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Ruin Probability for Some Mixed Linear Exponential Family in Classical Risk Process

Khanchit Chuarkham[1], **Arthit Intarasit**[2], **Pakwan Riyapan**[2,*]

[1]Faculty of Commerce and Management, Prince of Songkla University, Thailand
[2]Department of Mathematics and Computer Science, Faculty of Science and Technology, Prince of Songkla University, Thailand

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Abstract This article presents the probability of ruin for the classical risk process by including the density function of claims which satisfies a mixed linear exponential family. This can be defined as

$$f(y) = e^{r(\alpha)y} \sum_{i=1}^{n} b_i \frac{p_i(y)}{q_i(\alpha)}$$

where $y > 0$, $\alpha > 0$, n is a positive integer with $n \geq 2$, $\sum_{i=1}^{n} b_i = 1$ with $0 < b_i < 1$, $p_i(y) = \frac{y^{i-1}}{(i-1)!}$, $q_i(\alpha) = \alpha^{-i}$, and $r(\alpha)$ is the canonical parameter. The main results show that the ordinary differential equation for the probability of ruin in the general case by using chain rule and mathematical induction technique is given in Theorem 2.2, the ordinary differential equation for some mixed linear exponential family when $r(\alpha) = -\alpha, b_1 = b, p_1(y) = 1, q_1(\alpha) = \alpha^{-1}, b_n = 1 - b, p_n(y) = q_n(\alpha) = \alpha^{-n}$ is demonstrated in Theorem 2.3, and an explicit solution for the probability of ruin when the mixed linear exponential family satisfies the conditions which are $n = 2$, $r(\alpha) = -\alpha$, $b_1 = b$ with $0 < b < 1$, and $b_n = 1 - b$ is indicated in Theorem 2.4. Finally, we use MATLAB to generate the numerical simulations for the probability of ruin in the risk process that the number of claims is a Poisson process and the density function of claims satisfies a mixed linear exponential family and a gamma distribution under the conditions of Theorem 2.4 with the parameters $\alpha = 1$ and $b = 0.2$. The numerical results reveal that the relative frequency of the ruin and the ruin probability also satisfy the Lundberg inequality which is the necessary condition for the ruin probability. In addition, the absolute values of its differences are small in order to confirm that the main results are correct.

Keywords Differential Equations, Linear Exponential Family, Risk Process, Ruin Probability

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1 Introduction

In recent years, the ruin theory plays a key role in insurance [1] for describing a weakness of insurer to ruin by using mathematical models. In such models, the ruin probability, distribution of excess immediately before being destroyed surplus immediately prior to ruin, and deficit at time of ruin are described to understand concepts in insurance companies.

In the classical risk model [2], all processes are given in a probability space $(\Omega, \Im, P)$. Now, let $\{U(t)\}_{t\geq 0}$ be the risk process defined as follows:

$$U(0) = u, \quad U(t) = u + ct - S(t) \tag{1}$$

where a constant u is an initial capital with $u \geq 0$, c is a premium rate, $N(t)$ is a number of claim occurring in $[0,t]$, $\{Y_i\}_{i\geq 1}$ is a sequence of iid claims, and $S(t) = \sum_{i=1}^{N(t)} Y_i$ represents the outflow of capital due to payments for claims occurring in [0,t]. Moreover, we assume that Y_i and $N(t)$ are independence for all $i \geq 1$ and $t \geq 0$.

There were many researchers who studied the probability of ruin in the risk process (1). In 2002, Rongming et al. [3] applied the Laplace transform to generate the probability of ruin in the risk model when a non-Poisson process is given by $N(t)$ and the mixed of two exponential density functions is represented by the sequence of claims. Yuanjiang et al.[4] obtained the ordinary differential equation for the probability of non-ruin in the risk model when a Poisson process is presented by $N(t)$ and the Gamma distribution $G(n, \frac{1}{\alpha})$ where n is an integer with $n \geq 2$ and $\alpha > 0$ is given by the sequence of claims. Slaulys et al. [5] demonstrated the Gerber-Shiu discounted penalty function of the risk process when a Poisson process is presented by $N(t)$, and the geometric distribution and the Gamma distribution $G(2, \frac{1}{\alpha})$ where $\alpha > 0$ are described by the sequence of claims. Kwan et al. [6] studied the probability of ruin when the Poisson process is defined by $N(t)$, and the claim size Y_i satisfies the following: Y_i follows a distribution F_1 if T_i is less than a threshold level a and Y_i follows a distribution F_2 if otherwise. An explicit solution for the probability of ruin where F_1 and F_2 are exponential distribution is obtained. Willmot et al. [7] displayed the Gerber-Shiu discounted penalty function of the model using Laplace transform where a renewal process is given by $N(t)$ and the mixed Erlang density functions are presented by the sequence of claims.

A random variable X has a distribution function from **the linear exponential family** if its density function satisfies the following:

$$f(x) = \frac{p(x)e^{r(\theta)x}}{q(\theta)}$$

where $x > 0$, $p(x)$ is the function depending on x, and the functions $r(\theta), q(\theta)$ are the canonical parameter and a normalizing constants depend on a constant θ, respectively. We can see that the exponential or gamma density function of the claim has been getting attention by all authors who studied the probability of ruin and the Gerber-Shiu discounted penalty function in which both functions are the part of the linear exponential family density function.

Therefore, the structure of this article is provided by the following details. We give the main results that are presented by a lemma and three theorems with their proofs in section 2. The numerical simulation is given in section 3. Finally, the conclusions are shown.

2 Main results

Here, we analyze the risk model (1) where a Poisson process is given by $N(t)$ with intensity $\lambda > 0$ and the sequence of iid claims $\{Y_i\}_{i\geq 1}$ has the density function satisfying the mixed linear exponential family which can be expressed as

$$f(y) = e^{r(\alpha)y} \sum_{i=1}^{n} b_i \frac{p_i(y)}{q_i(\alpha)}$$

where $y > 0$, $\alpha > 0$, n is an integer which is $n \geq 2$, $\sum_{i=1}^{n} b_i = 1 (0 < b_i < 1)$, $p_i(y) = \frac{y^{i-1}}{(i-1)!}$, $q_i(\alpha) = \alpha^{-i}$ and $r(\alpha)$ is the canonical parameter. Let F and μ be the distribution function and a mean of Y_1, i.e., $F(y) = P(Y_1 \leq y)$ and $E[Y_1] = \mu$. Let $u \geq 0$ be an initial capital. The **probability of ruin** at time $t > 0$ is represented by

$$\psi(u) = P(U(t) < 0 \text{ for some } t > 0).$$

We call $\Phi(u) = 1 - \psi(u)$ that the **probability of non - ruin** at time t which $\Phi(u) = 0$, if $u < 0$. Let $0 < \rho < 1$ be the safety loading of the insurer. From equation (1), we obtain $ct = (1+\rho)E[S(t)]$. Since $E[S(t)] = \lambda t E[Y_1] = \lambda t \mu$, then $0 < \rho = \frac{c - \lambda\mu}{\lambda\mu}$.

Lemma 2.1. *Suppose that a Poisson process is represented by $N(t)$ with intensity λ. If a claim Y_i has a distribution function $F(y)$, then*

$$D\Phi(u) = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_{0\leq y\leq u} \Phi(u-y)dF(y),$$

where D is the differential operator [8] denoted by $D = \frac{d}{du}$.

Proof. The proof of this Lemma can be followed by the works of Feller [9] (p.183), Grandell [10] (p.4) and Mikosch [11] (p.164). □

Theorem 2.2. *Suppose that a Poisson process is given by $N(t)$ with intensity λ. If $\{Y_i\}_{i\geq 1}$ is the sequence of claims with the density function satisfying the mixed linear exponential family which can be defined as*

$$f(y) = e^{r(\alpha)y}\sum_{i=1}^{n} b_i \frac{p_i(y)}{q_i(\alpha)},$$

where $y > 0$, $\alpha > 0$, n is an integer which is $n \geq 2$, $\sum_{i=1}^{n} b_i = 1 (0 < b_i < 1)$, $p_i(y) = \frac{y^{i-1}}{(i-1)!}$, $q_i(\alpha) = \alpha^{-i}$ and $r(\alpha)$ is the canonical parameter, then the ordinary differential equation for the ruin probability $\psi(u)$ satisfies the following:

$$[D - r(\alpha)]^n D\psi(u) - \frac{\lambda}{c}[D - r(\alpha)]^n \psi(u) + \frac{\lambda}{c}[-r(\alpha)]^n = -\frac{\lambda}{c}\sum_{i=1}^{n}(i-1)!a_i[D - r(\alpha)]^{n-i}[\psi(u) - 1]$$

where $a_i = \frac{b_i\alpha^i}{(i-1)!}$ and D^n is the n^{th} differential operator.

Proof. From Lemma 2.1,

$$D\Phi(u) = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_{0\leq y\leq u}\Phi(u-y)dF(y). \tag{2}$$

Let $a_i = \frac{b_i\alpha^i}{(i-1)!}$, then $f(y) = e^{r(\alpha)y}\sum_{i=1}^{n} a_i y^{i-1}$. Substituting $dF(y) = f(y)dy$ in to equation (2), we get

$$D\Phi(u) = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_{0\leq y\leq u}\Phi(u-y)\sum_{i=1}^{n} a_i y^{i-1} e^{r(\alpha)y}dy. \tag{3}$$

Let $y = u - w$, then equation (3) can be written as follows:

$$D\Phi(u) = \frac{\lambda}{c}\Phi(u) - \frac{\lambda}{c}\int_0^u \Phi(w)\sum_{i=1}^{n} a_i(u-w)^{i-1}e^{r(\alpha)(u-w)}dw. \tag{4}$$

From equation (4), let

$$A(u;n) = \int_0^u \Phi(w)\sum_{i=1}^{n} a_i(u-w)^{i-1}e^{r(\alpha)(u-w)}dw. \tag{5}$$

Now, let $g_n(\zeta(u),\eta(u)) = \int_0^{\zeta(u)}\Phi(w)\sum_{i=1}^{n} a_i(\eta(u)-w)^{i-1}e^{r(\alpha)(\eta(u)-w)}dw$, where $\zeta(u) = u$ and $\eta(u) = u$. Next, we show that

$$\begin{aligned}[D - r(\alpha)]^k g_n(\zeta(u),\eta(u)) &= \sum_{i=1}^{k}(i-1)!a_i[D - r(\alpha)]^{k-i}\Phi(\zeta(u)) \\ &\quad + \int_0^{\zeta(u)}\Phi(w)\sum_{i=k+1}^{n}\frac{(i-1)!}{(i-(k+1))!}a_i(\eta(u)-w)^{i-(k+1)}e^{r(\alpha)(\eta(u)-w)}dw\end{aligned} \tag{6}$$

for $1 \leq k < n$. By mathematical induction, we obtain

$$[D - r(\alpha)]g_n(\zeta(u),\eta(u)) = a_1\Phi(\zeta(u)) + \int_0^{\zeta(u)}\Phi(w)\sum_{i=2}^{n}\frac{(i-1)!}{(i-2)!}a_i(\eta(u)-w)^{i-2}e^{r(\alpha)(\eta(u)-w)}dw.$$

This shows that equation (6) holds for $k = 1$. Next, we assume that equation (6) is true for all $1 < k < n$. Then

$$\begin{aligned}[D - r(\alpha)]^{k+1} g_n(\zeta(u),\eta(u)) &= \sum_{i=1}^{k}(i-1)!a_i[D - r(\alpha)]^{k-i+1}\Phi(\zeta(u)) \\ &\quad + [D - r(\alpha)]\int_0^{\zeta(u)}\Phi(w)\sum_{i=k+1}^{n}\frac{(i-1)!}{(i-(k+1))!}a_i(\eta(u)-w)^{i-(k+1)}e^{r(\alpha)(\eta(u)-w)}dw.\end{aligned} \tag{7}$$

From equation (7), let

$$\begin{aligned}\bar{g}_n(\zeta(u),\eta(u)) &= \int_0^{\zeta(u)} \Phi(w) \sum_{i=k+1}^{n} \frac{(i-1)!}{(i-(k+1))!} a_i(\eta(u)-w)^{i-(k+1)} e^{r(\alpha)(\eta(u)-w)} dw \\ &= \int_0^{\zeta(u)} \Phi(w) \left[\frac{k!}{0!}a_{k+1} + \frac{(k+1)!}{1!}a_{k+2}(\eta(u)-w)^1 + \frac{(k+2)!}{2!}a_{k+3}(\eta(u)-w)^2\right. \\ &\quad + \frac{(k+3)!}{3!}a_{k+4}(\eta(u)-w)^3 + \cdots + \frac{(n-2)!}{(n-1-(k+1))!}a_{n-1}(\eta(u)-w)^{n-1-(k+1)} \\ &\quad \left. + \frac{(n-1)!}{(n-(k+1))!}a_n(\eta(u)-w)^{n-(k+1)}\right] e^{r(\alpha)(\eta(u)-w)} dw. \end{aligned} \tag{8}$$

By the chain rules, we obtain the differentiation of equation (8) as follows:

$$\begin{aligned}D\bar{g}_n(\zeta(u),\eta(u)) &= k!a_{k+1}\Phi(\zeta(u)) + r(\alpha)\bar{g}_n(\zeta(u),\eta(u)) \\ &\quad + \int_0^{\zeta(u)} \Phi(w) \left[(k+1)!a_{k+2} + \frac{(k+2)!}{1!}a_{k+3}(\eta(u)-w)^1\right. \\ &\quad + \frac{(k+3)!}{2!}a_{k+4}(\eta(u)-w)^2 + \cdots + \frac{(n-2)!}{(n-2-(k+1))!}a_{n-1}(\eta(u)-w)^{n-2-(k+1)} \\ &\quad \left. + \frac{(n-1)!}{(n-1-(k+1))!}a_n(\eta(u)-w)^{n-1-(k+1)}\right] e^{r(\alpha)(\eta(u)-w)} dw. \end{aligned} \tag{9}$$

This means that

$$\begin{aligned}[D-r(\alpha)]\bar{g}_n(\zeta(u),\eta(u)) &= k!a_{k+1}\Phi(\zeta(u)) \\ &\quad + \int_0^{\zeta(u)} \Phi(w) \sum_{i=k+2}^{n} \frac{(i-1)!}{(i-(k+2))!} a_i(\eta(u)-w)^{i-(k+2)} e^{r(\alpha)(\eta(u)-w)} dw. \end{aligned} \tag{10}$$

From equations (7), (8), (9) and (10), we get

$$\begin{aligned}[D-r(\alpha)]^{k+1} g_n(\zeta(u),\eta(u)) &= \sum_{i=1}^{k+1} (i-1)!a_i[D-r(\alpha)]^{(k+1)-i}\Phi(\zeta(u)) \\ &\quad + \int_0^{\zeta(u)} \Phi(w) \sum_{i=k+2}^{n} \frac{(i-1)!}{(i-(k+2))!} a_i(\eta(u)-w)^{i-(k+2)} e^{r(\alpha)(\eta(u)-w)} dw. \end{aligned}$$

This proves equation (6). From equations (5) and (6), we have

$$\begin{aligned}[D-r(\alpha)]^k A(u;n) &= \sum_{i=1}^{k} (i-1)!a_i[D-r(\alpha)]^{k-i}\Phi(u) \\ &\quad + \int_0^{u} \Phi(w) \sum_{i=k+1}^{n} \frac{(i-1)!}{(i-(k+1))!} a_i(u-w)^{i-(k+1)} e^{r(\alpha)(u-w)} dw \end{aligned}$$

for $1 \le k < n$. This gets

$$\begin{aligned}[D-r(\alpha)]^n A(u;n) &= [D-r(\alpha)]([D-r(\alpha)]^{n-1}A(u;n)) \\ &= [D-r(\alpha)] \left(\sum_{i=1}^{n-1} (i-1)!a_i[D-r(\alpha)]^{(n-1)-i}\Phi(u)\right. \\ &\quad \left. + (n-1)!a_n \int_0^{u} \Phi(w) e^{r(\alpha)(u-w)} dw\right) \\ &= \sum_{i=1}^{n} (i-1)!a_i[D-r(\alpha)]^{n-i}\Phi(u). \end{aligned} \tag{11}$$

From equations (4), (5) and (11), we obtain

$$\begin{aligned}[D-r(\alpha)]^n D\Phi(u) &= \frac{\lambda}{c}[D-r(\alpha)]^n\Phi(u) \\ &\quad -\frac{\lambda}{c}[D-r(\alpha)]^n\int_0^u \Phi(w)\sum_{i=1}^n a_i(u-w)^{i-1}e^{r(\alpha)(u-w)}dw \\ &= \frac{\lambda}{c}[D-r(\alpha)]^n\Phi(u)-\frac{\lambda}{c}\sum_{i=1}^n (i-1)!a_i[D-r(\alpha)]^{n-i}\Phi(u). \end{aligned} \tag{12}$$

Since $[D-r(\alpha)]^n=\sum_{k=0}^n \frac{n!}{k!(n-k)!}[-r(\alpha)]^k D^{n-k}$ and $\Phi(u)=1-\psi(u)$, then equation (12) satisfies the following equation:

$$[D-r(\alpha)]^n D\psi(u)-\frac{\lambda}{c}[D-r(\alpha)]^n\psi(u)+\frac{\lambda}{c}[-r(\alpha)]^n=-\frac{\lambda}{c}\sum_{i=1}^n (i-1)!a_i[D-r(\alpha)]^{n-i}[\psi(u)-1].$$

This proves Theorem 2.2. □

Theorem 2.3. *Suppose that a Poisson process is given by $N(t)$ with intensity λ. If $\{Y_i\}_{i\geq 1}$ is the sequence of claims with the density function satisfying the mixed linear exponential family which can be defined as follows*

$$f(y)=b\alpha e^{-\alpha y}+(1-b)\alpha^n y^{n-1}e^{-\alpha y}$$

where $y>0$, $\alpha>0$, $0<b<1$ and n is an integer which is $n\geq 2$, then the ordinary differential equation for the probability of ruin $\psi(u)$ is presented by

$$[D+\alpha]^n D\psi(u)-\frac{\lambda}{c}[D+\alpha]^n\psi(u)+\frac{b\lambda\alpha}{c}[D+\alpha]^{n-1}\psi(u)+\frac{(1-b)\lambda\alpha^n}{c}\psi(u)=0$$

where D^n is the n^{th}-order differential operator.

Proof. It's not difficult to see that $r(\alpha)=-\alpha, b_1=b, p_1(y)=1, q_1(\alpha)=\alpha^{-1}$, $b_n=1-b$, $p_n(y)=\frac{y^{n-1}}{(n-1)!}$ and $q_n(\alpha)=\alpha^{-n}$. By Theorem 2.2, we obtain

$$\begin{aligned}[D+\alpha]^n D\psi(u)-\frac{\lambda}{c}[D+\alpha]^n\psi(u)+\frac{\lambda}{c}\alpha^n &= -\frac{\lambda}{c}\left(a_1[D+\alpha]^{n-1}[\psi(u)-1]+(n-1)!a_n[\psi(u)-1]\right) \\ &= -\frac{\lambda}{c}\left(b\alpha[D+\alpha]^{n-1}[\psi(u)-1]+(1-b)\alpha^n[\psi(u)-1]\right) \\ &= -\frac{b\lambda\alpha}{c}[D+\alpha]^{n-1}\psi(u)-\frac{(1-b)\lambda\alpha^n}{c}\psi(u)+\frac{\lambda}{c}\alpha^n. \end{aligned}$$

Therefore,

$$[D+\alpha]^n D\psi(u)-\frac{\lambda}{c}[D+\alpha]^n\psi(u)+\frac{b\lambda\alpha}{c}[D+\alpha]^{n-1}\psi(u)+\frac{(1-b)\lambda\alpha^n}{c}\psi(u)=0.$$

The proof is complete. □

Theorem 2.4. *Suppose that a Poisson process is described by $N(t)$ with intensity λ. If $\{Y_i\}_{i\geq 1}$ is the sequence of claims with the density function satisfying the mixed linear exponential family which can be defined as follows.*

$$f(y)=b\alpha e^{-\alpha y}+(1-b)\alpha^2 ye^{-\alpha y}$$

where $y>0$, $\alpha>0$ and $0<b<1$, then the ordinary differential equation for the probability of ruin $\psi(u)$ can be shown by

$$D^3\psi(u)+\left[2\alpha-\frac{\lambda}{c}\right]D^2\psi(u)+\left[\alpha^2-\frac{(2-b)\lambda\alpha}{c}\right]D\psi(u)=0$$

and

$$\psi(u)=a_1e^{\nu_1 u}+a_2e^{\nu_2 u} \tag{13}$$

where $a_1=\frac{\nu_2(\nu_1+\alpha)^2}{\alpha^2(\nu_2-\nu_1)}, a_2=\frac{\nu_1(\nu_2+\alpha)^2}{\alpha^2(\nu_1-\nu_2)}, \nu_1=\frac{\lambda-2c\alpha-\sqrt{\lambda^2+4(1-b)c\alpha\lambda}}{2c}$ *and* $\nu_2=\frac{\lambda-2c\alpha+\sqrt{\lambda^2+4(1-b)c\alpha\lambda}}{2c}$.

Proof. By Theorem 2.3, we obtain

$$[D+\alpha]^2 D\psi(u) - \frac{\lambda}{c}[D+\alpha]^2\psi(u) + \frac{b\lambda\alpha}{c}[D+\alpha]\psi(u) + \frac{(1-b)\lambda\alpha^2}{c}\psi(u) = 0.$$

Obviously,

$$D^3\psi(u) + \left[2\alpha - \frac{\lambda}{c}\right] D^2\psi(u) + \left[\alpha^2 - \frac{(2-b)\lambda\alpha}{c}\right] D\psi(u) = 0. \tag{14}$$

Let $\psi(u) = a_0 e^{\nu_0 u} + a_1 e^{\nu_1 u} + a_2 e^{\nu_2 u}$ be the general solution of equation (14), where a_0, a_1, a_2 are constants and ν_0, ν_1, ν_2 are the solution of a characteristic equation

$$\nu^3 + \left[2\alpha - \frac{\lambda}{c}\right]\nu^2 + \left[\alpha^2 - \frac{(2-b)\lambda\alpha}{c}\right]\nu = 0.$$

Then, we have $\nu\left(\nu^2 + \left[2\alpha - \frac{\lambda}{c}\right]\nu + \left[\alpha^2 - \frac{(2-b)\lambda\alpha}{c}\right]\right) = 0$. So that $\nu_0 = 0$,

$$\nu_1 = \frac{-2\alpha + \frac{\lambda}{c} - \sqrt{\left[2\alpha - \frac{\lambda}{c}\right]^2 - 4\left[\alpha^2 - \frac{(2-b)\lambda\alpha}{c}\right]}}{2} = \frac{\lambda - 2c\alpha - \sqrt{\lambda^2 + 4(1-b)c\alpha\lambda}}{2c}$$

and

$$\nu_2 = \frac{-2\alpha + \frac{\lambda}{c} + \sqrt{\left[2\alpha - \frac{\lambda}{c}\right]^2 - 4\left[\alpha^2 - \frac{(2-b)\lambda\alpha}{c}\right]}}{2} = \frac{\lambda - 2c\alpha + \sqrt{\lambda^2 + 4(1-b)c\alpha\lambda}}{2c}.$$

Next, we will show that $\nu_1 < 0$ and $\nu_2 < 0$. It's suffice to show that $\nu_2 < 0$. Since $\frac{c-\lambda\mu}{\lambda\mu} = \rho > 0$ and $\mu = \frac{2-b}{\alpha}$, then $c - \frac{(2-b)\lambda}{\alpha} = c - \lambda\mu > 0$. That is $\alpha^2 c > (2-b)\lambda\alpha$. Obviously, $\alpha^2 - \frac{(2-b)\lambda\alpha}{c} > 0$. Hence, $-4\left[\alpha^2 - \frac{(2-b)\lambda\alpha}{c}\right] < 0$. This gives the following expression:

$$0 \le \left[2\alpha - \frac{\lambda}{c}\right]^2 - 4\left[\alpha^2 - \frac{(2-b)\lambda\alpha}{c}\right] < \left[2\alpha - \frac{\lambda}{c}\right]^2.$$

Thus, $0 \le \sqrt{\left[2\alpha - \frac{\lambda}{c}\right]^2 - 4\left[\alpha^2 - \frac{2\lambda\alpha}{c} + \frac{b\lambda\alpha}{c}\right]} < 2\alpha - \frac{\lambda}{c}$. It's clear that $\nu_2 < 0$. Let $u \to \infty$. From above, we obtain $a_0 = \psi(\infty) = 0$. Finally, we try to show that $a_1 = \frac{\nu_2(\nu_1+\alpha)^2}{\alpha^2(\nu_2-\nu_1)}$ and $a_2 = \frac{\nu_1(\nu_2+\alpha)^2}{\alpha^2(\nu_1-\nu_2)}$. From equation (4) and the general solution of equation (14), we get

$$\begin{aligned}
a_1\nu_1 e^{\nu_1 u} + a_2\nu_2 e^{\nu_2 u} &= -\frac{\lambda}{c}(1 - a_1 e^{\nu_1 u} - a_2 e^{\nu_2 u}) \\
&\quad + \frac{\lambda}{c}\int_0^u (1 - a_1 e^{\nu_1 w} - a_2 e^{\nu_2 w})\left[b\alpha + (1-b)\alpha^2(u-w)\right] e^{-\alpha(u-w)} dw \\
&= -\frac{\lambda}{c}(1 - a_1 e^{\nu_1 u} - a_2 e^{\nu_2 u}) \\
&\quad + \frac{\lambda}{c} e^{-\alpha u}\int_0^u \left[b\alpha + (1-b)\alpha^2(u-w)\right] e^{\alpha w} dw \\
&\quad - \frac{\lambda}{c} a_1 e^{-\alpha u}\int_0^u \left[b\alpha + (1-b)\alpha^2(u-w)\right] e^{(\alpha+\nu_1)w} dw \\
&\quad - \frac{\lambda}{c} a_2 e^{-\alpha u}\int_0^u \left[b\alpha + (1-b)\alpha^2(u-w)\right] e^{(\alpha+\nu_2)w} dw \\
&= -\frac{\lambda}{c}\left(1 - a_1 e^{\nu_1 u} - a_2 e^{\nu_2 u}\right) + \frac{\lambda}{c} e^{-\alpha u}\left(e^{\alpha u} - [1-b]\alpha u - 1\right) \\
&\quad - \frac{\lambda}{c} a_1 e^{-\alpha u}\left(\left[\frac{b\alpha}{\nu_1+\alpha} + \frac{(1-b)\alpha^2}{(\nu_1+\alpha)^2}\right] e^{(\nu_1+\alpha)u} - \frac{(1-b)\alpha^2 u}{\nu_1+\alpha} - \frac{b\alpha}{\nu_1+\alpha} - \frac{(1-b)\alpha^2}{(\nu_1+\alpha)^2}\right) \\
&\quad - \frac{\lambda}{c} a_2 e^{-\alpha u}\left(\left[\frac{b\alpha}{\nu_2+\alpha} + \frac{(1-b)\alpha^2}{(\nu_2+\alpha)^2}\right] e^{(\nu_2+\alpha)u} - \frac{(1-b)\alpha^2 u}{\nu_2+\alpha} - \frac{b\alpha}{\nu_2+\alpha} - \frac{(1-b)\alpha^2}{(\nu_2+\alpha)^2}\right).
\end{aligned}$$

Hence,

$$\left(\nu_1 - \frac{\lambda}{c} + \frac{\lambda}{c}\left[\frac{b\alpha}{\nu_1+\alpha} + \frac{(1-b)\alpha^2}{(\nu_1+\alpha)^2}\right]\right) a_1 e^{\nu_1 u} + \left(\nu_2 - \frac{\lambda}{c} + \frac{\lambda}{c}\left[\frac{b\alpha}{\nu_2+\alpha} + \frac{(1-b)\alpha^2}{(\nu_2+\alpha)^2}\right]\right) a_2 e^{\nu_2 u}$$
$$= \frac{\lambda}{c}\left[\left([b-1]\alpha + \frac{(1-b)\alpha^2}{\nu_1+\alpha}a_1 + \frac{(1-b)\alpha^2}{\nu_2+\alpha}a_2\right) u + \left(\frac{b\alpha}{\nu_1+\alpha} + \frac{(1-b)\alpha^2}{(\nu_1+\alpha)^2}\right) a_1 + \left(\frac{b\alpha}{\nu_2+\alpha} + \frac{(1-b)\alpha^2}{(\nu_2+\alpha)^2}\right) a_2 - 1\right] e^{-\frac{\alpha}{b}u}. \tag{15}$$

Since $\nu_1 - \frac{\lambda}{c} + \frac{\lambda}{c}\left[\frac{b\alpha}{\nu_1+\alpha} + \frac{(1-b)\alpha^2}{(\nu_1+\alpha)^2}\right] = 0$ and $\nu_2 - \frac{\lambda}{c} + \frac{\lambda}{c}\left[\frac{b\alpha}{\nu_2+\alpha} + \frac{(1-b)\alpha^2}{(\nu_2+\alpha)^2}\right] = 0$, then we consider all terms the right hand side in equation (15) satisfying the following expressions:

$$[b-1]\alpha + \frac{(1-b)\alpha^2}{\nu_1+\alpha}a_1 + \frac{(1-b)\alpha^2}{\nu_2+\alpha}a_2 = 0 \text{ and}$$
$$\left(\frac{b\alpha}{\nu_1+\alpha} + \frac{(1-b)\alpha^2}{(\nu_1+\alpha)^2}\right) a_1 + \left(\frac{b\alpha}{\nu_2+\alpha} + \frac{(1-b)\alpha^2}{(\nu_2+\alpha)^2}\right) a_2 - 1 = 0. \tag{16}$$

From the previous expressions in (16), we solve them in order to find a_1 and a_2. This gives $a_1 = \frac{\nu_2(\nu_1+\alpha)^2}{\alpha^2(\nu_2-\nu_1)}$ and $a_2 = \frac{\nu_1(\nu_2+\alpha)^2}{\alpha^2(\nu_1-\nu_2)}$. The proof is now complete. □

3 The Simulation Study

Here, we show simulations for the probability of ruin in the risk process (1), where the number of claim is a Poisson process given by $N(t)$ with intensity $\lambda = 1$ and the sequence of claims $\{Y_i\}_{i\geq 1}$ follows Theorem 2.4. We begin by choosing $\alpha = 1, b = 0.2$ and a safety loading of the insurer $\rho = 0.3$ and 0.5, resulting in $c = 2.34, 2.70$, respectively. This brings about $\{\nu_1, \nu_2\} = \{-1.4088, -0.1638\}, \{-1.3898, -0.2398\}$ and $\{a_1, a_2\} = \{-0.0220, 0.7912\}, \{-0.0317, 0.6984\}$, respectively. Moreover, we know that *the Lundberg inequality* is the necessary condition for the probability of ruin $\psi(u)$,i.e.,

$$\psi(u) \leq e^{-Ru}$$

where R is *the Lundberg exponent* which is the positive solution of

$$\int_0^\infty e^{Ry} dF(y) - 1 = \frac{cR}{\lambda} = cR.$$

Hence,

$$\int_0^\infty e^{Ry} f(y) dy - 1 = cR. \tag{17}$$

Substituting the density function of claim $f(y) = 0.2e^{-y} + 0.8ye^{-y}$ $(y > 0)$ into equation (17), we obtain

$$\int_0^\infty [0.2 + 0.8y]\, e^{-(1-R)y} dy - 1 = cR.$$

Obviously, the integration holds for $0 < R < 1$ and

$$\frac{0.2}{1-R} + \frac{0.8}{(1-R)^2} - 1 = cR.$$

An equivalent form is $cR^2 + (1-2c)R + (c-1.8) = 0$. For $c = 2.34$, this results in $R = 0.1638$ and for $c = 2.70$ we get $R = 0.239845$.

Next, we find out a level for slimming the ruin from the simulation. Since $e^{-0.239845u} < e^{-0.1638u}$ for all $u > 0$, then it's suffice to choose $R = 0.1638$ for the Lundberg inequality. Given that $u > 80$, we get

$$\psi(u) \leq e^{-0.1638\times 80} = 2.03707 \times 10^{-6}$$

which it's sufficiently small. So, 80 is a suitable level for fitting the shape of the ruin from the simulation. That is, if the ruin occurs after the total capital is more than 80, we will obtain a few ruins. This means that the ruin does not occur if the total capital falls between 0 to 80.

Let reconsider the density function of claim

$$f(y) = 0.2e^{-y} + 0.8ye^{-y} \text{ for } y > 0,$$

then the distribution function of claim is

$$F(y) = 0.2F_{expo}(y) + 0.8F_{gamma}(y) \tag{18}$$

where F_{expo}, F_{gamma} satisfy an exponential distribution $Exp(1)$ and a gamma distribution $G(2, 1)$, respectively. So, the random variate of claim is sampled by the concept of the last equation.

For this simulation, MATLAB2018 on PC is used for establishing the risk process $10,000$ times and computing the ruin probability. MATLAB code and the results are shown as follows: Code;

```
u = 3; r = 0; nr = 0; f = 0; E = 0; lambda = 1; alpha = 1;
b = 0.2; R = 0.1638; selfty = 0.3; mean = (2 - b)/alpha;
c = (1 + selfty) * mean;
v1 = (-2 * alpha + (lambda/c) - sqrt((2 * alpha -
     (lambda/c))^2 - 4 * (alpha^2 - ((2 - b) * lambda * alpha)/c)))/2;
v2 = (-2 * alpha + (lambda/c) + sqrt((2 * alpha -
     (lambda/c))^2 - 4 * (alpha^2 - ((2 - b) * lambda * alpha)/c)))/2;
a1 = (v2 * (v1 + alpha)^2)/((alpha^2) * (v2 - v1));
a2 = (v1 * (v2 + alpha)^2)/((alpha^2) * (v1 - v2));
Q = a1 * exp(v1 * u) + a2 * exp(v2 * u);
option = optimoptions('fsolve','display','off');
% ================ u = 3 ================ %
tic
for i = 1 : 10000
F = rand(1);
Y = fsolve(@(x)0.2 * cdf('Exponential', x, 1)
    + 0.8 * cdf('Gamma', x, 2, 1) - F, 0, option);
U = u + c * exprnd(lambda) - Y;
while (U > 0)&&(U < 80)
F = rand(1);
Y = fsolve(@(x)0.2 * cdf('Exponential', x, 1)
    + 0.8 * cdf('Gamma', x, 2, 1) - F, 0, option);
U = U + c * exprnd(lambda) - Y;
end
toc
if U < 0
r = r + 1;
else if U >= 80
nr = nr + 1;
end
end
end
% ================ End U ================ %
r;
nr;
f = r/(r + nr);
E = f - Q;
T = exp(-R * u);
% ====================================== %
fprintf('c = %.2f, v1 = %.4f, v2 = %.4f, a1 = %.4f, a2 = %.4f\n', c, v1, v2, a1, a2)
disp(' ')
disp('========================================')
disp('c     u     r     f     Q     T     E')
disp('========================================')
```

c	u	r	f	Q	T	E
2.34	1	6699	0.6699	0.666302807	0.848911787	0.003597193
2.34	3	4840	0.4840	0.483724731	0.611769317	0.000275269
2.34	5	3527	0.3527	0.348809244	0.440872306	0.003890756
2.34	10	1442	0.1442	0.153788981	0.194368391	-0.009588981
2.34	15	660	0.0660	0.067801400	0.085691641	-0.001801400
2.34	20	277	0.0277	0.029891799	0.037779071	-0.002191799
2.34	25	127	0.0127	0.013178484	0.016655746	-0.000478484
2.34	30	57	0.0057	0.005810036	0.007343057	-0.000110036
2.34	35	22	0.0022	0.002561487	0.003237351	-0.000361487
2.34	40	7	0.0007	0.001129290	0.001427258	-0.000429290
2.34	50	1	0.0001	0.000219499	0.000277414	-0.000119499
2.34	60	1	0.0001	0.000042664	0.000053920	0.000057336
2.34	70	0	0.0000	0.000008293	0.000010480	-0.000008293
2.34	80	0	0.0000	0.000001612	0.000002037	-0.000001612
..	..	..	..	..	..	..
2.70	1	5436	0.5436	0.541536188	0.786749798	0.002063812
2.70	3	3374	0.3374	0.339593810	0.486978648	-0.002193810
2.70	5	2123	0.2123	0.210472876	0.301427728	0.001827124
2.70	10	611	0.0611	0.063451377	0.090858675	-0.002351377
2.70	15	169	0.0169	0.019125977	0.027387324	-0.002225977
2.70	20	54	0.0054	0.005765089	0.008255299	-0.000365089
2.70	25	18	0.0018	0.001737754	0.002488376	0.000062246
2.70	30	6	0.0006	0.000523806	0.000750066	0.000076194
2.70	35	3	0.0003	0.000157889	0.000226091	0.000142111
2.70	40	0	0.0000	0.000047592	0.000068150	-0.000047592
2.70	50	0	0.0000	0.000004324	0.000006192	-0.000004324
2.70	60	0	0.0000	0.000000393	0.000000563	-0.000000393
2.70	70	0	0.0000	0.000000036	0.000000051	-0.000000036
2.70	80	0	0.0000	0.000000003	0.000000005	-0.000000003

Table 1. Approximate r, f, Q, T and E.

```
disp(' ')
fprintf('1.%.2f%.0f%.0f%.4f%.9f%.9f%.9f\n',c,u,r,f,Q,T,E);
disp('=========================================')
run
c = 2.34, v1 = -1.4088, v2 = -0.1638, a1 = -0.0220, a2 = 0.7912
=========================================
  c   u   r    f       Q          T          E
=========================================
1. 2.34  3  4879  0.4879  0.483724731  0.611769317  0.004175269
=========================================
```

The numerical results in Table 1 show the number of ruin r satisfying risk process (1); the relative frequency of ruin f; the ruin probability Q as mentioned in equation (13) where given $\lambda = 1, \alpha = 1, b = 0.2$; $T = e^{-Ru}$ and $E = f - Q$ in the case of the premium rate $c = 2.34, 2.70$, respectively. As shown in the table, we can see that the relative frequency for ruin and the probability of ruin also satisfy $the\ Lundberg\ inequality$. Moreover, the absolute values of its difference have a small value which confirms that Theorem 2.4 is correct.

4 Conclusions

In this paper, we have presented the ordinary differential equation for ruin probability when the mixed linear exponential family is included by the density of claims. In specific cases, this paper proposes a close form for the probability of ruin when the density of claims is mixed between the exponential and gamma functions as a part of the mixed linear exponential family. This is different from the past where the density function would be exponential or gamma. The simulation of the specific case is also presented when $n = 2$.

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