dt} = -\alpha S \qquad (12)$$

Then we apply *Proposition 1*, we fractionalize the latter by considering the Caputo fractional derivative of $S(t)$ with respect to $g(t) = e^{\beta t}$. which is strictly increasing with $g'(t) \neq 0$. the model equation will be

$$cD_{0+}^{\theta,g}S_\theta(t) = -\alpha S_\theta(t).\ \theta \in (0.1) \qquad (13)$$

With initial condition$S_\theta(0) = S_0 > 0$

$$cD_{0+}^{\theta,g}S_\theta(t) = \frac{1}{\Gamma(1-\theta)}\int_0^t \left(\frac{e^{-\beta\tau} - e^{-\beta t}}{\beta}\right)^{-\theta}\frac{dS_\theta}{d\tau}d\tau. \qquad (14)$$

The solution is given by

$$S_\theta(t) = S_0E_\theta\left[\frac{-\alpha}{\beta^\theta}\left(1 - e^{-\beta t}\right)^\theta\right] \qquad (15)$$

Proposition 3

"The Caputo fractional derivate of $X(t)$ with respect to$e^{\beta t}$"

We applied the same modifications as in Proposition 2, we also considered the case when the substrate is sufficiently small, or simply $S(t) \rightarrow 0$ then (10) can be viewed as

$$\frac{dX}{dt} = M(S)X \rightarrow e^{\beta t}\frac{dX}{dt} = S - k_dX = -k_dX.\ \theta \in (0.1) \qquad (16)$$

This leads to yield a new model of equation

$$cD_{0+}^{\theta,g}X_\theta(t) = -k_dX_\theta(t).\quad \theta \in (0.1) \qquad (17)$$

Following the similar approach, we receive

$$X_\theta(t) = X_0E_\theta\left[\frac{-k_d}{\beta^\theta}\left(1 - e^{-\beta t}\right)^\theta\right] \qquad (18)$$

Remark 3

- $\lim_{t\rightarrow +\infty} S_\theta(t) = S_0E_0\left[-\frac{\alpha}{\beta^\theta}\right]$.
- $\lim_{t\rightarrow +\infty} X_\theta(t) = X_0E_0\left[\frac{-k_d}{\beta^\theta}\right]$.
- $\frac{dS}{dX} = -\alpha \rightarrow S = -\alpha X + c$
- $S = e^{\beta t}M(S) \rightarrow X = \left(\frac{2}{\alpha}e^{\beta t}M(S) + c\right)^{1/2}$

The profiles of the solutions (15) and (18) for the time are illustrated in Figures 1–4 for different values of time θ, β, α and k_d.

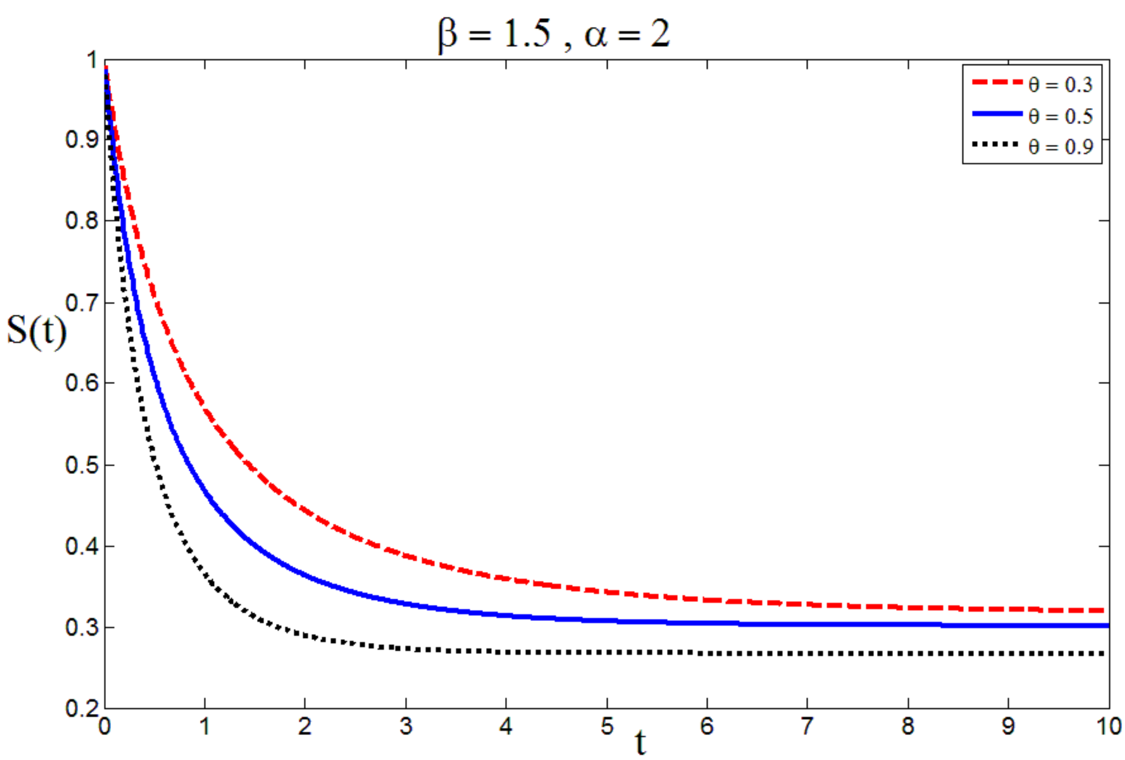

Figure 1. The profiles of the function $S_\theta(t)$ for the values $\beta = 1.5$, $\alpha = 2$ and θ takes the values: $\theta = 0.3$ (red), 0.5 (blue), 0.9 (black)

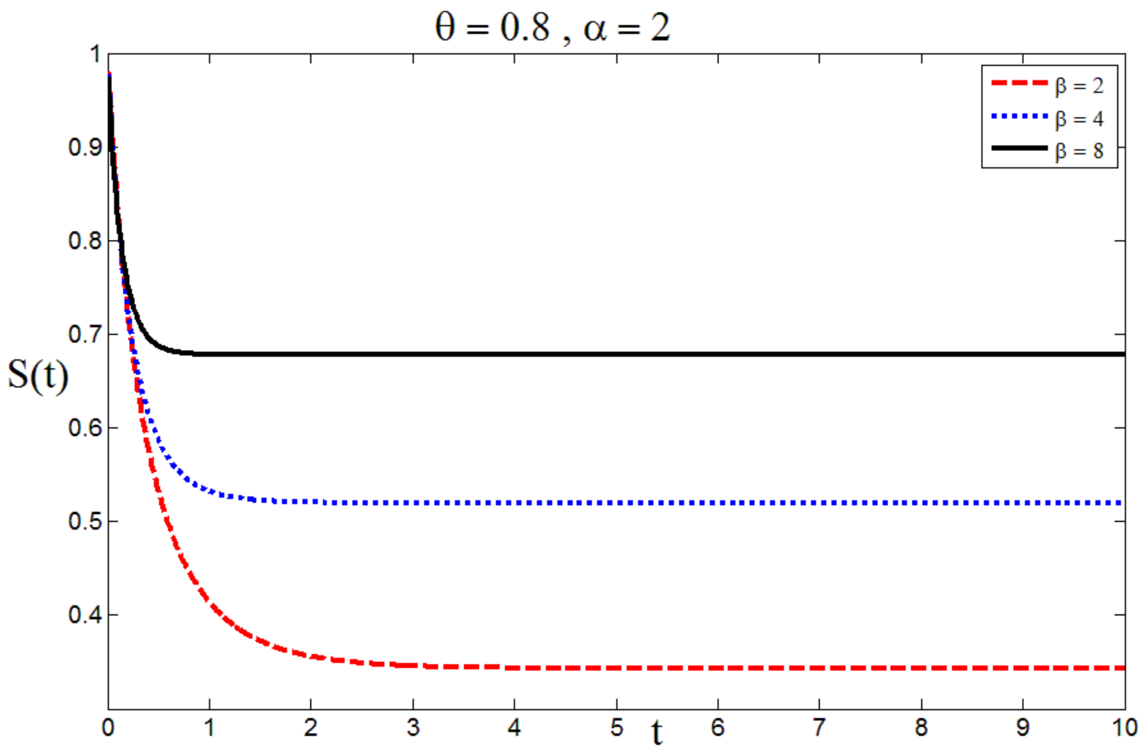

Figure 2. The profiles of the function $S_\theta(t)$ for the values $\theta = 0.8$, $\alpha = 2$ and β takes the values: $\beta = 2$ (red), 4 (blue), 8 (black)

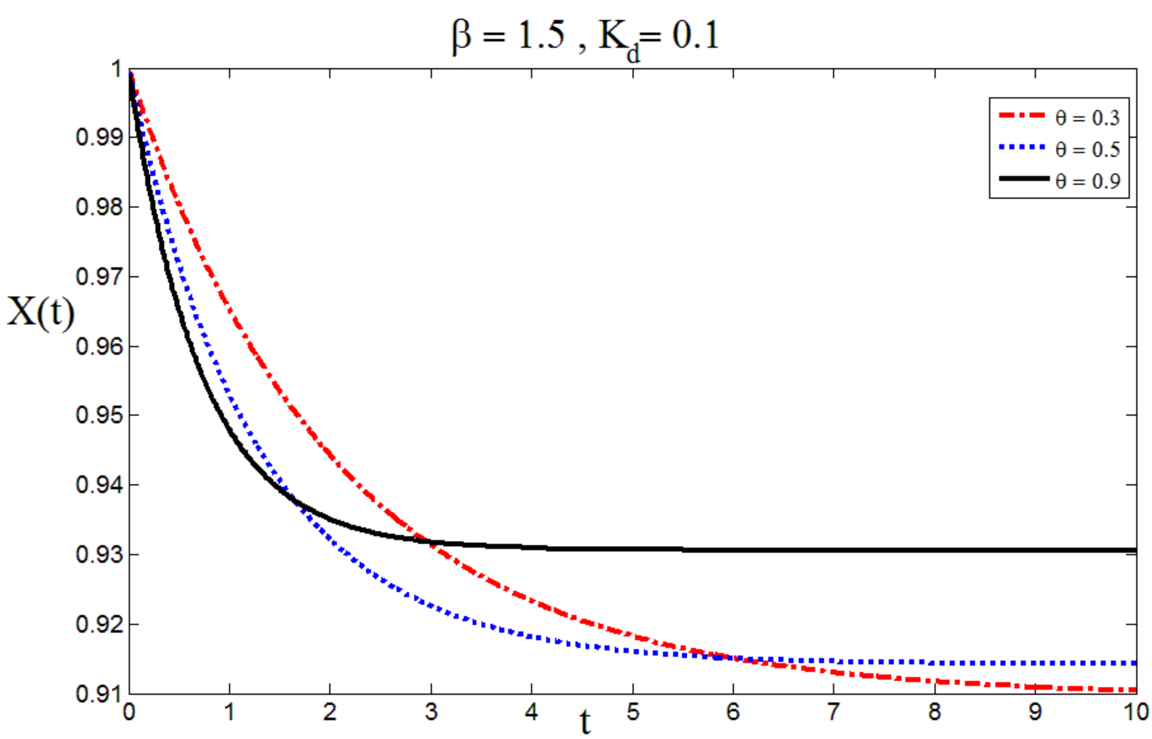

Figure 3. The profiles of the function $X_\theta(t)$ for the values $\beta = 1.5$, $\alpha = 2$ and θ takes the values: $\theta = 0.3$ (red), 0.5 (blue), 0.9 (black)

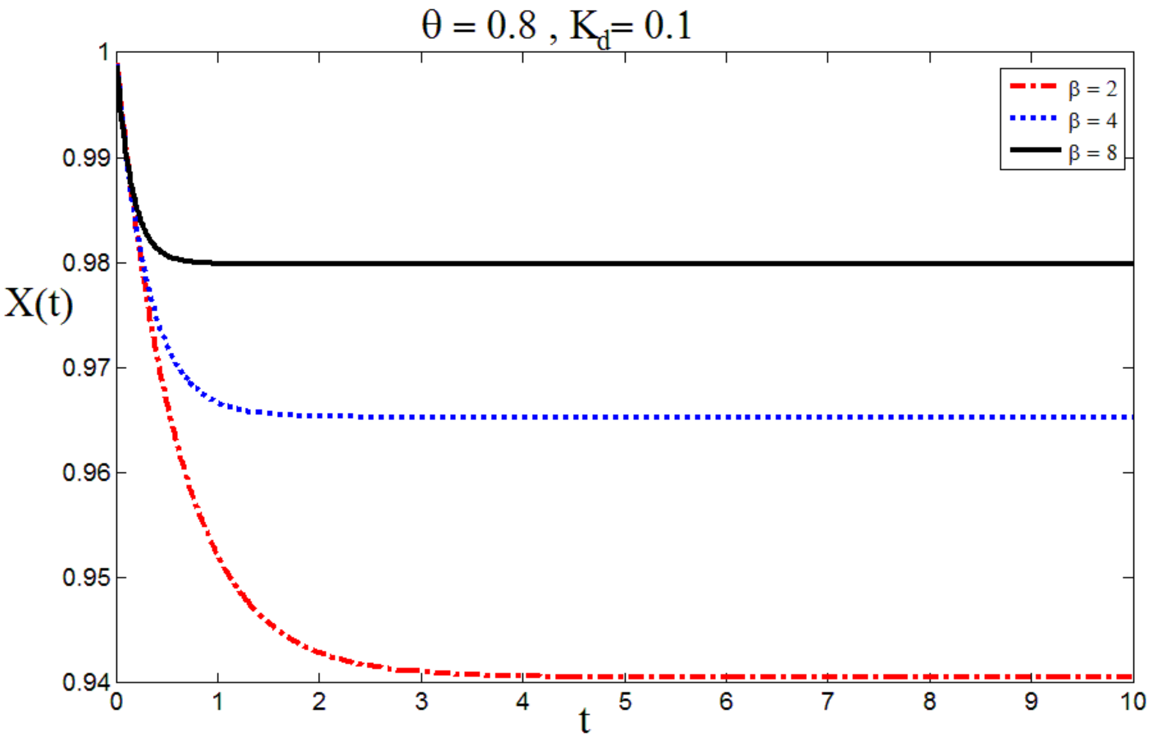

Figure 4. The profiles of the function $X_{\theta}(t)$ for the values $\theta = 0.8$, $\alpha = 2$ and β takes the values: $\beta = 2$ (red), 4 (blue), 8 (black)

2. Natural FC Solution Models for Vegetables

We consider the kinetic equation associated with the D model [22]

$$\frac{d\phi}{dt} = -\frac{1}{\tau}\,\phi(t) \quad (19)$$

having the solution

$$\phi(t) = e^{-t/\tau} \quad (20)$$

Proposition 4

Let $g(t) = -\frac{1}{\tau} = e^{-\beta t}$. then (19) can be viewed as

$$e^{\beta t}\frac{d\phi}{dt} = \phi(t). \quad (21)$$

After fractionalizing, we deduce

$$cD_{0+}^{\theta,\,g}\phi_\theta(t) = \phi_\theta(t).\ \theta \in (0.1) \quad (22)$$

With solution

$$\phi_\theta(t) = \phi_0 E_\theta\left[\beta^{-\theta}\left(1 - e^{-\beta t}\right)^{\theta}\right].\theta \in (0.1) \quad (23)$$

The profiles of the solution (23) for the time are illustrated in Figures 5 and 6 for different values of time θ and β. We observe that the solution increased in the range $t \in [0,10]$ when the values of θ increased for fixed value of β (Figure 5). The same property appears in Figure 6 when the value of β is fixed and three different values of θ. The functions became constant for large values of time.

3. Biological Phospholipids Pathway

We explore the singularly perturbed system introduced in [23].

$$\varepsilon\frac{dx}{dt} = -xz \quad (24)$$

$$y + z = c.\ c \in R \quad (25)$$

The variables *x*, *y*, *z* correspond to the concentrations of agonist activates membrane bound receptor, G-protein activated, and G-protein produces effectors.

Proposition 5

We assume that $z(t)$ is an exponential decaying function of t and we write $z(t) = e^{-\beta t}$ this leads to modify (24) as follows

$$\varepsilon e^{\beta t}\frac{dx}{dt} = -x \quad (26)$$

Now, applying the fractionalization as in previous sections, we derive

$$cD_{0+}^{\theta,\,g}x_\theta(t) = -x.\,g(t) = e^{\beta t}.\ \beta > 1.\ \theta \in (0.1) \quad (27)$$

with solution

$$x_\theta(t) = x_0 E_\theta\left[-\frac{1}{\varepsilon}.\ \frac{1}{\beta^\theta}\left(1 - e^{-\beta t}\right)^{\theta}\right].\theta \in (0.1) \quad (28)$$

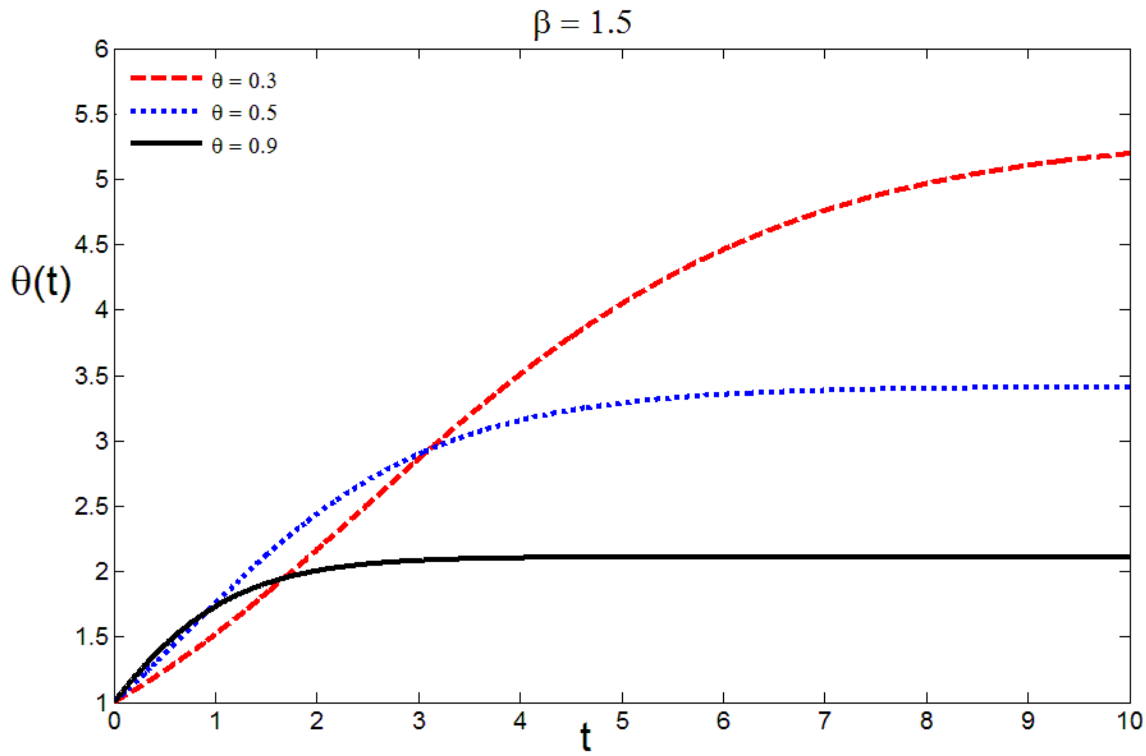

Figure 5. The profiles of the function $\phi_{\theta}(t)$ for the value $\beta = 1.5$, and θ takes the values: $\theta = 0.3$ (red), 0.5 (blue), 0.9 (black)

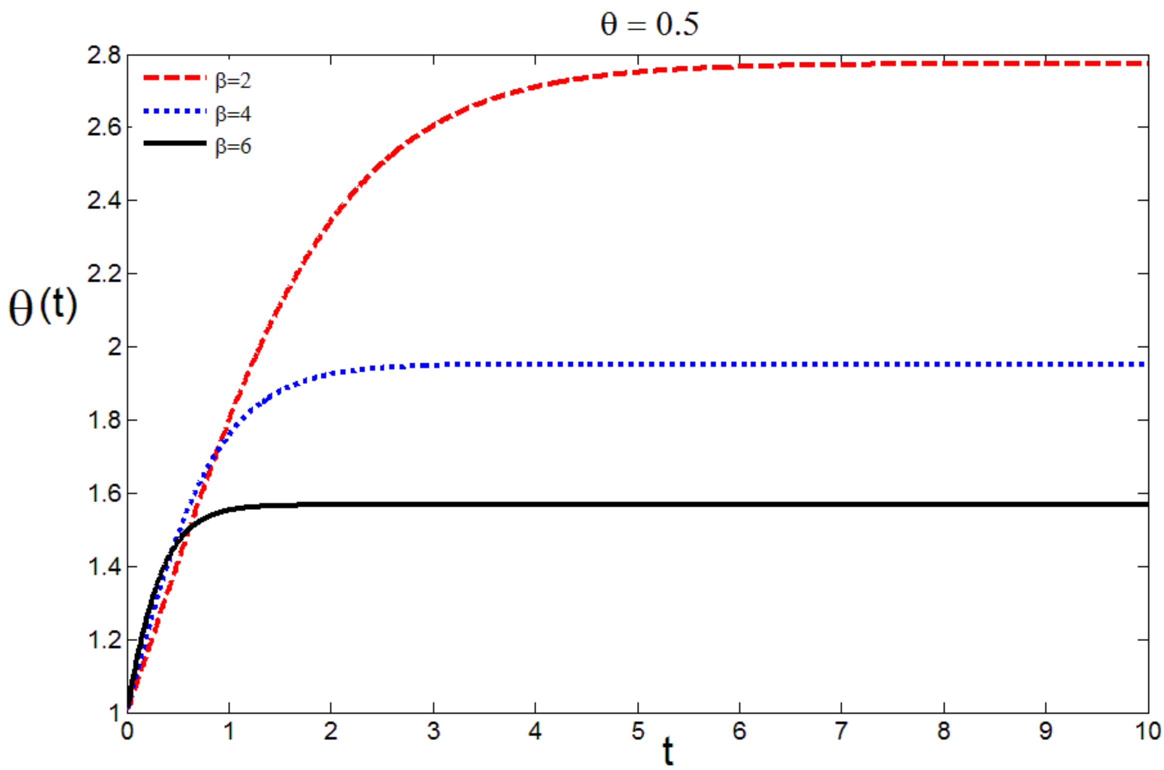

Figure 6. The profiles of the function $\phi_{\theta}(t)$ for the value $\theta = 0.5$, and β takes the values: $\beta = 2$ (red), 4 (blue), 8 (black)

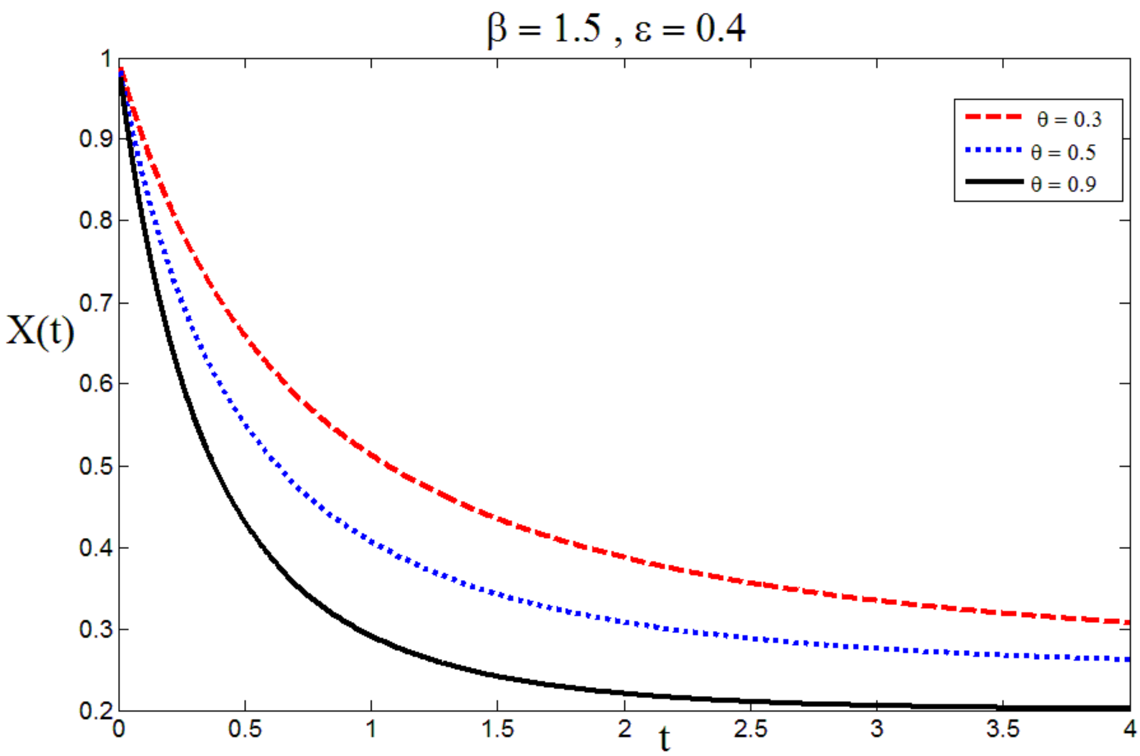

Figure 7. The profiles of the function $x_{\theta}(t)$ for the values $\beta = 1.5$, $\varepsilon = 0.4$ and θ takes the values: $\theta = 0.3$ (red), 0.5 (blue), 0.9 (black)

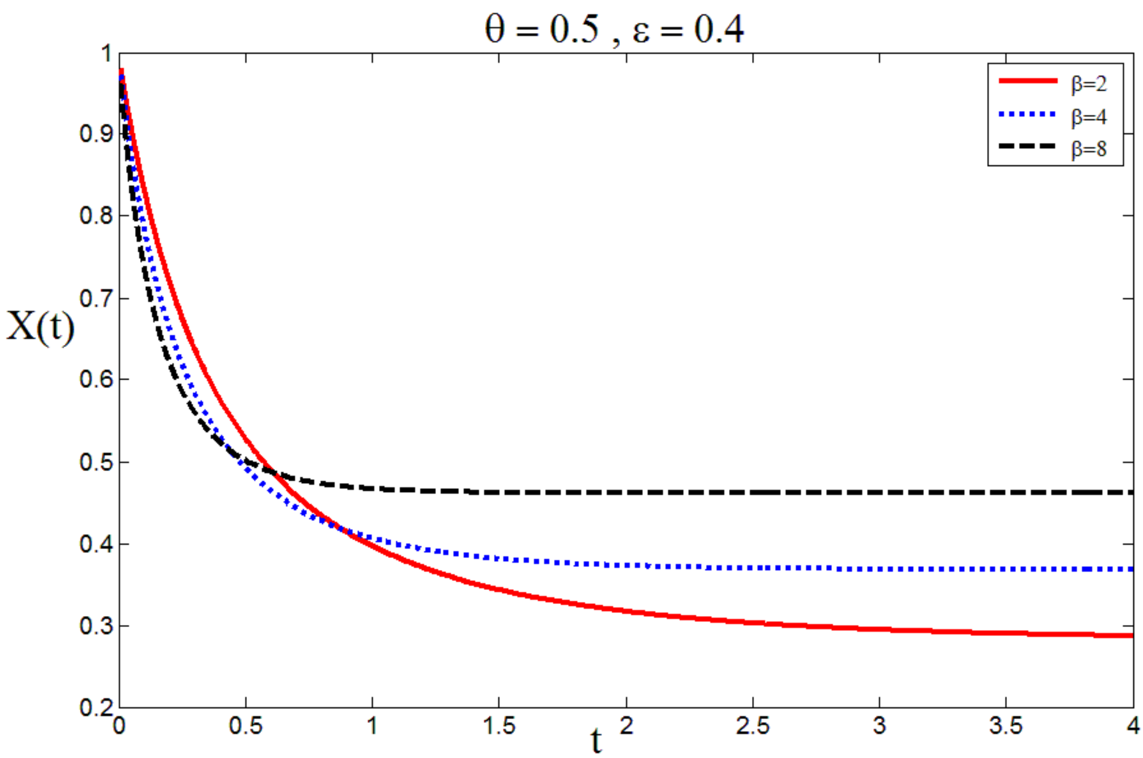

Figure 8. The profiles of the function $x_{\theta}(t)$ for the values $\theta = 0.5$, $\varepsilon = 0.4$ and β takes the values: $\beta = 2$ (red), 4 (blue), 8 (black)

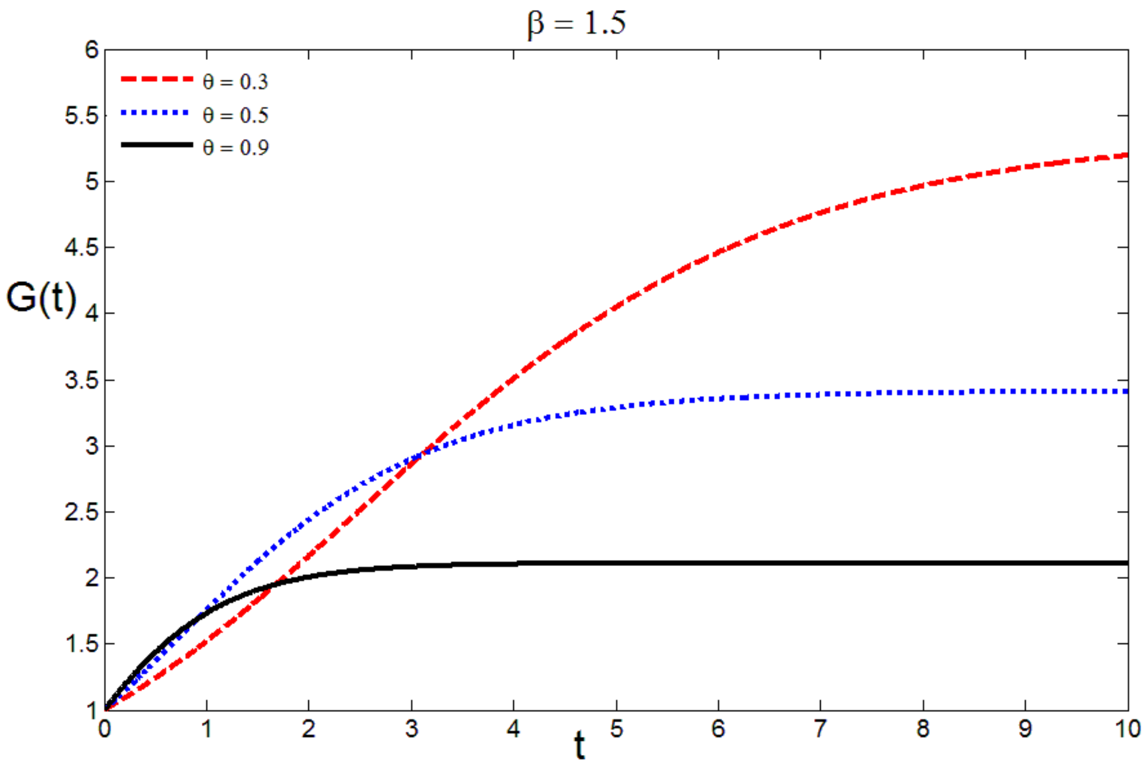

Figure 9. The profiles of the function $G_{\theta}(t)$ for the value $\beta = 1.5$ and θ takes the values: $\theta = 0.3$ (red), 0.5 (blue), 0.9 (black)

The profiles of the solution (28) for the time are illustrated in Figures 7 and 8 for different values of time for representative value of θ, ε and β. We observed that the solution decreased when the time increased to a certain value. The functions became constant for large values of time.

4. Glucose Absorption by the Body

Glucose is absorbed by the body at the rate proportional to the amount of glucose present in the bloodstream. Let λ denote the (positive) constant of proportionality. Suppose there are G_0units of glucose in the bloodstream when $t = 0$, and let $G(t)$ be the number of units in the bloodstream at time t. Then, since the glucose being absorbed by the body is leaving the bloodstream, $G(t)$ satisfies the following equality.

$$\frac{dG}{dt} = -\lambda G \tag{29}$$

With solution

$$G(t) = G_0 e^{-\lambda t} \tag{30}$$

Proposition 6

From Proposition 1 and by letting $-\lambda = e^{-\beta t}$ we receive the new model

$$cD_{0+}^{\theta,g} G_\theta(t) = G_\theta(t).\ g(t) = e^{\beta t}.\ \beta > 1.\ \theta \in (0.1) \tag{31}$$

That holds the following solution

$$G_\theta(t) = G_0 E_\theta\left[\frac{1}{\beta^\theta}\left(1 - e^{-\beta t}\right)^\theta\right].\ \theta \in (0.1) \tag{32}$$

The profiles of the solution (32) for the time are illustrated in Figures 9 and 10 for different values of time and for representative value of θ and β.

5. The Spread of Epidemics

One model for the spread of epidemics assumes that the number of people infected changes at a rate proportional to the product of the number of people already infected and the number of people who are vulnerable, but not yet infected. Therefore, if $\boldsymbol{S}$ denotes the total population of susceptible people and $\boldsymbol{I}(\boldsymbol{t})$ denotes the number of infected people at time $\boldsymbol{t}$, then $\boldsymbol{S} - \boldsymbol{I}$ is the number of people who are prone, but not yet infected. Thus, the rate of change in the number of infected people with respect to time $\boldsymbol{t}$ is given by

$$\frac{dI}{dt} = rI(S - I).\ r > 0 \tag{33}$$

That solves as

$$I = \frac{SI_0}{I_0 + (S - I_0)e^{-rSt}} \tag{34}$$

Note that $\lim_{t\to\infty} I(t) = S$ predicting all susceptible people eventually become infected.

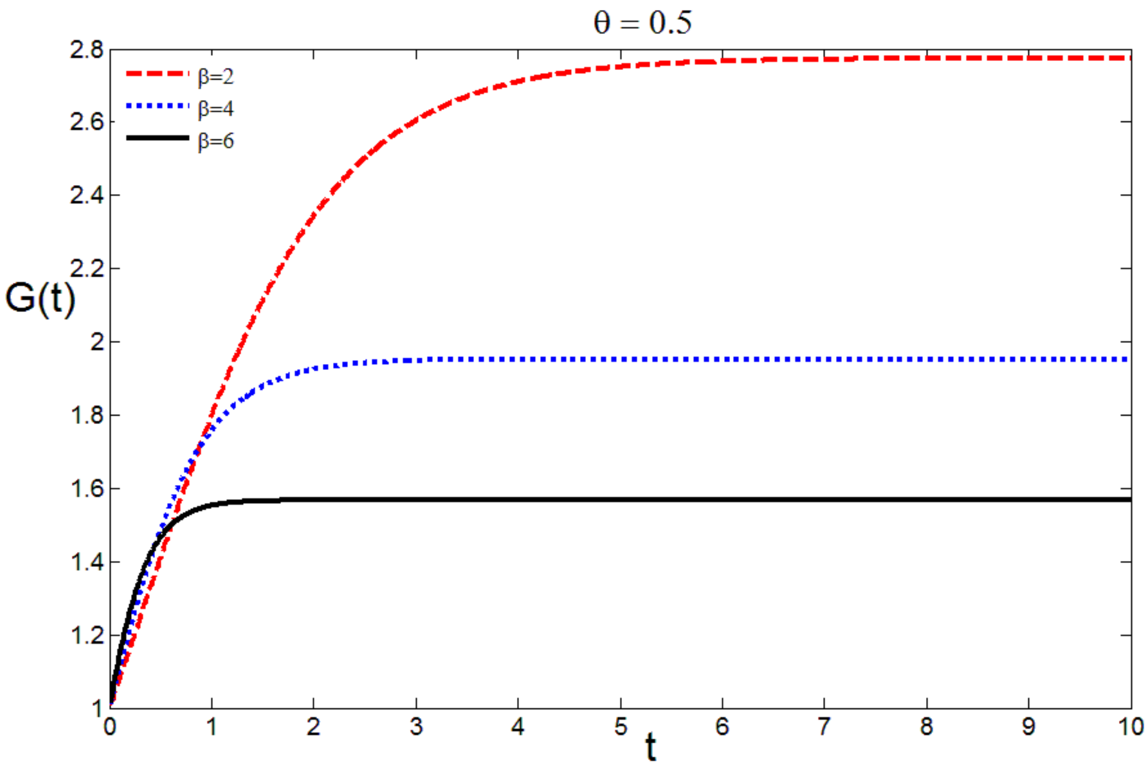

Figure 10. The profiles of the function $G_\theta(t)$ for the value $\theta = 0.5$ and β takes the values: $\beta = 2$ (red), 4 (blue), 8 (black)

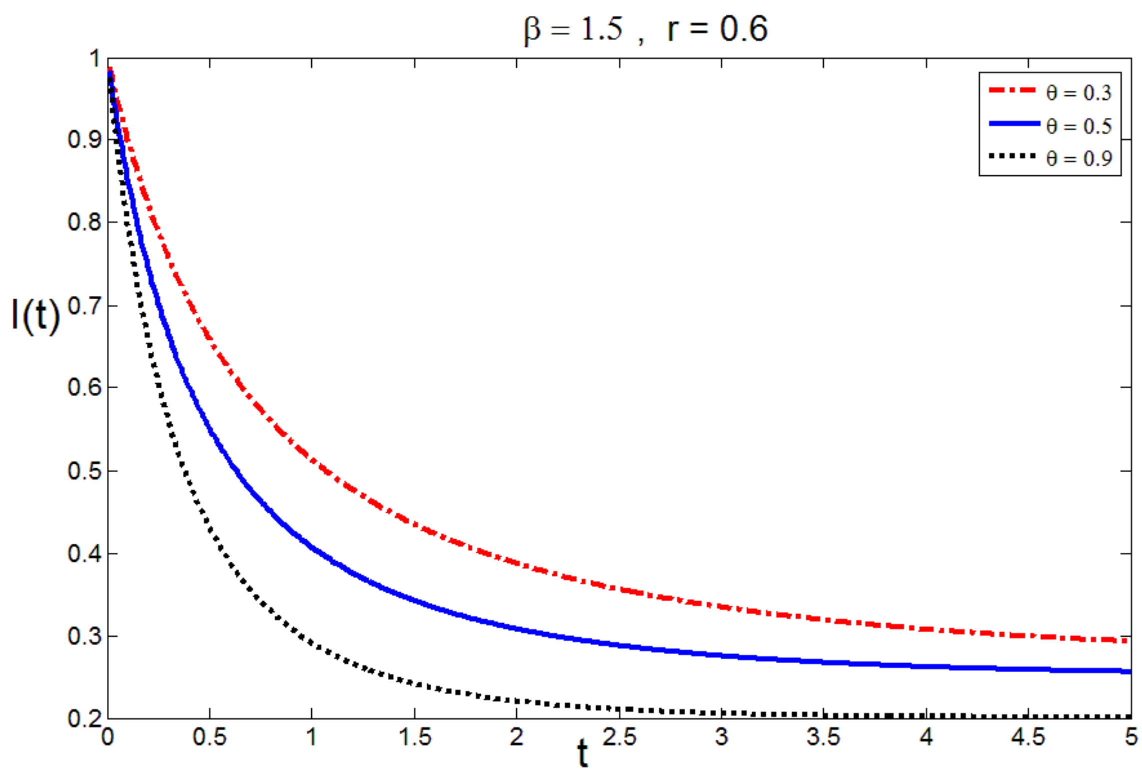

Figure 11. The profiles of the function $I_\theta(t)$ for the values $\beta = 1.5$, $r = 0.6$ and θ takes the values: $\theta = 0.3$ (red), 0.5 (blue), 0.9 (black)

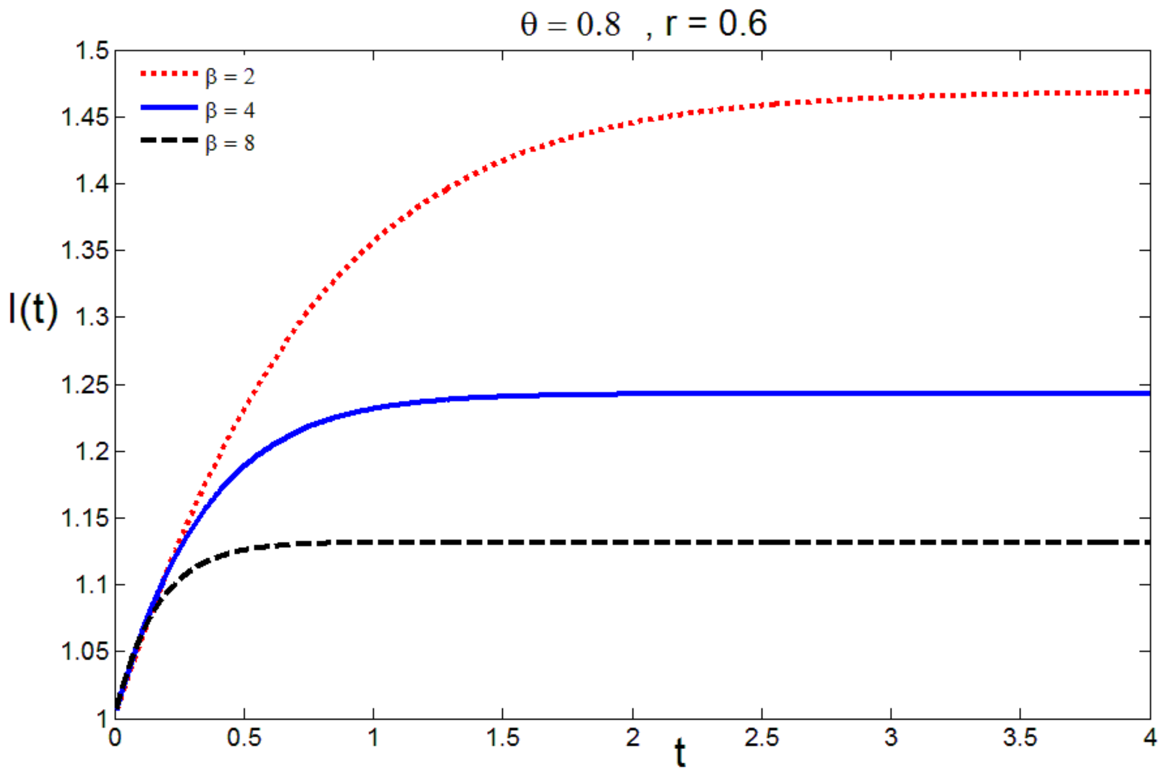

Figure 12. The profiles of the function $I_\theta(t)$ for the values $\theta = 0.8$, $r = 0.6$ and β takes the values: $\beta = 2$ (red), 4 (blue), 8 (black)

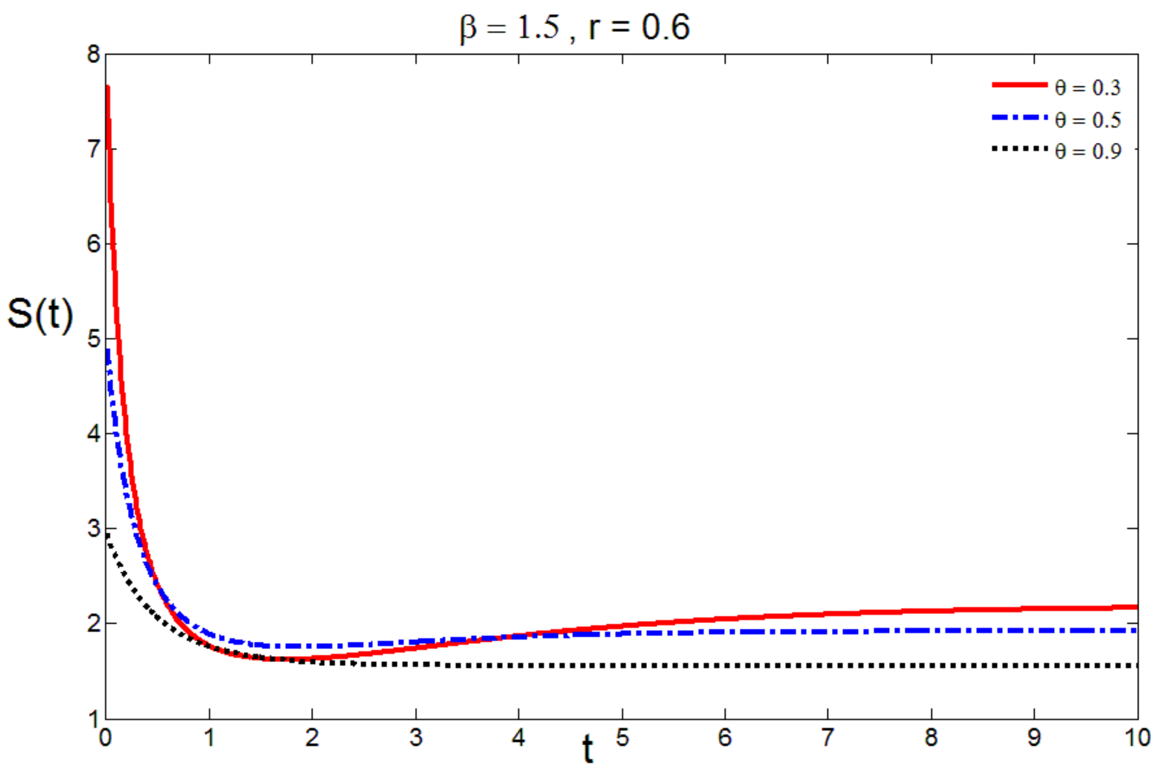

Figure 13. The profiles of the function $S_\theta(t)$ for the values $\beta = 1.5$, $r = 0.6$ and θ takes the values: $\theta = 0.3$ (red), 0.5 (blue), 0.9 (black)

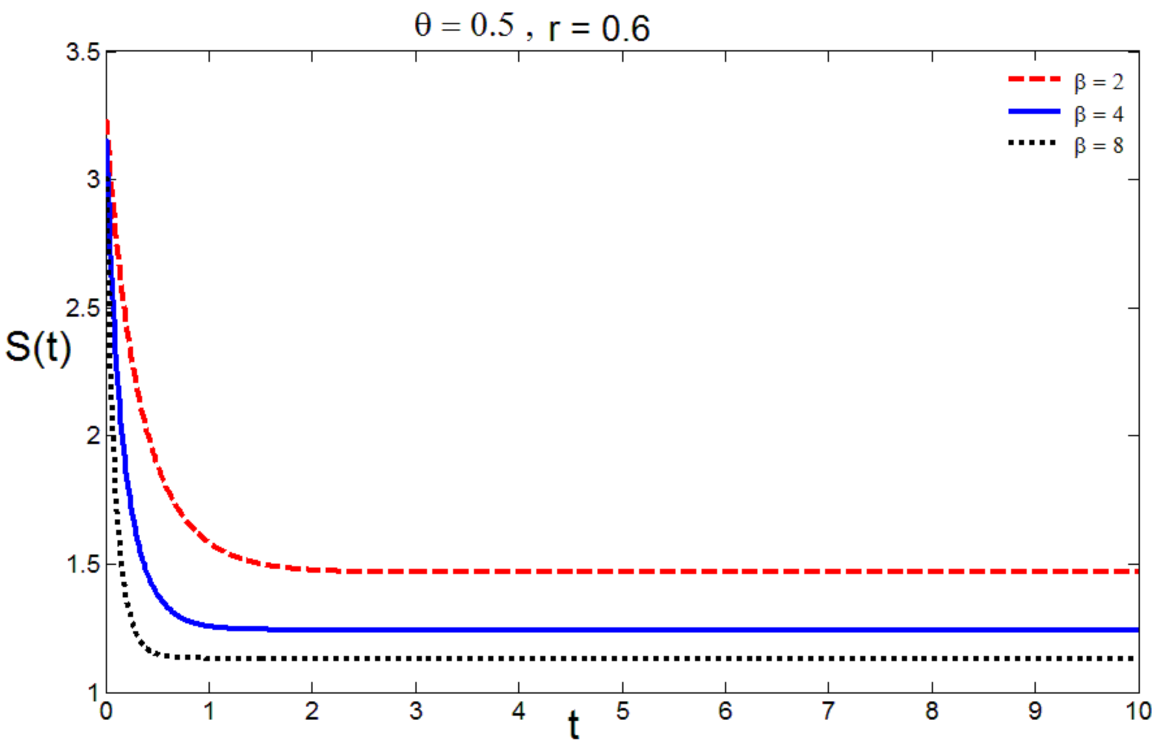

Figure 14. The profiles of the function $S_{\theta}(t)$ for the values $\theta = 0.5$, $r = 0.6$ and β takes the values: $\beta = 2$ (red), 4 (blue), 8 (black)

Proposition 7

If we let $S - I = e^{-\beta t}. \beta > 1$ this also predicts that all susceptible people eventually become infected. However, the new model by the means of Proposition 1 will be given as

$$e^{\beta t}\frac{dI_\theta}{dt} = rI_\theta \quad (35)$$

$$cD_{0+}^{\theta\cdot\, g} I_\theta(t) = rI_\theta(t).\, g(t) = e^{\beta t}.\beta > 1.\ \theta \in (0.1) \quad (36)$$

That has the solution as follow:

$$I_\theta(t) = I_0 E_\theta\left[\frac{r}{\beta^\theta}\left(1 - e^{-\beta t}\right)^\theta\right].\ \theta \in (0.1) \quad (37)$$

$$S_\theta(t) \approx\ I_0 E_\theta\left[\frac{r}{\beta^\theta}\left(1 - e^{-\beta t}\right)^\theta\right] +\ e^{-\beta t} \quad (38)$$

Finally,

$$\lim_{t\to\infty} I_\theta(t) = \lim_{t\to\infty} S_\theta(t) = I_0 E_\theta\left[\frac{r}{\beta^\theta}\right]$$

The profiles of the solutions (37) and (38) for the time are illustrated in Figures 11 - 14 for different values of time and for representative value of θ, r and β.

3. Results

In response to the problem statements and applying some modifications we received new fractional calculus models that match the original ODE models in terms of increasing (decreasing). However, with an almost logarithmic way which might be more accurate and realistic comparing to the original models that are strictly increasing (decreasing) and might fail to interpret the relevant phenomena wen time tends to infinity.

4. Conclusions

In this work, we have applied some modifications on the well-known mathematical models of biology that are associated with ordinary differential equations. This is to apply fractionalizations by the means of Caputo derivative of a function with respect to certain exponential function. Consequently. we derived seven propositions on the given models such as the bacterial growth, the natural FC solution models for vegetables, the biological phospholipids pathway, the glucose absorption by the body and the spread of epidemics. One of them can readily deduce that the main differences between the original model and the modified and one in each can apply the solution functions (Exponential in the original versus Mittag-Leffler in the modified). However, as shown by graphs at some parameters, all results and main characteristics are consistent, i.e. a slight difference between the original and the modified model.

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