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Parameter Estimation for Weibull Burr Type X Model with Right Censored Data

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Abstract Studies have considered generalizing statistical distributions in the past. These were aimed at making such distributions more flexible and suitable for describing real-world phenomena. In this study, we considered exploring the Weibull Burr Type X distribution, which extends the Burr Type X distribution using the Weibull generator. Particularly, the performance of the maximum likelihood estimators for its parameters encompassing the right censored dataset was explored and compared. On the performance of its estimators with respect to bias and root mean square error, we considered the Monte Carlo simulation study to make a comparison using varying sample sizes and censored percentages. We illustrated the usefulness and potentials of the Weibull Burr Type X distribution using a right censored dataset. We considered comparing the fitness of this model to its sub-models using real world dataset. The result showed that the Weibull Burr Type X distribution provides a better fit than other competing models. This indicates that the distribution is flexible and competitive. The Weibull Burr Type X distribution exhibits unimodal and decreasing shapes. The extra parameter in the distribution varies the model's tail weight and introduces skewness into the model. We introduced this model as an alternative to other existing models for modelling right censored data in various research fields and areas of study.

Keywords Weibull Burr Type X Model, Right Censored, Maximum Likelihood Estimator, Monte Carlo Simulations, Goodness of Fit

1. Introduction

Investigating new families of lifetime distributions and generalizing more flexible lifetime distributions have been of interest to a certain group of statisticians. Burr [1] introduced twelve different forms of cumulative distribution functions for modelling datasets by using the method of differential equations. Many applications of Burr distribution have been illustrated in the literature for the different twelve forms of this distribution. However, the one-parameter Burr Type X model has received great attention as it provides a new flexible model for modelling lifetime datasets. Surles and Padgett [2] proposed and generalized the Burr Type X model with one parameter by adding one scale parameter as a result they obtained a model called the two parameters Burr Type X distribution. It was observed that the Burr Type X distribution with two parameters is effective in modelling the strength data as well as the lifetime data. Several researchers have studied the properties and illustrated the applications of the two parameters Burr Type X distribution in various fields such as health, agricultural, and biological using complete datasets and different types of censored datasets (e. g., Surles and Padgett [2,3], Raqab and Kundu [4], Abd-Elfattah [5], Abd EL-Baset et al., [6], Khaleel et al., [7] and Khaleel [8]). In addition, the two parameters Burr Type X distribution is a generalized Rayleigh distribution and also considered as a special case (or sub-model) of the exponentiated Weibull distribution that was introduced by Mudholkar and Sirvastava [9].

The two parameters Burr Type X distribution has the cumulative distribution function (cdf) and the probability density function (pdf) defined by Surles and Padgett [2] as;

$$G(t,\lambda,\theta) = \left(1 - e^{-(\lambda t)^2}\right)^{\theta},\ \ t > 0, \lambda > 0, \theta > 0 \quad (1)$$

and

$$g(t,\lambda,\theta) = 2\,\lambda^2\,\theta\,t\,e^{-(\lambda t)^2}\left(1 - e^{-(\lambda t)^2}\right)^{\theta-1},\ \ t > 0 \quad (2)$$

where T is a random variable, λ and θ are the scale and shape parameters respectively.

Bourguignon et al., [10] studied and proposed the generalized family of a univariate distribution with two additional shape parameters using the Weibull generator which was applied to the odds ratio $\frac{G(t,\phi)}{[1-G(t,\phi)]}$. The term "generator" means having a different distribution F for each baseline distribution G. Suppose $G(t,\phi)$ and $g(t,\phi)$ are the cumulative and the density functions of the baseline model with parameter vector ϕ, respectively. Let $F(t,\alpha,\beta)$ be the cdf of the Weibull distribution with two parameters given by $F(t,\alpha,\beta) = 1 - e^{-\alpha\,(t)^{\beta}}$, for $t > 0$, where α and β are positive parameters. Based on this density, by replacing t with $\frac{G(t,\phi)}{[1-G(t,\phi)]}$, the cdf and pdf of the Weibull - G family of distributions with two extra parameters α and β are defined by Bourguignon et al., [10] as:

$$F(t,\alpha,\beta,\phi) = \int_0^{\frac{G(t,\phi)}{[1-G(t,\phi)]}} \alpha\,\beta\,t^{\beta-1}e^{-\alpha(t)^{\beta}}dt =$$

$$= 1 - e^{-\alpha\left(\frac{G(t,\phi)}{[1-G(t,\phi)]}\right)^{\beta}},\ \ t,\alpha,\beta > 0 \quad (3)$$

and

$$f(t,\alpha,\beta,\phi) =$$

$$= \alpha\ \beta\ g(t,\phi)\ \frac{[\,G(t,\phi)]^{\beta-1}}{[1-\,G(t,\phi)]^{\beta+1}}\,e^{-\alpha\left(\frac{G(t,\phi)}{[1-G(t,\phi)]}\right)^{\beta}}, t > 0 \quad (4)$$

where $G(t,\phi)$ is the baseline cdf, which depends on the parameter vector ϕ. The cdf in Equation (3) is a special case of the T-X family which was introduced by Alzaatreh et al., [11].

Ibrahim et al., [12] developed a continuous distribution called the Weibull-Burr Type X (WBX) distribution which extends the Burr Type X distribution using the Weibull generator and studied the mathematical properties. The flexibility of the WBX model was demonstrated in the analysis of uncensored dataset. By implementing Equation (1) in Equation (3), this resulted in forming the cdf of the WBX model with four parameters. It was defined by Ibrahim et al., [12] as following:

$$F(t,\alpha,\beta,\lambda,\theta) = 1 - e^{-\alpha\left(\frac{\left(1-e^{-(\lambda t)^2}\right)^{\theta}}{\left[1-\left(1-e^{-(\lambda t)^2}\right)^{\theta}\right]}\right)^{\beta}} =$$

$$1 - e^{-\alpha\frac{\left(1-e^{-(\lambda t)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda t)^2}\right)^{\theta}\right]^{\beta}}}. \quad (5)$$

By taking the first derivative of the cdf in Equation (5) with respect to the variable t ; $\frac{\partial}{\partial t}F(t,\alpha,\beta,\lambda,\theta) = f(t,\alpha,\beta,\lambda,\theta)$, then the pdf of the WBX model is obtained as:

$$f(t,\alpha,\beta,\lambda,\theta) =$$

$$2\,\alpha\ \beta\,\lambda^2\,\theta\,t\,e^{-(\lambda t)^2}\ \frac{\left(1-e^{-(\lambda t)^2}\right)^{\theta\beta-1}\ e^{-\alpha\left(\frac{\left(1-e^{-(\lambda t)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda t)^2}\right)^{\theta}\right]^{\beta}}\right)}}{\left[1-\left(1-e^{-(\lambda t)^2}\right)^{\theta}\right]^{\beta+1}}, \quad (6)$$

where $t > 0, \alpha > 0, \beta > 0, \lambda > 0$ and $\theta > 0$. α and β are two additional shape parameters. T~WBX ($\alpha,\beta,\lambda,\theta$) denotes a continuous random variable from WBX model. The graphical presentation of the WBX probability density function at varying parameter values is shown in Figure 1.

From Figure 1, it can be easily noticed that the WBX distribution is skewed, and exhibits unimodal and decreasing shapes. This also implies that the model can be used to describe or represent real life events with unimodal and decreasing shapes, and can be used to fit events that are heavily skewed.

Another way of describing a probability model is through the reliability analysis, in this case they are survival function and the hazard function.

The survival function is the fundamental quantity in survival analysis that is used to describe the failure time of a phenomena and the probability that an individual survives beyond time t. The survival function of the WBX distribution was given by Ibrahim et al., [12] as:

$$S(t,\alpha,\beta,\lambda,\theta) = e^{-\alpha\frac{\left(1-e^{-(\lambda t)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda t)^2}\right)^{\theta}\right]^{\beta}}}\ \ t > 0. \quad (7)$$

The graphical illustration of the WBX survival function for some parameter values is presented in Figure 2.

From Figure 2, as expected, the plots of the survival function for the WBX distribution follow the usual pattern of survival function that is monotonically decreasing by time.

The hazard function (or the failure rate) is also fundamental in survival analysis. The hazard function of the WBX distribution is obtained by taking the ratio of the pdf in Equation (6) and the survival function in Equation (7).

$$h(t,\alpha,\beta,\lambda,\theta) =$$

$$= 2\,\alpha\,\beta\,\lambda^2\,\theta\,t\,e^{-(\lambda t)^2}\;\frac{\left(1-e^{-(\lambda t)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda t)^2}\right)^{\theta}\right]^{\beta+1}}, \mathrm{t} > 0. \quad (8)$$

The graphical illustration of the hazard function for the WBX model is presented in Figure 3.

The plots in Figure 3 show that the shape of the hazard function for the WBX model has different shapes, such as increasing, decreasing, and bathtub. This indicates that the WBX model is more flexible than the baseline model (Two parameters Burr Type X) in modelling lifetime datasets. Hence, the WBX model can be used quite effectively in analyzing real lifetime datasets. The different shapes of the hazard plots were one of the reasons that motivated this study.

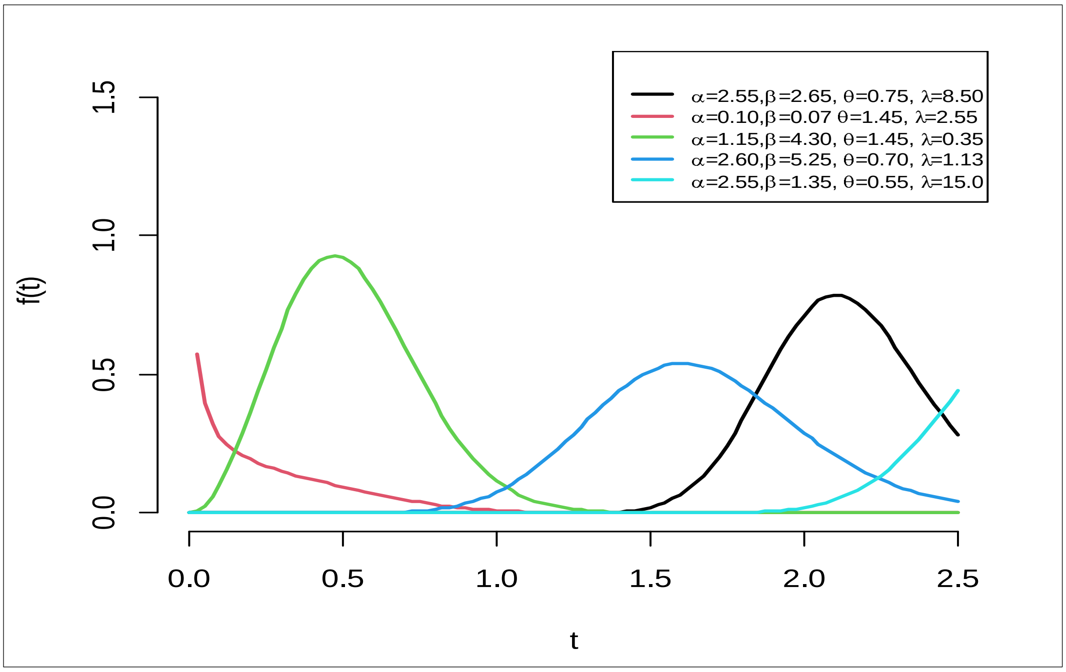

Figure 1. Plots of the WBX probability density function for some parameter values

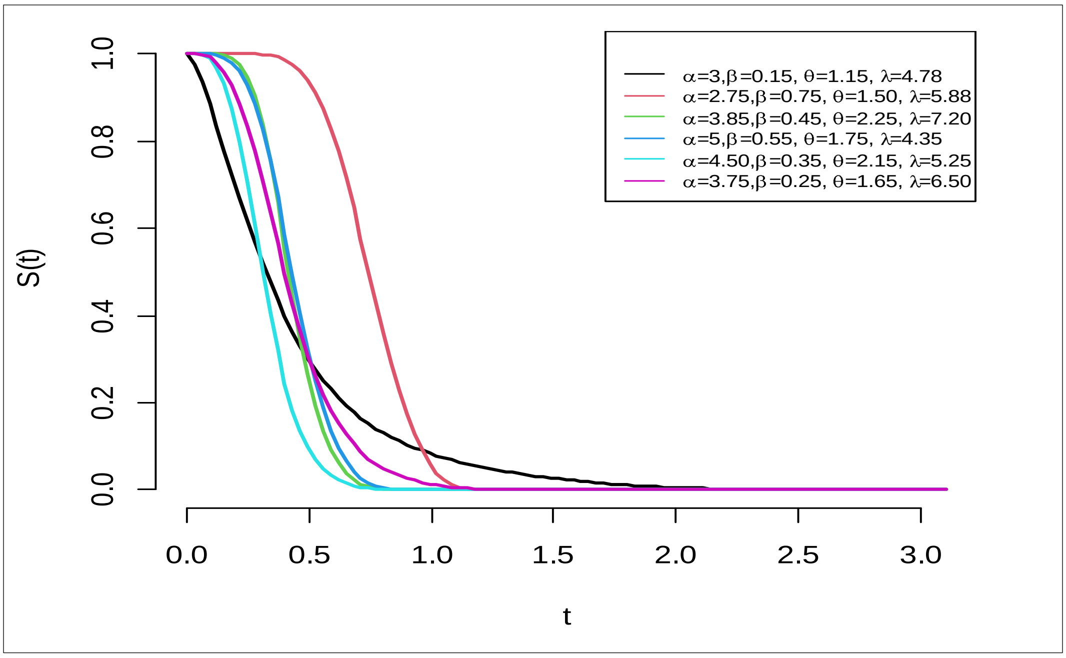

Figure 2. Plots of the WBX survival function for some parameter values

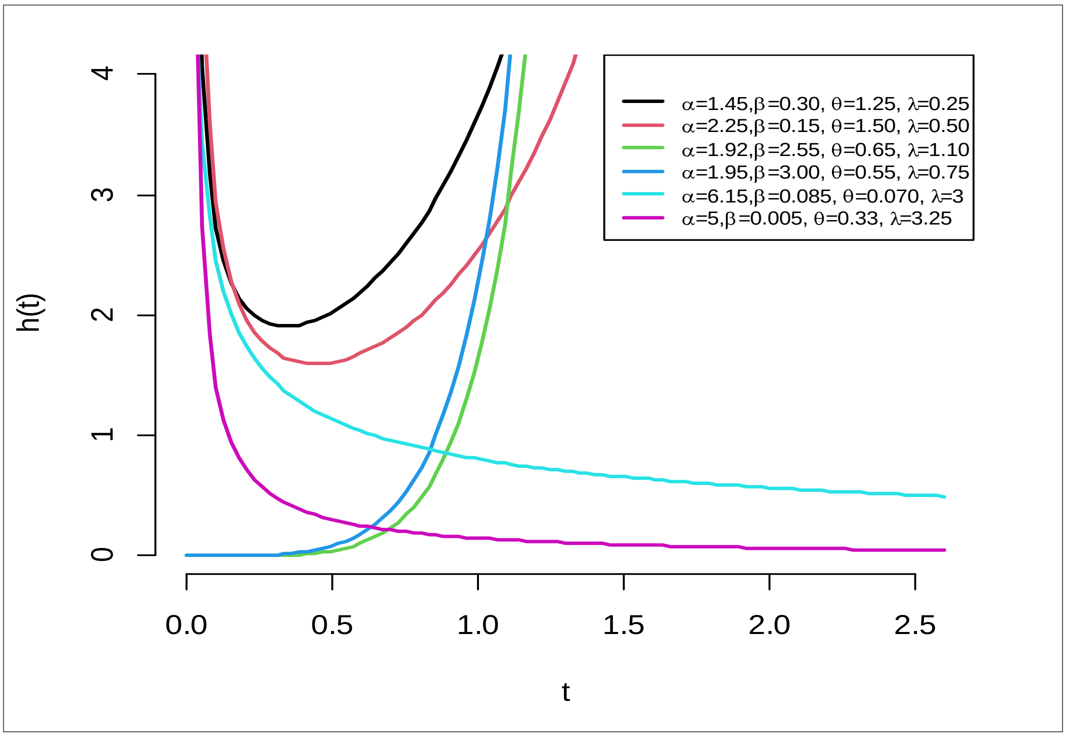

Figure 3. Plots of the WBX hazard function for some parameter values.

The aim of this research is to explore and compare the performance of the maximum likelihood estimation for the WBX model parameters encompassing right censored dataset by means of the Monte Carlo simulation, and to assess the flexibility of the WBX model by comparing it with its sub models in the presence of right censoring using real life application.

2. Methodology

2.1. The Model and Estimation Method

The method of the maximum likelihood estimation is used to estimate the unknown parameters of the WBX distribution under right censored time. Maximum likelihood estimation (MLE) is one of the most important methods of estimating parameters for various models [13]. It is a very popular method and has very wide applicability.

2.2. The Description of the Model

Suppose $t_1, t_2, \ldots, t_n$ is a set of the random lifetime taken from the Weibull Burr Type X distribution with probability density function $f(t_i, \alpha, \beta, \lambda, \theta)$ defined in Equation (6), and let $c_1, c_2, \ldots, c_n$ be a set of the random lifetime that represents the censoring mechanism independent of the Weibull Burr Type X distribution. Based on Lawless [14], the observed right censored lifetime is of the form $(y_i,\ \delta_i)$, where

$$y_i = \min(t_i,\ c_i), (i = 1, \ldots, n). \quad (9)$$

$$\delta_i = \begin{cases} 1, & t_i \le c_i \quad \text{subject is uncensored} \\ 0, & t_i > c_i \quad \text{subject right censored}' \end{cases}$$

$$(i = 1, \cdots, n). \quad (10)$$

and, δ_i is the event indicator.

2.3. The Estimation Method

It is well known that in the case of right censored lifetime, the likelihood contribution of these censored observations can be incorporated by using the survival function of the interest model denoted by $S(.)$. Thus, based on Lawless [14], the likelihood function for any lifetime model with parameter vector ϕ gives the vector of observations under right censored data is obtained as the following:

$$L(\phi, y_1, \ldots, y_n) = \prod_{i=1}^{n} f(y_i, \phi)^{\delta_i} S(y_i, \phi)^{1-\delta_i}, \quad (11)$$

where $f(.)$ is the probability density function of the interest model.

Therefore, the corresponding likelihood function for the WBX model with parameter vector $\phi = (\alpha, \beta, \lambda, \theta)^T$ is given as:

$$L(\phi, y_1, \dots, y_n) = \prod_{i=1}^{n}\left[\left(2\,\alpha\,\beta\,\lambda^2\,\theta\,y_i\,e^{-(\lambda y_i)^2}\,\frac{\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta-1}}{\left[1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right]^{\beta+1}}\,e^{-\alpha\left(\frac{\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right]^{\beta}}\right)}\right)^{\delta_i}\left(e^{-\alpha\frac{\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left[1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right]^{\beta}}}\right)^{1-\delta_i}\right]. \quad (12)$$

The ML estimator is obtained by finding the value of $\hat{\phi} = \left(\hat{\alpha},\ \hat{\beta},\ \hat{\lambda},\ \hat{\theta}\right)^T$ in the parameter space that maximizes the log-likelihood function. This is obtained by equating the first partial derivatives of the log-likelihood function to zero and solving for $\phi = (\alpha, \beta, \lambda, \theta)^T$ simultaneously as follows:

First, the log-likelihood function for the WBX model with parameter vector $\phi = (\alpha, \beta, \lambda, \theta)^T$ in Equation (12) is given as following:

$$\begin{aligned}\log L(\phi, y_1, \dots, y_n) &= \sum_{i=1}^{n}\delta_i\,[\log(2\alpha\beta\lambda^2\theta y_i)] - \sum_{i=1}^{n}\delta_i\,(\lambda y_i)^2 + (\theta\beta-1)\sum_{i=1}^{n}\delta_i\log\left(1-e^{-(\lambda y_i)^2}\right)\\ &- (\beta+1)\sum_{i=1}^{n}\delta_i\log\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right) - \alpha\sum_{i=1}^{n}\delta_i\left[\frac{\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\right]\\ &- \alpha\sum_{i=1}^{n}(1-\delta_i)\left[\frac{\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\right].\end{aligned} \quad (13)$$

Then the first partial derivatives of the log-likelihood function with respect to the parameters $\phi = (\alpha, \beta, \lambda, \theta)$ in Equation (13) are equated to zero to give the following:

$$\frac{\partial \log L(\phi, y_1, \dots, y_n)}{\partial \alpha} = -\sum_{i=1}^{n}(1-\delta_i)\left[\frac{\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\right] - \sum_{i=1}^{n}\delta_i\left[\frac{\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\right] + \sum_{i=1}^{n}\frac{\delta_i}{\alpha} = 0\,. \quad (14)$$

$$\begin{aligned}&\frac{\partial \log L(\phi, y_1, \dots, y_n)}{\partial \beta}\\ &= -\alpha\sum_{i=1}^{n}\left(\left[\frac{(1-\delta_i)\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\right]\theta\log\left(1-e^{-(\lambda y_i)^2}\right) - \log\Big[1\right.\\ &\left. - \left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\Big]\left[\frac{(1-\delta_i)\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\right]\right) + \sum_{i=1}^{n}\frac{\delta_i}{\beta} + \theta\sum_{i=1}^{n}\delta_i\log\left(1-e^{-(\lambda y_i)^2}\right)\\ &- \sum_{i=1}^{n}\delta_i\log\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\sum_{i=1}^{n}\left(\left[\frac{\alpha\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\delta_i\theta\log\left(1-e^{-(\lambda y_i)^2}\right)\right]\right.\\ &\left. - \log\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\left[\frac{\alpha\delta_i\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}}\right]\right) = 0.\end{aligned} \quad (15)$$

$$\begin{aligned}
&\frac{\partial \log L(\phi, y_1, \dots, y_n)}{\partial \lambda} \\
&= -2\alpha\lambda\theta\beta \sum_{i=1}^{n} \left(\frac{(1-\delta_i)\left(1-e^{-(\lambda y_i)^2}\right)^{-1+\theta+\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{1+\beta}}\, {y_i}^2 e^{-(\lambda y_i)^2} \right. \\
&\left. + \frac{(1-\delta_i)\, {y_i}^2 e^{-(\lambda y_i)^2}\left(1-e^{-(\lambda y_i)^2}\right)^{-1+\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}} \right) + \sum_{i=1}^{n} \frac{2\delta_i}{\lambda} - 2\lambda \sum_{i=1}^{n} \delta_i {y_i}^2 \\
&+ 2\lambda(\theta\beta - 1) \sum_{i=1}^{n} \frac{\delta_i {y_i}^2 e^{-(\lambda y_i)^2}}{1-e^{-(\lambda y_i)^2}} - 2\lambda\theta(\beta+1) \sum_{i=1}^{n} - \frac{\delta_i {y_i}^2 e^{-(\lambda y_i)^2}\left(1-e^{-(\lambda y_i)^2}\right)^{-1+\theta}}{\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)} \\
&- \alpha \sum_{i=1}^{n} \left(2\theta\beta\lambda \frac{\delta_i {y_i}^2 e^{-(\lambda y_i)^2}\left(1-e^{-(\lambda y_i)^2}\right)^{-1+\theta+\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{1+\beta}} 2\theta\beta\lambda \frac{\delta_i {y_i}^2 e^{-(\lambda y_i)^2}\left(1-e^{-(\lambda y_i)^2}\right)^{-1+\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}} \right) = 0.
\end{aligned} \tag{16}$$

$$\begin{aligned}
&\frac{\partial \log L(\phi, y_1, \dots, y_n)}{\partial \theta} \\
&= \sum_{i=1}^{n} \left(\left[\frac{\alpha(1-\delta_i)\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}} \right] \beta \log\left(1-e^{-(\lambda y_i)^2}\right) \right. \\
&\left. - \beta \log\left(1-e^{-(\lambda y_i)^2}\right) \left[\frac{\alpha(1-\delta_i)\left(1-e^{-(\lambda y_i)^2}\right)^{\theta+\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{1+\beta}} \right] \right) \\
&+ \sum_{i=1}^{n} \frac{\delta_i}{\theta} + \beta \sum_{i=1}^{n} \delta_i \log\left(1-e^{-(\lambda y_i)^2}\right) \\
&- (1+\beta) \sum_{i=1}^{n} - \frac{\delta_i\left(1-e^{-(\lambda y_i)^2}\right)^{\theta} \log\left(1-e^{-(\lambda y_i)^2}\right)}{1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}} \\
&- \sum_{i=1}^{n} \left(\left[\frac{\alpha\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta+\theta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{1+\beta}} \right] \delta_i \beta \log\left(1 \right. \right. \\
&\left. - e^{-(\lambda y_i)^2}\right) \\
&\left. + \delta_i\, \beta \log\left(1-e^{-(\lambda y_i)^2}\right) \left[\frac{\alpha\left(1-e^{-(\lambda y_i)^2}\right)^{\theta\beta}}{\left(\left(1-\left(1-e^{-(\lambda y_i)^2}\right)^{\theta}\right)\right)^{\beta}} \right] \right) \\
&= 0.
\end{aligned} \tag{17}$$

Equations (14) to (17) are non-linear equations that cannot be solved analytically. Thus, an iterative method is required to obtain the ML estimators.

3. Results

3.1. Numerical Studies

The simulation study under right censored lifetime samples from the WBX model was carried out to investigate the performance of the maximum likelihood estimators of α, β, λ and θ and to investigate the effect of varying sample sizes and varying censoring percentages on the parameter estimates. In addition, the developed model is used to illustrate its applicability and flexibility on a right censored real dataset.

3.2. Monte Carlo Simulations

A Monte Carlo simulation study was carried out for three different sample sizes; $n = 25, 50, 100$ depicting small, moderate, and large samples respectively. Three different values of the approximate censoring proportions (cp) were selected such as, $0.0, 0.30, 0.50$. These censoring proportions refer to complete samples, moderate censored proportion, and heavy censored proportion respectively. The process of generating a right censored lifetime from the WBX model is described below using a fairly similar procedure that was applied by Martinez et al., [15], this procedure can be used to generate a right censored sample from any continuous lifetime distribution. The following steps were employed to generate a right censored data from the WBX model:

Step 1: Initial values of the four parameters were chosen as $\alpha = 1.2, \beta = 3, \lambda = 2$ and $\theta = 0.5$.

Step 2: A sequence of the random numbers from Uniform distribution, $u_i \sim unif(n, 0, 1)$ was generated to produce a set of the random lifetime, t_i , $(i = 1, \dots, n)$ from the WBX distribution as follows:

$$t_i = \frac{1}{\lambda}\left[-\log\left(1-\left\{\frac{\left[\frac{-\log(u_i)}{\alpha}\right]^{\frac{1}{\beta}}}{1+\left[\frac{-\log(u_i)}{\alpha}\right]^{\frac{1}{\beta}}}\right\}^{\frac{1}{\theta}}\right)\right]^{\frac{1}{2}}, \quad (18)$$

where t_i in this study is obtained by inverting the survival function of the WBX model defined in Equation (7), t_i can also be obtained by inverting the cdf of the WBX model defined in Equation (5).

Step 3: A set of the random censored lifetime $c_i, (i = 1, \dots, n)$ was generated from the exponential distribution with rate μ ($c_i \sim$ Exp (μ)).

Step 4: The observed right censored lifetime y_i was generated according to $y_i = \min(t_i,\ c_i)$.

Step 5: Pairs of values (y_1, δ_1), (y_2, δ_2) , $\cdots$, (y_n, δ_n), were obtained. δ_i is the event indicator which is defined in Equation (10).

Step 6: The approximate censoring proportion in the sample was calculated as follows: $cp = 1 - \frac{\sum_{i=1}^{n} \delta_i}{n}$ where $\sum_{i=1}^{n} \delta_i$ is the sum of uncensored status, and the value of μ in step 3 would be adjusted to obtain the desired approximate censoring proportion in the data.

Step 7: Repeat steps (1 to 6) 1000 times aim to generate 1000 random samples from the WBX model under right censoring lifetime.

For each generated sample, the estimates of (α , β , λ and θ) can be obtained by solving the likelihood Equations (14 to 17) using any iterative procedure for solving nonlinear equations. In this research, the maximum likelihood estimators for the parameters were computed using the Newton Raphson iterative method, which was implemented using the R software (package: MaxLik) that developed by Henningsen et al., [16]. The initial values for the Newton-Raphson iteration must be chosen carefully as the Newton-Raphson method is sensitive to the initial values. In other words, the behavior of this numerical iterative technique depends on the initial guess [17].

The mean, bias, standard error (SE) and root mean squared error (RMSE) of the maximum likelihood estimator for the parameter values were calculated as the measures to obtain information about the performance of the maximum likelihood estimator for the parameters of WBX model:

$$\text{Mean}(\hat{\phi}) = \frac{\sum_{i=1}^{1000} \hat{\phi}_i}{1000}.$$

$$\text{Bias}(\hat{\phi}) = \text{Mean}(\hat{\phi}) - \phi.$$

$$\text{SE}(\hat{\phi}) = \sqrt{\sum_{i=1}^{1000} \frac{\left[\hat{\phi}_i - mean(\hat{\phi})\right]^2}{1000}}.$$

$$\text{RMSE}(\hat{\phi}) = \sqrt{\sum_{i=1}^{1000} \frac{\left[\hat{\phi}_i - \phi\right]^2}{1000}}.$$

where $\hat{\phi} = \left(\hat{\alpha},\ \hat{\beta},\ \hat{\lambda},\ \hat{\theta}\right)^T$ is the maximum likelihood estimator of $\phi = (\alpha, \beta, \lambda, \theta)^T$. Table 1 presents the summary statistics for maximum likelihood estimators for α, β, λ and θ of different sample sizes n and various censoring proportions cp.

According to the investigation of the right censored time, Table 1 shows that the mean of the maximum likelihood estimators is satisfactorily closed to the initial values as the sample size increases. This is according to the consistent property of the maximum likelihood estimator [18]. With regards to the bias of the estimators, it generally tends to decrease with increasing in sample size. This is through the asymptotic unbiasedness properties of the MLE [19]. The SE, and RMSE values for the estimators both decrease with the increasing of the sample size due to the large sample properties of the MLE. While at fixed sample size n, the SE and RMSE values for the estimators both increase when the censoring proportions increase due to the loss of the information in the sample caused by censoring [20]; This is expected because the increase of censoring proportion means less data with complete failure times. Thus, the likelihood contribution depends more on the survival function and censored times instead of the density function and the exact failure times.

Table 1. Summary statistics for maximum likelihood estimator for α, β, λ and θ

cp=0.0	α = 1.20					β = 3			
	n	Mean($\hat{\alpha}$)	Bias($\hat{\alpha}$)	SE($\hat{\alpha}$)	RMSE($\hat{\alpha}$)	Mean($\hat{\beta}$)	Bias($\hat{\beta}$)	SE($\hat{\beta}$)	RMSE($\hat{\beta}$)
	25	1.2975	0.0975	0.2600	0.2777	3.1240	0.1240	0.4566	0.4731
	50	1.2950	0.0950	0.2054	0.2263	3.0594	0.0594	0.3222	0.3277
	100	1.2941	0.0941	0.1466	0.1742	3.0270	0.0270	0.2194	0.2211
	λ = 2					θ = 0.50			
	n	Mean($\hat{\lambda}$)	Bias($\hat{\lambda}$)	SE($\hat{\lambda}$)	RMSE($\hat{\lambda}$)	Mean($\hat{\theta}$)	Bias($\hat{\theta}$)	SE($\hat{\theta}$)	RMSE($\hat{\theta}$)
	25	2.0417	0.0417	0.1166	0.1238	0.5195	0.0195	0.0309	0.0365
	50	2.0190	0.0190	0.0715	0.0740	0.5145	0.0145	0.0197	0.0245
	100	2.0054	0.0054	0.0413	0.0417	0.5099	0.0099	0.0142	0.0173
cp=0.30	α = 1.20					β = 3			
	n	Mean($\hat{\alpha}$)	Bias($\hat{\alpha}$)	SE($\hat{\alpha}$)	RMSE($\hat{\alpha}$)	Mean($\hat{\beta}$)	Bias($\hat{\beta}$)	SE($\hat{\beta}$)	RMSE($\hat{\beta}$)
	25	1.2996	0.0996	0.2633	0.2815	3.1401	0.1401	0.4861	0.5059
	50	1.2980	0.0980	0.2276	0.2478	3.0733	0.0733	0.3227	0.3309
	100	1.2965	0.0965	0.1531	0.1810	3.0347	0.0347	0.2203	0.2230
	λ = 2					θ = 0.50			
	n	Mean($\hat{\lambda}$)	Bias($\hat{\lambda}$)	SE($\hat{\lambda}$)	RMSE($\hat{\lambda}$)	Mean($\hat{\theta}$)	Bias($\hat{\theta}$)	SE($\hat{\theta}$)	RMSE($\hat{\theta}$)
	25	2.0629	0.0629	0.1195	0.1350	0.5208	0.0208	0.0337	0.0396
	50	2.0420	0.0420	0.0681	`0.0800	0.5203	0.0203	0.0215	0.0269
	100	2.0176	0.0176	0.0524	0.0553	0.5120	0.0120	0.0156	0.0197
cp=0.50	α = 1.20					β = 3			
	n	Mean($\hat{\alpha}$)	Bias($\hat{\alpha}$)	SE($\hat{\alpha}$)	RMSE($\hat{\alpha}$)	Mean($\hat{\beta}$)	Bias($\hat{\beta}$)	SE($\hat{\beta}$)	RMSE($\hat{\beta}$)
	25	1.3028	0.1028	0.2664	0.2855	3.1510	0.1510	0.4869	0.5098
	50	1.3020	0.1020	0.2280	0.2598	3.0843	0.0843	0.3242	0.3350
	100	1.2984	0.0984	0.1990	0.2220	3.0377	0.0377	0.2249	0.2280
	λ = 2					θ = 0.50			
	n	Mean($\hat{\lambda}$)	Bias($\hat{\lambda}$)	SE($\hat{\lambda}$)	RMSE($\hat{\lambda}$)	Mean($\hat{\theta}$)	Bias($\hat{\theta}$)	SE($\hat{\theta}$)	RMSE($\hat{\theta}$)
	25	2.0830	0.0830	0.1211	0.1468	0.5309	0.0309	0.0391	0.0498
	50	2.0478	0.0478	0.0766	0.0903	0.5301	0.0301	0.0234	0.0381
	100	2.0372	0.0372	0.0534	0.0651	0.5221	0.0221	0.0195	0.0295

3.3. Real Data Analysis

In this study, the flexibility of the WBX distribution analyzing a censored dataset is illustrated and the right censored dataset is used. The study compared the fits of the WBX distribution with some of its sub-models. The WBX distribution reduces to the following distributions: the Weibull distribution with two parameters, particularly when $\lambda = \theta = 1$; the Burr Type X (BX) distribution with two parameters when $\alpha = \beta = 1$, the Burr Type X distribution with one parameter (BX1) when $\alpha = \beta = \lambda = 1$, and it reduces to the Rayleigh distribution when $\alpha = \beta = \theta = 1$ [12]. In this application, the unknown parameters of the distributions are estimated by the maximum likelihood method using R language with the aid of 'fitdistrplus' package which was developed by Delignette-Muller and Dutang [21]. This package can be used to find the best fit of a parametric distribution with non-censored or censored data.

The heart transplant dataset has been used as an example of the right censored dataset. The data collected in Stanford California consists of 69 patients receiving heart transplant. This dataset consists of the survival time (T_i) in days (how long patients survive after receiving a heart transplant) and the survival status: 1 = dead, and 0 = alive (censored) [22]. Table 2 presents the heart transplant dataset.

Table 3 displays the summary statistics of the dataset.

According to Table 3, the percentage of the right censored observations in this dataset is 34.78% which indicates a moderate censored percentage. For the comparison of the distributions, some measures of the goodness of fit are calculated including Akaike information criterion (AIC) by Akaike [23], Consistent Akaike information criterion ($CAIC$) by Bozdogan [24], and Bayesian information criterion (BIC) by Schwarz [25]. These measures are given as:

$$AIC = -2l + 2k,$$

$$CAIC = AIC + 2k(k + 1) / (n - k - 1),$$

$$BIC = -2l + kln(n),$$

where k is the number of parameters in the statistical model, n is the sample size and $l(.)$ is the maximized value of the log-likelihood function under the considered model. Whereas, smaller values of these statistics (AIC, BIC, and CAIC) indicate a better fit. Table 4 compares the fit of the WBX distribution with some of its sub-models based on the values of AIC, CAIC and BIC.

Table 2. Heart transplant dataset

(T_i, δ_i)	(T_i, δ_i)	(T_i, δ_i)	(T_i, δ_i)	(T_i, δ_i)	(T_i, δ_i)	(T_i, δ_i)
(5, 1)	(51, 1)	(78, 1)	(165, 1)	(334, 1)	(596, 0)	(995, 1)
(16, 1)	(53, 1)	(80, 1)	(180, 0)	(340, 0)	(630, 0)	(1032, 1)
(16, 1)	(58, 1)	(81, 1)	(186, 1)	(342, 1)	(670, 0)	(1141, 0)
(17, 1)	(61, 1)	(90, 1)	(188, 1)	(370, 0)	(675, 1)	(1321, 0)
(28, 1)	(66, 1)	(96, 1)	(207, 1)	(397, 0)	(733, 1)	(1386, 1)
(30, 1)	(68, 1)	(100, 1)	(219, 1)	(445, 0)	(841, 0)	(1407, 0)
(39, 1)	(68, 1)	(109, 0)	(265, 0)	(482, 0)	(852, 1)	(1571, 0)
(39, 0)	(72, 1)	(110, 1)	(285, 1)	(515, 0)	(915, 0)	(1586, 0)
(43, 1)	(72, 1)	(1316, 0)	(285, 1)	(545, 0)	(941, 0)	(1799, 0)
(45, 1)	(77, 1)	(153, 1)	(308, 1)	(583, 1)	(979, 1)	

δ_i is the censored indicator: 1 is uncensored and 0 is right censored observation

Table 3. Descriptive statistics of heart transplant dataset

n	n.censored	RC%	Mean	Median	Std.Dev.	Min	Max
69.000	24.000	34.7826	209.2460	80.000	314.6738	5.000	1799.000

RC% is the percentage of the right censored observations in the dataset

Table 4. ML estimates, (l): Log-likelihood, SE, k, AIC, CAIC, and BIC for heart transplant dataset

Model	ML Estim.	SE	l	k	AIC	CAIC	BIC
WBX	$\hat{\alpha}$ = 0.264181 $\hat{\beta}$ = 1.702232 $\hat{\lambda}$ = 5.431571 $\hat{\theta}$ = 0.033721	0.086463 0.574340 0.231588 0.008001	-318.012	4	644.024	644.649	652.960
BX	$\hat{\lambda}$ = 9.149871 $\hat{\theta}$ = 0.5734543	7.223631 0.078274	-323.125	2	650.250	650.432	654.718
Weibull	$\hat{\alpha}$ = 644.256851 $\hat{\beta}$ = 0.669053	144.288657 0.082110	-329.066	2	662.132	662.314	666.600
Rayleigh	$\hat{\lambda}$ = 6.00340	5.049200	-332.969	1	667.938	667.998	670.172
BX1	$\hat{\theta}$ = 10.85443	6.555800	-368.022	1	738.044	738.104	740.278

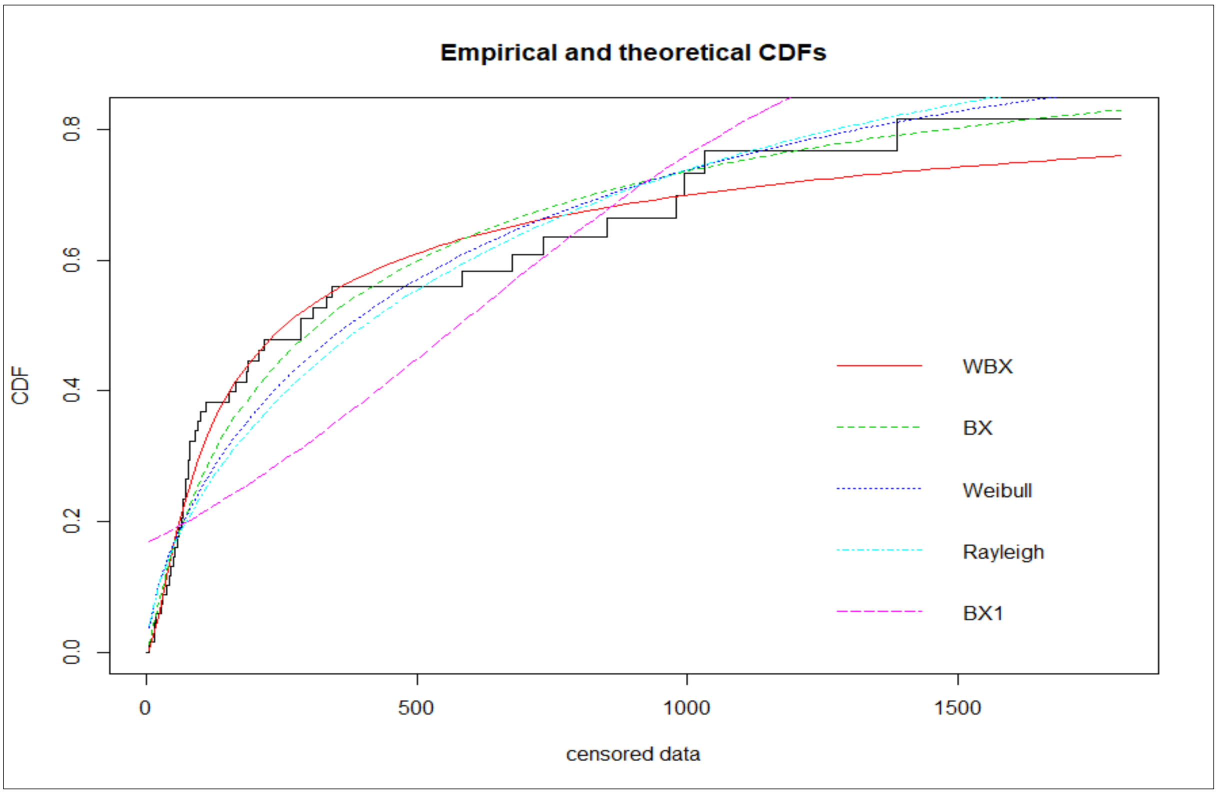

Figure 4. Goodness-of-fit cdf plots for the fits of the WXB, BX, Weibull, Rayleigh, and BX1 distributions for heart transplant censored dataset

Table 5. LRT for heart transplant dataset

Full Model	Reduced Model	H0 versus H1	ψ	d.f	$\chi^2_{df,0.05}$	p-value
WBX	BX	$H0: \alpha = \beta = 1$ $H1: \alpha \neq \beta \neq 1$	10.226	2	5.991	0.00601800
	Weibull	$H0: \lambda = \theta = 1$ $H1: \lambda \neq \theta \neq 1$	22.108	2	5.991	1.582373e-05
	Rayleigh	$H0: \alpha = \beta = \theta = 1$ $H1: \alpha \neq \beta \neq \theta \neq 1$	29.914	3	7.82	1.438753e-06
	BX1	$H0: \alpha = \beta = \lambda = 1$ $H1: \alpha \neq \beta \neq \lambda \neq 1$	100.020	3	7.82	1.538846e-21

According to Table 4, the results show that the WBX distribution has the smallest AIC, CAIC, and BIC values compared to the BX, Weibull, Rayleigh, and BX1 distributions. This indicates that the WBX is the model with the best fit for the heart transplant dataset. Moreover, the quality of the fit can be judged using the goodness of fit cdf plot that can compare the fit of various distributions with the same censored dataset [21]. This is illustrated in Figure 4.

Figure 4 shows that the plot of the cdf of the WBX distribution is very close to the plot of the empirical cdf. Thus, this indicates that WBX model is the best fit for this right censored dataset.

Likelihood ratio statistic (LR) is used to check if the fit using the WBX distribution is statistically "superior" to a fit of sub-distributions for a given dataset. Generally, the hypothesis form of the LR statistic is $H_0 : \gamma = \gamma_0$ versus $H_1 : \gamma \neq \gamma_0$ and LR statistic for testing this hypothesis is computed as, $\psi = -2[l(\hat{\gamma}_0) - l(\hat{\gamma})]$ where the $l(\hat{\gamma}_0)$ is the maximized value of the log-likelihood function of the reduced model under the null hypothesis and $l(\hat{\gamma})$ is the maximized value of the log-likelihood function of the full model under the alternative hypothesis respectively. The null hypothesis is rejected when test statistics (ψ) exceeds the predetermined critical value $\chi^2_{df,\alpha}$ where df (degree of freedom) is the number of parameters in full model minus the number of parameters in the reduced model [26]. Table 5 illustrates the likelihood ratio statistics for testing the goodness of fit for the full model WBX versus the reduced models: BX, Weibull, Rayleigh, and BX1 distributions using the heart transplant dataset.

From Table 5, the results show that all the p-values of the likelihood ratio statistics are less than the significant level $\alpha = 0.05$ and all the test statistics values are greater than its corresponding critical value ${\chi^2}_{df,0.05}$.Therefore, for all the test statistics, the null hypothesis (H_0) are rejected and the alternative hypothesis (H_1) are accepted. The LR results also indicate that the WBX model is a very good fit for the heart transplant dataset.

4. Conclusions

This paper studied the performance of the maximum likelihood estimators for the parameters of the Weibull Burr Type X distribution under right censored dataset and compared under different sample sizes with varying censoring percentages. From the simulation study, the results indicated that the standard error and the root mean squared error of the estimators decrease when the sample size increases, which suggests that the parameters of the Weibull Burr Type X model are stable. Hence, the maximum likelihood estimation is an appropriate method for estimating the parameters of the Weibull Burr Type X model under right censoring times. In the real data analysis using the AIC, CAIC, BIC and the LR test, the results showed that the Weibull Burr Type X distribution outperformed its sub-models in terms of flexibility. It can be deduced from this finding of the fact that this distribution can serve as an alternative model to other models available in the literature for modelling right censored data.

The work on this research was limited to test the algorithm of the Weibull Burr Type X distribution model without covariate. Future work may test the algorithm by adding different types of covariates to the model, such as the fixed and time-dependent covariates. In this way, one can further pursue the regression problem. Moreover, future research can also consider other methods of the parameter estimation for the WBX model such as the Empirical Bayes approach, Bayesian approach, weighted least square approach and the comparisons can be made between these approaches and the classical maximum likelihood approach.

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Conflict of Interest

No conflict of interest was declared.

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