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Bipolar Soft Limit Points in Bipolar Soft Generalized Topological Spaces

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Abstract The concept of soft set theory can be used as a mathematical tool for dealing with problems that contain uncertainty. Then, a new mixed mathematical model called the bipolar soft set is created by merging soft sets and bipolarity, which gave the concept of a binary model of grading. Bipolar soft set is characterized by two soft sets, one of which provides positive information and the other negative. Bipolar soft generalized topology is a generalization of bipolar soft topology. The importance of limit points in all branches of mathematics cannot be ignored. It forms one of the most significant and fundamental concepts in topology. On this basis, the derived set concept is required in the establishment and continuation of some properties. Accordingly, the limit point in bipolar soft generalized theory is defined. In this paper, we present the notion of bipolar soft generalized limit points. We explained the relation between the bipolar soft generalized derived and the bipolar soft generalized closure set. Added to that, we discussed some structures of a bipolar soft generalized topological space such as: $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-interior point, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-exterior point, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-boundary point, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-neighborhood point and basis on $\mathcal{BSGTS}$. Finally, we give comparisons among these concepts of bipolar soft generalized topological spaces ($\mathcal{BSGTS}$) by using bipolar soft point ($\mathcal{BSP}$). Each concept introduced in this paper is explained with clear examples.

Keywords $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-interior point, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-limit point, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-boundary point $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-neighborhood

1 Introduction

Reference to the set theory highlights it as a fundamental theory in mathematics that ranges from engineering to medical, and from medical to social field. Researchers in mathematics and other subjects have suggested theories such as the fuzzy set theory [2], rough set theory [3], game theory, soft set theory and graph theory to solve such problems. In 1999, [1] introduced the soft set as a novel mathematical strategy for dealing with ambiguity. Soft set theory has a wide range of potential applications. Many interesting applications of soft set theory can be found in [4] and [5]. Maji et al. [6] and Ali et al. [7] developed soft set operations and some concepts of soft sets. Soft set theory is becoming increasingly popular; (see [8]-[12]).

In 2014, Thomas and John [13] introduced a new soft topological space called soft generalized topological spaces ($\mathcal{SGTS}$) and studied some types of soft spaces such as soft compact spaces and soft separation axioms via soft generalized open sets. The generalized topology is different from topology by its axioms. According to Császár, a collection of subsets of $\mathcal{U}$ is a generalized topology on $\mathcal{U}$ if it contains the empty set and arbitrary union of its members. But the soft generalized topological spaces are based on soft sets theory and not sets.

In 2011, Shabir and Naz [14] defined the concept of soft topological spaces. The topological properties, characteristics, and results in soft topologies have since been submitted by a number of researchers; (see [15]-[27]).

In 2013, Shabir and Naz [28] explained that the bipolar soft set structure has clearer and more general results than the soft set structure. A year later, in 2014, the notion of soft generalized topological space was put forward by Thomas and John [13], who defined it as an initial universe with a fixed set of parameters containing the soft union of any soft sets and soft null sets. There have been different studies on the concept of bipolar soft structures (see [30]-[41]), yet studies on the limit point concept were required by mathematicians so as to bring about more developments in mathematics.

Bipolarity is significant for characterization between positive and negative information for excellence which is to be sensible occurrence and fail it. Öztürk [42] introduced the point concept on bipolar soft set structure, fundamental properties for this point concept are presented. In 2022, Saleh, Asaad and Mohammed [43] defined a bipolar soft generalized topological space ($\mathcal{BSGTS}$) and they introduced some topological structures such as $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-boundary sets.

In this study, the concept of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior point over an initial universe with fixed set of parameters has been firstly introduced. This is followed by the presentation of the main results of $\mathcal{BSGTS}$ on an initial $\mathcal{BSS}$. Then, the concepts of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure ($\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-adherent) point, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior point, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-boundary point, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-neighborhood point have been investigated along the investigation of their results and the presentation of the relation between such notions. Finally, bipolar soft generalized basis ($\mathcal{BSGB}$) and bipolar soft generalized topological subspace ($\mathcal{BSGTSS}$) have been explained. To support the results, the properties with examples have been introduced.

2 Preliminaries

In this section, we introduce some basic concepts about bipolar soft sets and bipolar soft points. Throughout the present paper, $\Upsilon(\Omega)$ be the class of all subsets of an initial universe Ω. Let ϖ be a set of parameters and $\varrho, \sigma \subseteq \varpi$. Let $\mathcal{BSS}(\Omega)$ be the set of all bipolar soft sets over Ω with parameters ϖ. We recall some definitions and results that can be used in the sequel.

Definition 2.1 *[6] The* ***Not*** *set of a set of parameters $\varrho = \{\varsigma_1, \varsigma_2, ..., \varsigma_n\}$ is denoted by $\neg\varrho$ and is defined as $\neg\varrho = \{\neg\varsigma_1, \neg\varsigma_2, ..., \neg\varsigma_n\}$ where $\neg\varsigma_i$ = Not ς_i for all $i = 1, 2, ..., n$.*

Definition 2.2 *[28] A triple $(\Lambda, \Theta, \varrho)$ is said to be a bipolar soft set on Ω, denoted by $\mathcal{BSS}$, where Λ and Θ are mappings defined by $\Lambda : \varrho \longrightarrow \Upsilon(\Omega)$ and $\Theta : \neg\varrho \longrightarrow \Upsilon(\Omega)$ in which $\Lambda(\varsigma) \cap \Theta(\neg\varsigma) = \phi$ for all $\varsigma \in \varrho$ and $\neg\varsigma \in \neg\sigma$.*
In other words, a $\mathcal{BSS}$ $(\Lambda, \Theta, \varrho)$ can be written as:

$$(\Lambda, \Theta, \varrho) = \{(\varsigma, \Lambda(\varsigma), \Theta(\neg\varsigma)) : \varsigma \in \varrho, \Lambda(\varsigma) \cap \Theta(\neg\varsigma) = \phi\}.$$

Definition 2.3 *[28] Let $(\Lambda_1, \Theta_1, \varrho)$ and $(\Lambda_2, \Theta_2, \sigma)$ be two $\mathcal{BSS}s$, then we say that $(\Lambda_1, \Theta_1, \varrho)$ is a bipolar soft subset of $(\Lambda_2, \Theta_2, \sigma)$, denoted by $(\Lambda_1, \Theta_1, \varrho) \mathrel{\widetilde{\widetilde{\subseteq}}} (\Lambda_2, \Theta_2, \varrho)$, if:*

1. *$\varrho \subseteq \sigma$,*
2. *$\Lambda_1(\varsigma) \subseteq \Lambda_2(\varsigma)$ and $\Theta_2(\neg\varsigma) \subseteq \Theta_1(\neg\varsigma)$ for all $\varsigma \in \varrho$ and $\neg\varsigma \in \neg\varrho$.*

Similarly, we say that $(\Lambda_1, \Theta_1, \varrho)$ is a bipolar soft superset of $(\Lambda_2, \Theta_2, \sigma)$, denoted by $(\Lambda_1, \Theta_1, \varrho) \mathrel{\widetilde{\widetilde{\supseteq}}} (\Lambda_2, \Theta_2, \sigma)$, if $(\Lambda_2, \Theta_2, \sigma)$ is a bipolar soft subset of $(\Lambda_1, \Theta_1, \varrho)$.

Definition 2.4 *[28] Two $\mathcal{BSS}s$ $(\Lambda_1, \Theta_1, \varrho)$ and $(\Lambda_2, \Theta_2, \sigma)$ are called equal, which is denoted by $(\Lambda_1, \Theta_1, \varrho) = (\Lambda_2, \Theta_2, \sigma)$, if $(\Lambda_1, \Theta_1, \varrho) \mathrel{\widetilde{\widetilde{\subseteq}}} (\Lambda_2, \Theta_2, \varrho)$ and $(\Lambda_2, \Theta_2, \varrho) \mathrel{\widetilde{\widetilde{\subseteq}}} (\Lambda_1, \Theta_1, \varrho)$.*

Definition 2.5 *[28] The complement of a $\mathcal{BSS}$ $(\Lambda, \Theta, \varrho)$ is denoted by $(\Lambda, \Theta, \varrho)^c$ and defined by $(\Lambda, \Theta, \varrho)^c = (\Lambda^c, \Theta^c, \varrho)$ where Λ^c and Θ^c are mappings having $\Lambda^c(\varsigma) = \Theta(\neg\varsigma)$ and $\Theta^c(\neg\varsigma) = \Lambda(\varsigma)$ for all $\varsigma \in \varrho$ and $\neg\varsigma \in \neg\varrho$.*

Definition 2.6 *[28] A $\mathcal{BSS}$ $(\Lambda, \Theta, \varrho)$ is called a relative null bipolar soft set, which is denoted by $(\Phi, \widetilde{\widetilde{\Omega}}, \varrho)$, if $\Lambda(\varsigma) = \phi$ and $\Theta(\neg\varsigma) = \Omega$ for all $\varsigma \in \varrho$ and for all $\neg\varsigma \in \neg\varrho$.*

Definition 2.7 *[28] A $\mathcal{BSS}$ $(\Lambda, \Theta, \varrho)$ is called a relative absolute bipolar soft set, which is denoted by $(\widetilde{\widetilde{\Omega}}, \Phi, \varrho)$, if $\Lambda(\varsigma) = \Omega$ and $\Theta(\neg\varsigma) = \phi$ for all $\varsigma \in \varrho$ and for all $\neg\varsigma \in \neg\varrho$.*

Definition 2.8 *[28] Let $(\Lambda_1, \Theta_1, \varrho)$ and $(\Lambda_2, \Theta_2, \sigma)$ be two $\mathcal{BSS}s$, then the bipolar soft intersection of these $\mathcal{BSS}s$ is the $\mathcal{BSS}$ (χ, Ψ, κ) where $\kappa = \varrho \cup \sigma$ is a non-empty set and for all $\varsigma \in \kappa$, we have*

$$\chi(\varsigma) = \begin{cases} \Lambda_1(\varsigma), & \varsigma \in \varrho - \sigma, \\ \Lambda_2(\varsigma), & \varsigma \in \sigma - \varrho, \\ \Lambda_1(\varsigma) \cap \Lambda_2(\varsigma), & \varsigma \in \varrho \cap \sigma \end{cases}$$

and

$$\Psi(\neg\varsigma) = \begin{cases} \Lambda_1(\neg\varsigma), & \neg\varsigma \in \neg\varrho - \neg\sigma, \\ \Lambda_2(\neg\varsigma), & \neg\varsigma \in \neg\sigma - \neg\varrho, \\ \Lambda_1(\neg\varsigma) \cup \Lambda_2(\neg\varsigma), & \neg\varsigma \in \neg\varrho \cap \neg\sigma. \end{cases}$$

It is denoted by $(\Lambda_1, \Theta_1, \varrho) \mathrel{\widetilde{\widetilde{\cap}}} (\Lambda_2, \Theta_2, \sigma) = (\chi, \Psi, \kappa)$.

Definition 2.9 *[28] Let $(\Lambda_1, \Theta_1, \varrho)$ and $(\Lambda_2, \Theta_2, \sigma)$ be two $\mathcal{BSS}s$, then the bipolar soft union of these $\mathcal{BSS}s$ is the $\mathcal{BSS}$ (χ, Ψ, κ) where $\kappa = \varrho \cup \sigma$ is a non-empty set and for all $\varsigma \in \kappa$, we have*

$$\chi(\varsigma) = \begin{cases} \Lambda_1(\varsigma), & \varsigma \in \varrho - \sigma, \\ \Lambda_2(\varsigma), & \varsigma \in \sigma - \varrho, \\ \Lambda_1(\varsigma) \cup \Lambda_2(\varsigma), & \varsigma \in \varrho \cap \sigma \end{cases}$$

and

$$\Psi(\neg\varsigma) = \begin{cases} \Lambda_1(\neg\varsigma), & \neg\varsigma \in \neg\varrho - \neg\sigma, \\ \Lambda_2(\neg\varsigma), & \neg\varsigma \in \neg\sigma - \neg\varrho, \\ \Lambda_1(\neg\varsigma) \cap \Lambda_2(\neg\varsigma), & \neg\varsigma \in \neg\varrho \cap \neg\sigma. \end{cases}$$

It is denoted by $(\Lambda_1, \Theta_1, \varrho) \mathrel{\widetilde{\widetilde{\cup}}} (\Lambda_2, \Theta_2, \sigma) = (\chi, \Psi, \kappa)$.

Definition 2.10 *[28] The restricted union of two $\mathcal{BSS}s$ $(\Lambda_1, \Theta_1, \varrho)$ and $(\Lambda_2, \Theta_2, \sigma)$ over the common universe Ω is the $\mathcal{BSS}$ (χ, ψ, κ) where $\kappa = \varrho \cap \sigma$ is a non-empty set and for all $\varsigma \in \kappa$, we have*

$$\chi(\varsigma) = \Lambda_1(\varsigma) \cup \Lambda_2(\varsigma) \text{ and } \psi(\neg\varsigma) = \Theta_1(\neg\varsigma) \cap \Theta_2(\neg\varsigma).$$

It is denoted by $(\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\cup}}_{\mathcal{R}}\, (\Lambda_2, \Theta_2, \sigma) = (\chi, \psi, \kappa)$.

Definition 2.11 *[28] The restricted intersection of two* $\mathcal{BSS}s$ $(\Lambda, \Theta_1, \varrho)$ *and* $(\Lambda_2, \Theta_2, \sigma)$ *over the common universe* Ω *is the* $\mathcal{BSS}$ (χ, ψ, κ) *where* $\kappa = \varrho \cap \sigma$ *is a non-empty set and for all* $\varsigma \in \kappa$*, we have*

$$\chi(\varsigma) = \Lambda_1(\varsigma) \cap \Lambda_2(\varsigma) \text{ and } \psi(\neg\varsigma) = \Theta_1(\neg\varsigma) \cup \Theta_2(\neg\varsigma).$$

It is denoted by $(\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\cap}}_{\mathcal{R}}\, (\Lambda_2, \Theta_2, \sigma) = (\chi, \psi, \kappa)$.

Definition 2.12 *[28] Let* $\pi \in \Omega$*. Then* $(\Lambda_\pi(\varsigma), \Theta_\pi(\varsigma), \varpi)$ *is denoted the* $\mathcal{BSS}$*, defined by* $\Lambda_\pi(\varsigma)=\{\pi\}$ *and* $\Theta_\pi(\neg\varsigma) = \Omega \setminus \{\pi\}$*, for each* $\varsigma \in \varpi$.

Proposition 2.1 *[28] If* $(\Lambda_1, \Theta_1, \varrho), (\Lambda_2, \Theta_2, \sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$*, then:*

1. $((\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\cup}}\, (\Lambda_2, \Theta_2, \sigma))^c = (\Lambda_1, \Theta_1, \varrho)^c \,\widetilde{\widetilde{\cap}}\, (\Lambda_2, \Theta_2, \sigma)^c$.
2. $((\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\cap}}\, (\Lambda_2, \Theta_2, \sigma))^c = (\Lambda_1, \Theta_1, \varrho)^c \,\widetilde{\widetilde{\cup}}\, (\Lambda_2, \Theta_2, \sigma)^c$.
3. $((\Lambda_1, \Theta_1, \varrho)^c)^c = (\Lambda_1, \Theta_1, \varrho)$.
4. $$\begin{aligned}(\Phi, \widetilde{\widetilde{\Omega}}, \varrho) &\,\widetilde{\widetilde{\subseteq}}\, (\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\cap}}\, (\Lambda_2, \Theta_2, \varrho)^c \\ &\,\widetilde{\widetilde{\subseteq}}\, (\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\cup}}\, (\Lambda_2, \Theta_2, \varrho)^c \\ &\,\widetilde{\widetilde{\subseteq}}\, (\widetilde{\widetilde{\Omega}}, \Phi, \varrho).\end{aligned}$$

Definition 2.13 *[36] Let* $(\Lambda_1, \Theta_1, \varrho)$ *and* $(\Lambda_2, \Theta_2, \sigma)$ *be two* $\mathcal{BSS}s$*, then the bipolar soft difference between these* $\mathcal{BSS}s$ *is the* $\mathcal{BSS}$ $(\Lambda, \Theta, \kappa)$*, where* $\kappa = \varrho \cup \sigma$*, which is defined as:*

$$(\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\setminus}}\, (\Lambda_2, \Theta_2, \sigma) = (\Lambda_1, \Theta_1, \varrho) \,\widetilde{\widetilde{\cap}}\, (\Lambda_2, \Theta_2, \sigma)^c.$$

Definition 2.14 *[43] Let* $\widetilde{\widetilde{\mathfrak{g}}}$ *be the class of* $\mathcal{BSS}s$*, then* $\widetilde{\widetilde{\mathfrak{g}}}$ *is called a bipolar soft generalized topology* $(\mathcal{BSGT})$ *on* Ω *if the following conditions are satisfying:*

1. $(\Phi, \widetilde{\widetilde{\Omega}}, \varrho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$.
2. *If* $(\Lambda_j, \Theta_j, \varrho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$ *for all* $j \in \mathcal{J}$*, then* $\widetilde{\widetilde{\bigcup}}_{j\in\mathcal{J}}(\Lambda_j, \Theta_j, \varrho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$.

We called $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *by a bipolar soft generalized topological space* $(\mathcal{BSGTS})$ *over* Ω.

Proposition 2.2 *[43] Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$*. Then the class* $\widetilde{\widetilde{\mathfrak{g}}}_\varsigma = \{\Lambda(\varsigma) : (\Lambda, \Theta, \varrho) \in \widetilde{\widetilde{\mathfrak{g}}}\}$ *for each* $\varsigma \in \varrho$*, defines a soft generalized topology on* Ω.

Definition 2.15 *[43] Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$*, then the members of* $\widetilde{\widetilde{\mathfrak{g}}}$ *are called bipolar soft* $\widetilde{\widetilde{\mathfrak{g}}}$*-open sets in* Ω*, denoted by* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-open set. Now, the complement of* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-open sets is* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-closed.*

It is clear that $(\Phi, \widetilde{\widetilde{\Omega}}, \varrho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set and $(\widetilde{\widetilde{\Omega}}, \Phi, \varrho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closed set. But neither $(\widetilde{\widetilde{\Omega}}, \Phi, \varrho)$ is $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open nor $(\Phi, \widetilde{\widetilde{\Omega}}, \varrho)$ is $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closed in general.

Definition 2.16 *[43] Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$ *and* $(\Lambda, \Theta, \varrho) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$*. Then*

1. *We denote* $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)$ *by* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-interior of* $(\Lambda, \Theta, \varrho)$*, which is the bipolar soft union of all* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-open subsets of* $(\Lambda, \Theta, \varrho)$*. In other words,*
$$i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) = \widetilde{\widetilde{\bigcup}}\{(\chi, \psi, \varrho) \text{: } (\chi, \psi, \varrho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}} \text{ and } (\chi, \psi, \varrho) \,\widetilde{\widetilde{\subseteq}}\, (\Lambda, \Theta, \varrho)\}.$$
2. *We denote* $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)$ *by* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-closure of* $(\Lambda, \Theta, \varrho)$*, which is the bipolar soft intersection of all* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-closed sets containing* $(\Lambda, \Theta, \varrho)$*. In other words,*
$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) = \widetilde{\widetilde{\bigcap}}\{(\chi, \psi, \varrho) : (\chi, \psi, \varrho)^c \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}} \text{ and } (\chi, \psi, \varrho) \,\widetilde{\widetilde{\supseteq}}\, (\Lambda, \Theta, \varrho)\}.$$
3. *We denote* $b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)$ *by* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-boundary of* $(\Lambda, \Theta, \varrho)$*, which is defined as*
$$b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) = c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) \,\widetilde{\widetilde{\cap}}\, c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)^c.$$
4. *We denote* $e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)$ *by* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-exterior of* $(\Lambda, \Theta, \varrho)$*, which is* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-interior of the complement of* $(\Lambda, \Theta, \varrho)$*. In the other words,*
$$e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) = i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)^c.$$

Definition 2.17 *[35] Let* $(\Lambda, \Theta, \varrho) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$ *and* θ *be a non-empty subset of* Ω*. Then we denote* $(^\theta\Lambda, ^\theta\Theta, \varrho)$ *by the sub bipolar soft set of* $(\Lambda, \Theta, \varrho)$ *over* θ*, which is defined as follows*

$$^\theta\Lambda(\varsigma) = \theta \cap \Lambda(\varsigma) \text{ and } ^\theta\Theta(\neg\varsigma) = \theta \cap \Theta(\neg\varsigma), \text{ for each } \varsigma \in \varrho \text{ and } \neg\varsigma \in \neg\varrho.$$

Proposition 2.3 *[35] Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$ *and* θ *be a non-empty subset of* Ω*. Then* $\widetilde{\widetilde{\mathfrak{g}}}_\theta = \{(^\theta\Lambda, ^\theta\Theta, \varrho) : (\Lambda, \Theta, \varrho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}\}$ *is a* $\mathcal{BSGTS}$ *on* Ω.

Theorem 2.1 *[43] Let* $(\Omega, \widetilde{\mathfrak{g}}, \varrho)$ *be a* $\mathcal{SGT}$*. Then* $\widetilde{\widetilde{\mathfrak{g}}}$ *is the class including* $\mathcal{BSS}s$ $(\Lambda, \Theta, \varrho)$ *in which* $(\Lambda, \varrho) \in \widetilde{g}$ *and* $\Theta(\neg\varrho) = \Omega \setminus \Lambda(\varsigma)$ *for all* $\varsigma \in \varrho$ *and* $\neg\varsigma \in \neg\varrho$ *defines a* $\mathcal{BSGT}$ *on* Ω.

Definition 2.18 *[42] Let* $(\Lambda, \Theta, \varrho) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$*. The* $\mathcal{BSS}$ $(\Lambda, \Theta, \varrho)$ *is called a bipolar soft point* $(\mathcal{BSP})$ *if there exist* $\pi, \upsilon \in \Omega, \pi \neq \upsilon, \varsigma \in \varrho$ *and* $\neg\varsigma \in \neg\varrho$ *such that*

$$\Lambda(\gamma) = \begin{cases} \{\pi\}, & \gamma = \varsigma, \\ \phi, & \gamma \in \varrho \setminus \{\varsigma\}. \end{cases}$$

$$\Theta(\gamma') = \begin{cases} \Omega \setminus \{\pi, \upsilon\}, & \gamma' = \neg\varsigma, \\ \Omega, & \gamma' \in \neg\varrho \setminus \{\neg\varsigma\}. \end{cases}$$

We denoted $\mathcal{BSP}$ $(\Lambda, \Theta, \varrho)$ briefly by π_υ^ς and denoted the family of all $\mathcal{BSP}s$ over Ω briefly by $\mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$.

Definition 2.19 *[42] Let $\mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$ be all $\mathcal{BSP}s$ with ϱ as the set of parameters. $|\Omega| = k$ be the number of elements in the set Ω and $|\varrho| = i$ be the number of elements in the set ϱ. Then, the number of elements of $\mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$ is denoted by*

$$|\mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}| = k(k-1)i.$$

Definition 2.20 *[42] Let π_υ^ς, $\pi'^{\varsigma'}_{\upsilon'} \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$. Then π_υ^ς and $\pi'^{\varsigma'}_{\upsilon'}$ are called different $\mathcal{BSP}s$ if $\pi \neq \pi'$ or $\varsigma \neq \varsigma'$.*

Definition 2.21 *[42] Let $(\Lambda, \Theta, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$ and $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$. Then π_υ^ς is said to be contained in $(\Lambda, \Theta, \varrho)$, which is denoted by $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} (\Lambda, \Theta, \varrho)$, if $\pi \in \Lambda(\varsigma)$ and $\upsilon \in \Omega \setminus \Theta(\neg\varsigma)$.*

Proposition 2.4 *[42] Let $(\Lambda, \Theta, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$. Then $(\Lambda, \Theta, \varrho)$ is the bipolar soft union of its $\mathcal{BSP}s(\Omega)_{(\varrho,\neg\varrho)}$. That is,*

$$(\Lambda, \Theta, \varrho) = \widetilde{\widetilde{\bigcup}} \{\pi_\upsilon^\varsigma : \pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} (\Lambda, \Theta, \varrho)\}.$$

Proposition 2.5 *[42] Let $\{(\Lambda_i, \Theta_i, \varrho) : i \in \mathcal{I}\}$ be a family of $\mathcal{BSS}$ over Ω. Then*

1. *$\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\bigcap}}_{i\in\mathcal{I}} (\Lambda_i, \Theta_i, \varrho)$ if and only if $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} (\Lambda_i, \Theta_i, \varrho)$ for each $i \in \mathcal{I}$. That is, $\pi \in \Lambda_i(\varsigma)$ and $\upsilon \in \Omega \setminus \Theta_i(\neg\varsigma)$ for every $i \in \mathcal{I}, \varsigma \in \varrho$ and $\neg\varsigma \in \neg\varrho$.*

2. *$\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\bigcup}}_{i\in\mathcal{I}}(\Lambda_i, \Theta_i, \varrho)$ if and only if there exists $i \in \mathcal{I}$ such that $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} (\Lambda_i, \Theta_i, \varrho)$. That is, there exists $i \in \mathcal{I}$ such that $\pi \in \Lambda_i(\varsigma)$ and $\upsilon \in \Omega \setminus \Theta_i(\neg\varsigma)$ for every $\varsigma \in \varrho$ and $\neg\varsigma \in \neg\varrho$.*

Proposition 2.6 *[42] Let $(\Lambda_1, \Theta_1, \varrho)$, $(\Lambda_2, \Theta_2, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$. Then $(\Lambda_1, \Theta_1, \varrho) \mathrel{\widetilde{\widetilde{\subseteq}}} (\Lambda_2, \Theta_2, \varrho)$ if and only if for each $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} (\Lambda_1, \Theta_1, \varrho)$ implies that $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} (\Lambda_2, \Theta_2, \varrho)$.*

Proposition 2.7 *[42] Let $(\Lambda, \Theta, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$ and $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$. If the $\mathcal{BSP}$ π_υ^ς belongs to $\mathcal{BSS}$ $(\Lambda, \Theta, \varrho)$, then the soft point π^ς also belongs to the soft set (Λ, ϱ).*

3 $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-limit points in $\mathcal{BSGTS}$

In [43], the concepts of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior set, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure set, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior set and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-boundary set are defined and some of their results are investigated. In this section, we will give more results among these concepts via $\mathcal{BSP}s$ on the $\mathcal{BSGTS}s$. In addition, we present $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-neighborhood and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-limit points. Some properties among these concepts will be obtained.

Definition 3.1 *Let π_υ^ς, $\pi'^{\varsigma'}_{\upsilon'} \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$ be two $\mathcal{BSP}s$. Then π_υ^ς and $\pi'^{\varsigma'}_{\upsilon'}$ are called not different $\mathcal{BSP}s$ if $\pi = \pi'$ and $\varsigma = \varsigma'$. Clearly $\upsilon = \upsilon'$ or $\upsilon \neq \upsilon'$.*

Lemma 3.1 *Let $\pi_\upsilon^\varsigma, \pi'^{\varsigma'}_{\upsilon'} \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$. If π_υ^ς and $\pi'^{\varsigma'}_{\upsilon'}$ are called not different, then their bipolar soft complements are different in case $\upsilon \neq \upsilon'$.*

Proof. Assume that the two $\mathcal{BSP}s$ π_υ^ς and $\pi'^{\varsigma'}_{\upsilon'}$ are not different. Then $\pi = \pi'$ and $\varsigma = \varsigma'$. In case $\upsilon \neq \upsilon'$. Thus, $\Lambda^c(\varsigma) = \Omega \setminus \{\pi, \upsilon\} \neq \Omega \setminus \{\pi, \upsilon'\} = \Lambda^c(\varsigma')$. Therefore, $\{\pi_\upsilon^\varsigma\}^c$ and $\{\pi'^{\varsigma'}_{\upsilon'}\}^c$ are different.

Definition 3.2 *Let $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ be a $\mathcal{BSGTS}$, $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$ and $(\Lambda, \Theta, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$. Then $(\Lambda, \Theta, \varrho)$ is called a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-neighborhood of π_υ^ς, denoted by, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-neighborhood if there exists $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\mathfrak{g}}}$ such as*

$$\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} (\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\subseteq}}} (\Lambda, \Theta, \varrho), \text{ which means that } \pi \in \chi(\varsigma) \subseteq \Lambda(\varsigma),$$

and

$$\upsilon \in \Omega \setminus \psi(\varsigma) \subseteq \Omega \setminus \Theta(\varsigma) \text{ for every } \varsigma \in \varrho \text{ and } \neg\varsigma \in \neg\varrho.$$

Then we denote $\mathfrak{N}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$ by the set of all $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-neighborhoods of π_υ^ς.

Definition 3.3 *Let $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ be a $\mathcal{BSGTS}$, $\pi_\upsilon^\varsigma \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$ and $(\Lambda, \Theta, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$. Then*

1. *π_υ^ς is called a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-interior point of $(\Lambda, \Theta, \varrho)$ if $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\subseteq}}} (\Lambda, \Theta, \varrho)$ for some $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathfrak{N}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$.*

2. *π_υ^ς is called a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-exterior point of $(\Lambda, \Theta, \varrho)$ if $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\subseteq}}} (\Lambda, \Theta, \varrho)^c$ for some $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathfrak{N}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$.*

3. *π_υ^ς is called a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closure (or bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-adherent) point of $(\Lambda, \Theta, \varrho)$ if $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\cap}}} (\Lambda, \Theta, \varrho) \neq (\Phi, \widetilde{\widetilde{\Omega}}, \varrho)$ for any $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathfrak{N}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$.*

4. *π_υ^ς is called a bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-boundary point of $(\Lambda, \Theta, \varrho)$ if $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\cap}}} (\Lambda, \Theta, \varrho) \neq (\Phi, \widetilde{\widetilde{\Omega}}, \varrho)$ and $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\cap}}} (\Lambda, \Theta, \varrho)^c \neq (\Phi, \widetilde{\widetilde{\Omega}}, \varrho)$ for any $(\chi, \psi, \varrho) \mathrel{\widetilde{\widetilde{\in}}} \mathfrak{N}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$.*

We denoted the bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-interior point, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-exterior point, bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-closure point (also bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-adherent point) and bipolar soft $\widetilde{\widetilde{\mathfrak{g}}}$-boundary point briefly by $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior point, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior point, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure point (also $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-adherent point) and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-boundary point respectively.

To explain the above definition, it is shown below.

Example 3.1 *Consider* $\Omega = \{\pi_1, \pi_2, \pi_3, \pi_4\}$, $\varrho = \{\varsigma_1, \varsigma_2\}$ *and*
$\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \varrho), (\Lambda_1, \Theta_1, \varrho), (\Lambda_2, \Theta_2, \varrho), (\Lambda_3, \Theta_3, \varrho)$ *is a* $\mathcal{BSGTS}$ *where*

$$(\Lambda_1, \Theta_1, \varrho) = \{(\varsigma_1, \{\pi_1, \pi_3\}, \{\pi_2\}), (\varsigma_2, \{\pi_2, \pi_3\}, \{\pi_1\})\},$$
$$(\Lambda_2, \Theta_2, \varrho) = \{(\varsigma_1, \{\pi_2, \pi_3\}, \{\pi_1\}), (\varsigma_2, \{\pi_1, \pi_3\}, \{\pi_2\})\},$$
$$(\Lambda_3, \Theta_3, \varrho) = \{(\varsigma_1, \{\pi_1, \pi_2, \pi_3\}, \phi), (\varsigma_2, \{\pi_1, \pi_2, \pi_3\}, \phi)\}.$$

Now, $\pi_{1_{\pi_3}}^{\varsigma_1}$, $\pi_{2_{\pi_3}}^{\varsigma_2}$ *are* $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$*-interior points of* $(\Lambda_1, \Theta, \varrho)$ *and* $\pi_{2_{\pi_4}}^{\varsigma_1}$, $\pi_{1_{\pi_4}}^{\varsigma_2}$ *are* $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$*-exterior points of* $(\Lambda_1, \Theta, \varrho)$*. So,* $\pi_{1_{\pi_3}}^{\varsigma_1}$ *is a* $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$*-closure point of* $(\Lambda_1, \Theta_1, \varrho)$ *and* $\pi_{4_{\pi_1}}^{\varsigma_1}$ *is* $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$*-boundary point of* $(\Lambda_1, \Theta_1, \varrho)$*.*

Proposition 3.1 *The* $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$*-neighborhood system* $\mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$ *in* $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *satisfies the follows*

1. *If* $(\Lambda, \Theta, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$*, then* $\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho)$*.*
2. *If* $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$ *and* $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_2, \Theta_2, \varrho)$*, then* $(\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$*.*
3. *If* $(\Lambda_1, \Theta_1, \varrho)$, $(\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$*, then* $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\cup}} (\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$*.*
4. *If* $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$*, then there exists* $(\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$ *such that* $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi'^{\varsigma'}_{\upsilon'})$ *for each* $\pi'^{\varsigma'}_{\upsilon'} \tilde{\tilde{\in}} (\Lambda_2, \Theta_2, \varrho)$*.*

Proof.

1. Suppose that $(\Lambda, \Theta, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$. Then there exists $(\chi, \psi, \varrho) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$ such that
$$\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi, \psi, \varrho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \varrho).$$
Thus, $\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho)$.
2. Assume that $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$ and $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_2, \Theta_2, \varrho)$. Since
$$\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi, \psi, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_2, \Theta_2, \varrho) \text{ for some } (\chi, \psi, \varrho) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}},$$
then
$$\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi, \psi, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_2, \Theta_2, \varrho).$$
Therefore $(\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$.
3. Let $(\Lambda_1, \Theta_1, \varrho)$, $(\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$, then there exist $(\chi_1, \psi_1, \varrho)$,$(\chi_2, \psi_2, \varrho) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$ such that
$$\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi_1, \psi_1, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_1, \Theta_1, \varrho)$$
and
$$\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi_2, \psi_2, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_2, \Theta_2, \varrho).$$
Since
$$(\chi_1, \psi_1, \varrho) \tilde{\tilde{\cup}} (\chi_2, \psi_2, \varrho) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$$
and
$$\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi_1, \psi_1, \varrho) \tilde{\tilde{\cup}} (\chi_2, \psi_2, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\cup}} (\Lambda_2, \Theta_2, \varrho).$$
Thus, $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\cup}} (\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$.
4. Let $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$, then there exists $(\chi, \psi, \varrho) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$ such that
$$\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi, \psi, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_1, \Theta_1, \varrho).$$
Now, $\pi'^{\varsigma'}_{\upsilon'} \tilde{\tilde{\in}} (\chi, \psi, \varrho) \tilde{\tilde{\subseteq}} (\Lambda_1, \Theta_1, \varrho)$ is satisfied, for every $\pi'^{\varsigma'}_{\upsilon'} \tilde{\tilde{\in}} (\chi, \psi, \varrho)$. If we replace $(\Lambda_2, \Theta_2, \varrho)$ instead of (χ, ψ, ϱ) then, the proof is completed.

Remark 3.1 *If* $(\Lambda_1, \Theta_1, \varrho)$, $(\Lambda_2, \Theta_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$*, then* $(\Lambda_1, \Theta_1, \varrho) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \varrho)$ *is not necessary to be in* $\mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_\upsilon^\varsigma)$*.*

Example 3.2 *If we take* $\tilde{\tilde{\mathfrak{g}}}$ *from Example 3.1. Then let*

$$(\chi_1, \psi_1, \varrho) = \{(\varsigma_1, \{\pi_1, \pi_3\}, \{\pi_2\}), (\varsigma_2, \{\pi_2, \pi_3, \pi_4\}, \{\pi_1\})\},$$
$$(\chi_2, \psi_2, \varrho) = \{(\varsigma_1, \{\pi_2, \pi_3\}, \{\pi_1\}), (\varsigma_2, \{\pi_1, \pi_3, \pi_4\}, \{\pi_2\})\}.$$

Clearly, $(\chi_1, \psi_1, \varrho)$, $(\chi_2, \psi_2, \varrho) \tilde{\tilde{\in}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_{3_{\pi_4}}^{\varsigma_1})$*.*
But $(\chi_1, \psi_1, \varrho) \tilde{\tilde{\cap}} (\chi_2, \psi_2, \varrho) = \{(\varsigma_1, \{\pi_3\}, \{\pi_1, \pi_2\}), (\varsigma_2, \{\pi_3, \pi_4\}, \{\pi_1, \pi_2\})\} \tilde{\tilde{\notin}} \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_{3_{\pi_4}}^{\varsigma_1})$*.*

Theorem 3.1 *A* $\mathcal{BSS}$ $(\Lambda, \Theta, \varrho)$ *is a* $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$*-open set if and only if* $(\Lambda, \Theta, \varrho)$ *is a* $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}$*-neighborhood of its each* $\mathcal{BSP}s$*.*

Proof. Suppose that $(\Lambda, \Theta, \varrho) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$ and $\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho)$. Then $\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \varrho)$. Thus, $(\Lambda, \Theta, \varrho)$ is $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}$-neighborhood of π_υ^ς.

Conversely, assume that $(\Lambda, \Theta, \varrho)$ is $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}$-neighborhood of its each $\mathcal{BSP}s$ and $\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho)$. Then there exists $(\chi, \psi, \varrho) \tilde{\tilde{\in}} \tilde{\tilde{\mathfrak{g}}}$ such that $\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\chi, \psi, \varrho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \varrho)$. Since we have

$$\begin{aligned}(\Lambda, \Theta, \varrho) &= \tilde{\tilde{\bigcup}}_{\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho)} \{\pi_\upsilon^\varsigma\} \\ &\tilde{\tilde{\subseteq}} \tilde{\tilde{\bigcup}}_{\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho)} (\chi, \psi, \varrho)_{\pi_\upsilon^\varsigma} \\ &\tilde{\tilde{\subseteq}} \tilde{\tilde{\bigcup}}_{\pi_\upsilon^\varsigma \tilde{\tilde{\in}} (\Lambda, \Theta, \varrho)} (\Lambda, \Theta, \varrho) \\ &= (\Lambda, \Theta, \varrho).\end{aligned}$$

Thus, $(\Lambda,\Theta,\varrho) = \widetilde{\widetilde{\bigcup}}_{\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\Lambda,\Theta,\varrho)} (\Lambda,\Theta,\varrho)$. It means that (Λ,Θ,ϱ) can be written as a bipolar soft union of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open sets, hence (Λ,Θ,ϱ) is $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open.

Theorem 3.2 *Let $(\Omega,\widetilde{\widetilde{\mathfrak{g}}},\varrho,\neg\varrho)$ be a $\mathcal{BSGTS}$ and $(\Lambda,\Theta,\varrho) \widetilde{\widetilde{\in}} \mathcal{BSS}(\Omega)$. Then*

1. $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) = \widetilde{\widetilde{\bigcup}} \{\pi_\upsilon^\varsigma : \pi_\upsilon^\varsigma$ *is a* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-interior point of* $(\Lambda,\Theta,\varrho)\}$.
2. $e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) = \widetilde{\widetilde{\bigcup}} \{\pi_\upsilon^\varsigma : \pi_\upsilon^\varsigma$ *is a* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-exterior point of* $(\Lambda,\Theta,\varrho)\}$.
3. $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) = \widetilde{\widetilde{\bigcup}} \{\pi_\upsilon^\varsigma : \pi_\upsilon^\varsigma$ *is a* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-closure point of* $(\Lambda,\Theta,\varrho)\}$.
4. $b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) = \widetilde{\widetilde{\bigcup}} \{\pi_\upsilon^\varsigma : \pi_\upsilon^\varsigma$ *is a* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-boundary point of* $(\Lambda,\Theta,\varrho)\}$.

Proof.

1. Since $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set and $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) \widetilde{\widetilde{\subseteq}} (\Lambda,\Theta,\varrho)$ for each $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$, then π_υ^ς is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior point of (Λ,Θ,ϱ).

   Conversely, let $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\Lambda,\Theta,\varrho)$ be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior point of (Λ,Θ,ϱ), then there exists a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) such that

   $$\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho) \widetilde{\widetilde{\subseteq}} (\Lambda,\Theta,\varrho).$$

   Since by the definition of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior set, we have $(\chi,\psi,\varrho) \widetilde{\widetilde{\subseteq}} i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$. Hence, every $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior point belongs to $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$.

2. Since $e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) = i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c$, then $i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set and $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c \widetilde{\widetilde{\subseteq}} (\Lambda,\Theta,\varrho)^c$ for each $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c$, then π_υ^ς is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior point of (Λ,Θ,ϱ).

   Conversely, let $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\Lambda,\Theta,\varrho)$ be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior point of (Λ,Θ,ϱ), then there exists a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) such that

   $$\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho) \widetilde{\widetilde{\subseteq}} (\Lambda,\Theta,\varrho)^c.$$

   Since by the properties of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-interior set, we have $(\chi,\psi,\varrho) \widetilde{\widetilde{\subseteq}} i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c$. From the definition of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior set, $(\chi,\psi,\varrho) \widetilde{\widetilde{\subseteq}} e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$. Therefore, every $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-exterior point belongs to $e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$.

3. From $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closed set, then $(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho))^c = i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c = e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set and $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) \widetilde{\widetilde{\cap}} (\Lambda,\Theta,\varrho) = (\Phi,\Theta,\varrho) \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho)$ for each $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$, then π_υ^ς is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure point of (Λ,Θ,ϱ).

   Conversely, let π_υ^ς be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure point of (Λ,Θ,ϱ), then there exists a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) in which

   $$\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho) \widetilde{\widetilde{\cap}} (\Lambda,\Theta,\varrho) \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho).$$

   From the properties of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure set, $(\chi,\psi,\varrho) \widetilde{\widetilde{\cap}} c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho)$. Therefore, every $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure point belongs to $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$.

4. From $b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) = c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) \widetilde{\widetilde{\cap}} c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c$. Then, $(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho))^c = i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c = e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ and $(c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c)^c = i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ are the $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open sets such that

   $$\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} e_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) \widetilde{\widetilde{\cap}} (\Lambda,\Theta,\varrho) = (\Phi,\Theta,\varrho) \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho)$$

   and $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} i_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) \widetilde{\widetilde{\cap}} (\Lambda,\Theta,\varrho) \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho)$. Therefore, π_υ^ς is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-boundary point of (Λ,Θ,ϱ).

   Conversely, let π_υ^ς be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-boundary point of (Λ,Θ,ϱ), then there exists a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) in which

   $$\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho) \widetilde{\widetilde{\cap}} (\Lambda,\Theta,\varrho) \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho)$$

   and

   $$\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho) \widetilde{\widetilde{\cap}} (\Lambda,\Theta,\varrho)^c \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho).$$

   From the properties of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure set, we have $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho) \widetilde{\widetilde{\cap}} c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho) \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho)$. Also, $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho) \widetilde{\widetilde{\cap}} c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)^c \neq (\Phi,\widetilde{\widetilde{\Omega}},\varrho)$. Therefore, every $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-boundary point belongs to $b_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$.

Now, two types of bipolar soft limit points on $\mathcal{BSGT}s$ over Ω will be discussed as mentioned in the following definition.

Definition 3.4 *Let $(\Omega,\widetilde{\widetilde{\mathfrak{g}}},\varrho,\neg\varrho)$ be a $\mathcal{BSGT}s$ defined on Ω and $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} \mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}$. Then*

1. *π_υ^ς is called $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-*limit point of (Λ,Θ,ϱ) if for every $(\chi,\psi,\varrho) \widetilde{\widetilde{\in}} \widetilde{\widetilde{\mathfrak{g}}}$ such that $\pi_\upsilon^\varsigma \widetilde{\widetilde{\in}} (\chi,\psi,\varrho)$, we have*

   $$(\Lambda,\Theta,\varrho) \widetilde{\widetilde{\cap}} (\chi,\psi,\varrho) \setminus \{\pi_\upsilon^\varsigma\} \neq (\Phi,\Theta,\varrho).$$

   In the other words, π_υ^ς is said to be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-*limit point of (Λ,Θ,ϱ) if every $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-neighborhood of π_υ^ς contains at least one $\mathcal{BSP}$ of (Λ,Θ,ϱ) other than π_υ^ς.
   We denote $\mathfrak{d}^*_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ by $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-*derived set of (Λ,Θ,ϱ) which is the set of all $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-*limit points of (Λ,Θ,ϱ).

2. *If we extend the definition of the $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit point to $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-limit point by making the last concept contain $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit points to gather with all $\mathcal{BSP}s$ which do not belong to each $\mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\varsigma})$. The set of all $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-limit points denoted by $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$.*
That is the set of all $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-limit points can be defined as

$$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)=\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\,\tilde{\tilde{\cup}}\,\{\pi_{\upsilon}^{\varsigma}\,\tilde{\tilde{\in}}\,\mathcal{BSP}(\Omega)_{(\varrho,\neg\varrho)}:\pi_{\upsilon}^{\varsigma}\,\tilde{\tilde{\notin}}\,\mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\varsigma})\}.$$

Remark 3.2 *Clearly, for any $(\Lambda,\Theta,\varrho)\,\tilde{\tilde{\in}}\,\mathcal{BSS}(\Omega)$, we have $\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$.*

Theorem 3.3 *Let $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varrho,\neg\varrho)$ be a $\mathcal{BSGTS}$ and $(\Lambda,\Theta,\varrho)\,\tilde{\tilde{\in}}\,\mathcal{BSS}(\Omega)$. Then*

1. *$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)=\tilde{\tilde{\bigcup}}\,\{\pi_{\upsilon}^{\varsigma}:\pi_{\upsilon}^{\varsigma}$ is a $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-limit point of $(\Lambda,\Theta,\varrho)\}$.*
2. *$\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)=\tilde{\tilde{\bigcup}}\,\{\pi_{\upsilon}^{\varsigma}:\pi_{\upsilon}^{\varsigma}$ is a $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit point of $(\Lambda,\Theta,\varrho)\}$.*

Proof.

1. From Definition 3.4, we have $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ which is the set of all $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}$-limit points of (Λ,Θ,ϱ). It is sufficient to show the following equality:
   For each $\varsigma\in\varrho$,
   $$\Lambda(\varsigma)=\bigcup\nolimits_{\pi_{\upsilon}^{\varsigma}\tilde{\tilde{\in}}\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)}\{\pi\}.$$
   Also, for each $\neg\varsigma\in\neg\varrho$,
   $$\Theta(\neg\varsigma)=\bigcap\nolimits_{\pi_{\upsilon}^{\varsigma}\tilde{\tilde{\in}}\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)}\{\Omega\setminus\{\pi,\upsilon\}\}.$$
   Clearly, the $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ is equal the union of all its $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}$-limit points.
2. Similar to point (1).

Theorem 3.4 *Let $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varrho,\neg\varrho)$ be a $\mathcal{BSGTS}$ and let $(\Lambda_1,\Theta_1,\varrho)$,$(\Lambda_2,\Theta_2,\varrho)\,\tilde{\tilde{\in}}\,\mathcal{BSS}(\Omega)$. Then*

1. *$\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Phi,\tilde{\tilde{\Omega}},\varrho)=(\Phi,\Omega,\varrho)$ and $\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\tilde{\tilde{\Omega}},\Phi,\varrho)\,\tilde{\tilde{\subseteq}}\,(\tilde{\tilde{\Omega}},\Phi,\varrho)$.*
2. *$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Phi,\tilde{\tilde{\Omega}},\varrho)\,\tilde{\tilde{\supseteq}}\,(\Phi,\Omega,\varrho)$ and $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\tilde{\tilde{\Omega}},\Phi,\varrho)=(\tilde{\tilde{\Omega}},\Phi,\varrho)$.*
3. *If $(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\subseteq}}\,(\Lambda_2,\Theta_2,\varrho)$, then*
   (a) $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.
   (b) $\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.
4. *(a) $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\Lambda_2,\Theta_2,\varrho))\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.*
   (b) $\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\Lambda_2,\Theta_2,\varrho))\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.
5. *(a) $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho))=\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.*
   (b) $\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho))=\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.

Proof.

1. The proof is trivial.
2. The proof is trivial.
3. (a) Suppose that $\pi_{\upsilon}^{\varsigma}\,\tilde{\tilde{\in}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)$, since $\pi_{\upsilon}^{\varsigma}$ is a $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-limit point of $(\Lambda_1,\Theta_1,\varrho)$. Then $(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\chi,\psi,\varrho)\setminus\{\pi_{\upsilon}^{\varsigma}\}\neq(\Phi,\Theta,\varrho)$ for each $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) containing $\pi_{\upsilon}^{\varsigma}$. Since $(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\subseteq}}\,(\Lambda_2,\Theta_2,\varrho)$, it leads to $(\Lambda_2,\Theta_2,\varrho)\,\tilde{\tilde{\cap}}\,(\chi,\psi,\varrho)\setminus\{\pi_{\upsilon}^{\varsigma}\}\neq(\Phi,\Theta,\varrho)$ for each $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) containing $\pi_{\upsilon}^{\varsigma}$. Therefore, $\pi_{\upsilon}^{\varsigma}\,\tilde{\tilde{\in}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.
   (b) Similar to point (3)(a).
4. (a) Since $(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\Lambda_2,\Theta_2,\varrho)\,\tilde{\tilde{\subseteq}}\,(\Lambda_1,\Theta_1,\varrho)$ and $(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\Lambda_2,\Theta_2,\varrho)\,\tilde{\tilde{\subseteq}}\,(\Lambda_2,\Theta,\varrho)$. By (3)(a) we get
   $$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\Lambda_2,\Theta_2,\varrho))\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)$$
   and
   $$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\Lambda_2,\Theta_2,\varrho))\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta,\varrho).$$
   Then $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,(\Lambda_2,\Theta_2,\varrho))\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cap}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.
   (b) Similar to point (4)(a).
5. (a) From $(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\subseteq}}\,(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho)$ and $(\Lambda_2,\Theta_2,\varrho)\,\tilde{\tilde{\subseteq}}\,(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho)$. By (3)(a) we get
   $$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho))$$
   and
   $$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho)).$$
   Thus, $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)\,\tilde{\tilde{\subseteq}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho))$.
   Now, let $\pi_{\upsilon}^{\varsigma}\,\tilde{\tilde{\in}}\,\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho))$. Then, $((\Lambda_1,\Theta_1,\varrho)\,\tilde{\tilde{\cup}}\,(\Lambda_2,\Theta_2,\varrho))\,\tilde{\tilde{\cap}}\,((\chi,\psi,\varrho)\setminus\{\pi_{\upsilon}^{\varsigma}\})$

$\neq (\Phi,\Theta,\varrho)$ for each a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) containing $\pi^{\varsigma}_{\upsilon}$. Thus, $(\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cap}}\ ((\chi,\psi,\varrho)\setminus\{\pi^{\varsigma}_{\upsilon}\})$ $\neq (\Phi,\Theta,\varrho)$ for each a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) containing $\pi^{\varsigma}_{\upsilon}$ or $(\Lambda_2,\Theta_2,\varrho)\ \tilde{\tilde{\cap}}\ ((\chi,\psi,\varrho)\setminus\{\pi^{\varsigma}_{\upsilon}\})$ $\neq (\Phi,\Theta,\varrho)$ for each a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set (χ,ψ,ϱ) containing $\pi^{\varsigma}_{\upsilon}$. This implies that

$$\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho) \text{ or } \pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho).$$

Therefore, $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cup}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$. So, this gives the equality.

(b) Similar to point (5)(a).

Remark 3.3 *The equalities in Theorem 3.4 (4) in general are not hold.*

Example 3.3 *Consider* $\Omega=\{\pi_1,\pi_2,\pi_3,\pi_4\}$, $\varrho=\{\varsigma_1,\varsigma_2\}$ *and*
$\tilde{\tilde{\mathfrak{g}}}=\{(\Phi,\tilde{\tilde{\Omega}},\varrho),(\Lambda_1,\Theta_1,\varrho),(\Lambda_2,\Theta_2,\varrho),\ (\Lambda_3,\Theta_3,\varrho)\}$ *is a* $\mathcal{BSGTS}$ *where*

$$(\Lambda_1,\Theta_1,\varrho)=\{(\varsigma_1,\{\pi_3\},\{\pi_1\}),(\varsigma_2,\{\pi_3\},\{\pi_1,\pi_2\})\},$$
$$(\Lambda_2,\Theta_2,\varrho)=\{(\varsigma_1,\phi,\{\pi_2,\pi_3\}),(\varsigma_2,\{\pi_1\},\{\pi_3\})\},$$
$$(\Lambda_3,\Theta_3,\varrho)=\{(\varsigma_1,\{\pi_3\},\phi),(\varsigma_2,\{\pi_1,\pi_3\},\phi)\},$$

and (χ_1,ψ_1,ϱ) *and* (χ_2,ψ_2,ϱ) *be* $\mathcal{BSS}s$ *defined as*

$$(\chi_1,\psi_1,\varrho)=\{(\varsigma_1,\{\pi_1\},\{\pi_3\}),(\varsigma_2,\{\pi_1\},\{\pi_3\})\} \text{ and}$$
$$(\chi_2,\psi_2,\varrho)=\{(\varsigma_1,\{\pi_3\},\{\pi_1\}),(\varsigma_2,\{\pi_3\},\{\pi_1,\pi_2\})\}.$$

Since, $(\chi_1,\psi_1,\varrho)\ \tilde{\tilde{\cap}}\ (\chi_2,\psi_2,\varrho)=\{(\varsigma_1,\phi,\{\pi_1,\pi_3\}),(\varsigma_2,\phi,\{\pi_1,\pi_2,\pi_3\})\}$.
Then

$$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varrho)=\{\pi^{\varsigma_1}_{1_{\pi_2}},\pi^{\varsigma_1}_{1_{\pi_3}},\pi^{\varsigma_1}_{1_{\pi_4}},\pi^{\varsigma_1}_{2_{\pi_1}},\pi^{\varsigma_1}_{2_{\pi_3}},\pi^{\varsigma_1}_{2_{\pi_4}},\pi^{\varsigma_1}_{3_{\pi_1}},\pi^{\varsigma_1}_{4_{\pi_1}},\pi^{\varsigma_1}_{4_{\pi_2}},\pi^{\varsigma_1}_{4_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_1}},\pi^{\varsigma_2}_{2_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_4}},\pi^{\varsigma_2}_{3_{\pi_2}},\pi^{\varsigma_2}_{4_{\pi_1}},\pi^{\varsigma_2}_{4_{\pi_2}},\pi^{\varsigma_2}_{4_{\pi_3}}\}.$$

Thus, $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varrho)=\{(\varsigma_1,\Omega,\phi),(\varsigma_2,\{\pi_2,\pi_3,\pi_4\},\phi)\}$ *and*

$$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varrho)=\{\pi^{\varsigma_1}_{1_{\pi_2}},\pi^{\varsigma_1}_{1_{\pi_3}},\pi^{\varsigma_1}_{1_{\pi_4}},\pi^{\varsigma_1}_{2_{\pi_1}},\pi^{\varsigma_1}_{2_{\pi_3}},\pi^{\varsigma_1}_{2_{\pi_4}},\pi^{\varsigma_1}_{3_{\pi_1}},\pi^{\varsigma_1}_{3_{\pi_2}},\pi^{\varsigma_1}_{3_{\pi_4}},\pi^{\varsigma_1}_{4_{\pi_1}},\pi^{\varsigma_1}_{4_{\pi_2}},\pi^{\varsigma_1}_{4_{\pi_3}},\pi^{\varsigma_2}_{1_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_1}},\pi^{\varsigma_2}_{2_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_4}},\pi^{\varsigma_2}_{3_{\pi_1}},\pi^{\varsigma_2}_{3_{\pi_2}},\pi^{\varsigma_2}_{3_{\pi_4}},\pi^{\varsigma_2}_{4_{\pi_1}},\pi^{\varsigma_2}_{4_{\pi_2}},\pi^{\varsigma_2}_{4_{\pi_3}}\},$$

Then, $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varrho)=\{(\varsigma_1,\Omega,\phi),(\varsigma_2,\Omega,\phi)\}=(\tilde{\tilde{\Omega}},\Phi,\varrho)$.
While

$$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\chi_2,\psi_2,\varrho)\ \tilde{\tilde{\cap}}\ (\chi_2,\psi_2,\varrho))=\{\pi^{\varsigma_1}_{1_{\pi_2}},\pi^{\varsigma_1}_{1_{\pi_3}},\pi^{\varsigma_1}_{1_{\pi_4}},\pi^{\varsigma_1}_{2_{\pi_1}},\pi^{\varsigma_1}_{2_{\pi_3}},\pi^{\varsigma_1}_{2_{\pi_4}},\pi^{\varsigma_1}_{4_{\pi_1}},\pi^{\varsigma_1}_{4_{\pi_2}},\pi^{\varsigma_1}_{4_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_1}},\pi^{\varsigma_2}_{2_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_4}},\pi^{\varsigma_2}_{4_{\pi_1}},\pi^{\varsigma_2}_{4_{\pi_2}},\pi^{\varsigma_2}_{4_{\pi_3}}\}.$$

$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\chi_2,\psi_2,\varrho)\ \tilde{\tilde{\cap}}\ (\chi_2,\psi_2,\varrho))=\{(\varsigma_1,\{\pi_1,\pi_2,\pi_4\},\phi),(\varsigma_2,\{\pi_2,\pi_4\},\phi)\}$.
Hence, $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cap}}\ (\Lambda_2,\Theta_2,\varrho)\ \neq\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cap}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.
Similarly, $\mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varrho)=\{\pi^{\varsigma_1}_{3_{\pi_1}},\pi^{\varsigma_2}_{3_{\pi_2}}\}=\{(\varsigma_1,\{\pi_3\},\{\pi_2,\pi_4\}),(\varsigma_2,\{\pi_3\},\{\pi_1,\pi_4\})\}$ *and* $\mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varrho)=\{\pi^{\varsigma_1}_{3_{\pi_1}},\pi^{\varsigma_1}_{3_{\pi_2}},\pi^{\varsigma_1}_{3_{\pi_4}},\pi^{\varsigma_2}_{1_{\pi_3}},\pi^{\varsigma_2}_{3_{\pi_1}},\pi^{\varsigma_2}_{3_{\pi_2}},\pi^{\varsigma_2}_{3_{\pi_4}}\}=\{(\varsigma_1,\{\pi_3\},\phi),(\varsigma_2,\{\pi_1,\pi_3\},\phi)\}$.

While $\mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}((\chi_2,\psi_2,\varrho)\ \tilde{\tilde{\cap}}\ (\chi_2,\psi_2,\varrho))=(\Phi,\tilde{\tilde{\Omega}},\varrho)$.

But, $\mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi_1,\varrho)\ \tilde{\tilde{\cap}}\ \mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varrho)=\{\pi^{\varsigma_1}_{3_{\pi_1}},\pi^{\varsigma_2}_{3_{\pi_2}}\}=\{(\varsigma_1,\{\pi_3\},\{\pi_2,\pi_4\}),(\varsigma_2,\{\pi_3\},\{\pi_1,\pi_4\})\}$.
Hence, $\mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cap}}\ (\Lambda_2,\Theta_2,\varrho)\ \neq\ \mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cap}}\ \mathfrak{d}^{*}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.

In the same method, we can take $(\chi_1,\psi_1,\varrho)\ \tilde{\tilde{\cup}}\ (\chi_2,\psi_2,\varrho)=\{(\varsigma_1,\{\pi_1,\pi_3\},\phi),(\varsigma_2,\{\pi_1,\pi_3\},\phi)\}$.
Then

$$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\chi_1,\psi_1,\varrho)\ \tilde{\tilde{\cup}}\ (\chi_2,\psi_2,\varrho))=\{\pi^{\varsigma_1}_{1_{\pi_2}},\pi^{\varsigma_1}_{1_{\pi_3}},\pi^{\varsigma_1}_{1_{\pi_4}},\pi^{\varsigma_1}_{2_{\pi_1}},\pi^{\varsigma_1}_{2_{\pi_3}},\pi^{\varsigma_1}_{2_{\pi_4}},\pi^{\varsigma_1}_{3_{\pi_1}},\pi^{\varsigma_1}_{3_{\pi_2}},\pi^{\varsigma_1}_{3_{\pi_4}},\pi^{\varsigma_1}_{4_{\pi_1}},\pi^{\varsigma_1}_{4_{\pi_2}},\pi^{\varsigma_1}_{4_{\pi_3}},\pi^{\varsigma_2}_{1_{\pi_2}},\pi^{\varsigma_2}_{1_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_1}},\pi^{\varsigma_2}_{2_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_4}},\pi^{\varsigma_2}_{3_{\pi_1}},\pi^{\varsigma_2}_{3_{\pi_2}},\pi^{\varsigma_2}_{3_{\pi_4}},\pi^{\varsigma_2}_{4_{\pi_1}},\pi^{\varsigma_2}_{4_{\pi_2}},\pi^{\varsigma_2}_{4_{\pi_3}}\}=(\tilde{\tilde{\Omega}},\Phi,\varrho) \text{ and}$$

$$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_1,\psi,\varrho)\ \tilde{\tilde{\cup}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\chi_2,\psi_2,\varrho)=\{\pi^{\varsigma_1}_{1_{\pi_2}},\pi^{\varsigma_1}_{1_{\pi_3}},\pi^{\varsigma_1}_{1_{\pi_4}},\pi^{\varsigma_1}_{2_{\pi_1}},\pi^{\varsigma_1}_{2_{\pi_3}},\pi^{\varsigma_1}_{2_{\pi_4}},\pi^{\varsigma_1}_{3_{\pi_1}},\pi^{\varsigma_1}_{3_{\pi_2}},\pi^{\varsigma_1}_{3_{\pi_4}},\pi^{\varsigma_1}_{4_{\pi_1}},\pi^{\varsigma_1}_{4_{\pi_2}},\pi^{\varsigma_1}_{4_{\pi_3}},\pi^{\varsigma_2}_{1_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_1}},\pi^{\varsigma_2}_{2_{\pi_3}},\pi^{\varsigma_2}_{2_{\pi_4}},\pi^{\varsigma_2}_{3_{\pi_1}},\pi^{\varsigma_2}_{3_{\pi_2}},\pi^{\varsigma_2}_{3_{\pi_4}},\pi^{\varsigma_2}_{4_{\pi_1}},\pi^{\varsigma_2}_{4_{\pi_2}},\pi^{\varsigma_2}_{4_{\pi_3}}\}=(\tilde{\tilde{\Omega}},\Phi,\varrho).$$

Hence, $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}((\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cup}}\ (\Lambda_2,\Theta_2,\varrho))=\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\varrho)\ \tilde{\tilde{\cup}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\varrho)$.

Theorem 3.5 *Let* $(\Omega,\tilde{\tilde{\mathfrak{g}}},\varrho,\neg\varrho)$ *be a* $\mathcal{BSGTS}$ *and* $(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\in}}\ \mathcal{BSS}(\Omega)$. *Then*

$$c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\subseteq}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\cup}}\ (\Lambda,\Theta,\varrho).$$

Proof. Suppose that $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$, then $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ (\Lambda,\Theta,\varrho)$ or $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\notin}}\ (\Lambda,\Theta,\varrho)$.
First if we consider $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ (\Lambda,\Theta,\varrho)$. Then $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ (\Lambda,\Theta,\varrho)\ \tilde{\tilde{\cup}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ and hence $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\subseteq}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\cup}}\ (\Lambda,\Theta,\varrho)$.

Now, if $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\notin}}\ (\Lambda,\Theta,\varrho)$. Then $(\Lambda,\Theta,\varrho)=(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\setminus}}\ \{\pi^{\varsigma}_{\upsilon}\}$. Since $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$ implies that

$$(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\cap}}\ (\chi,\psi,\varrho)\neq(\Phi,\Theta,\varrho) \text{ for every } (\chi,\psi,\varrho)\ \tilde{\tilde{\in}}\ \tilde{\tilde{\mathfrak{g}}} \text{ containing } \pi^{\varsigma}_{\upsilon}.$$

Then $(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\cap}}\ (\chi,\psi,\varrho)\ \tilde{\tilde{\setminus}}\ \{\pi^{\varsigma}_{\upsilon}\}\neq(\Phi,\Theta,\varrho)$. Hence, $\pi^{\varsigma}_{\upsilon}\ \tilde{\tilde{\in}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)$. Therefore, $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\subseteq}}\ \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\varrho)\ \tilde{\tilde{\cup}}\ (\Lambda,\Theta,\varrho)$.

Remark 3.4 *In the next example, we can show that the equality of Theorem 3.5 in general does not hold.*

Example 3.4 *If we take $\widetilde{\widetilde{\mathfrak{g}}}$ from Example 3.3. Then let* $(\Lambda, \Theta, \varrho) = \{(\varsigma_1, \{\pi_1\}, \{\pi_3\}), (\varsigma_2, \{\pi_1\}, \{\pi_3\})\}$. *Hence,*

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) = \{(\varsigma_1, \{\pi_1\}, \{\pi_3\}), (\varsigma_2, \{\pi_1, \pi_2\}, \{\pi_3\})\},$$

but, $\mathfrak{d}_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\cup}}\ (\Lambda, \Theta, \varrho) = (\widetilde{\widetilde{\Omega}}, \Phi, \varrho)$. *It means that,* $\mathfrak{d}_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\cup}}\ (\Lambda, \Theta, \varrho) \neq c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)$.

Remark 3.5 *The equality of Theorem 3.5 is independent for* $\mathfrak{d}^*_{\widetilde{\widetilde{\mathfrak{g}}}}$.

Example 3.5 *If we take $\widetilde{\widetilde{\mathfrak{g}}}$ from Example 3.3. Then let* $(\Lambda, \Theta, \varrho) = \{(\varsigma_1, \{\pi_1\}, \{\pi_3\}), (\varsigma_2, \{\pi_1\}, \{\pi_3\})\}$. *Hence,*

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) = \{(\varsigma_1, \{\pi_1\}, \{\pi_3\}), (\varsigma_2, \{\pi_1, \pi_2\}, \{\pi_3\})\},$$

but, $\mathfrak{d}^*_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\cup}}\ (\Lambda, \Theta, \varrho) = \{(\varsigma_1, \{\pi_1, \pi_3\}, \phi), (\varsigma_2, \{\pi_1, \pi_3\}, \phi)\}$.
It means that, $\mathfrak{d}^*_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\cup}}\ (\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\not\supseteq}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)$ *and* $\mathfrak{d}^*_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\cup}}\ (\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\not\subseteq}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \varrho)$.

4 Bipolar Soft Basis for $\mathcal{BSGTS}$

In this section, we introduce the bipolar soft generalized basis ($\mathcal{BSGB}$) for the $\mathcal{BSGT}$ $\widetilde{\widetilde{\mathfrak{g}}}$ defined on Ω and bipolar soft generalized topological subspace ($\mathcal{BSGTSS}$). Some properties of these concepts are discussed.

Definition 4.1 *Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$ *and* $\widetilde{\widetilde{\mathfrak{B}}} \ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\mathfrak{g}}}$. *Then* $\widetilde{\widetilde{\mathfrak{B}}}$ *is called a bipolar soft generalized basis for the* $\mathcal{BSGT}$ $\widetilde{\widetilde{\mathfrak{g}}}$, *denoted by,* $\mathcal{BSGB}$ *if every element in* $\widetilde{\widetilde{\mathfrak{g}}}$ *can be written as the bipolar soft union of elements of* $\widetilde{\widetilde{\mathfrak{B}}}$.

Theorem 4.1 *Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$ *and* $\widetilde{\widetilde{\mathfrak{B}}}$ *is a* $\mathcal{BSGB}$ *of* $\widetilde{\widetilde{\mathfrak{g}}}$. *Then,* $\widetilde{\widetilde{\mathfrak{g}}}$ *is equal to the class of all bipolar soft union of elements of* $\widetilde{\widetilde{\mathfrak{B}}}$.

Proof. Follows directly from Definition 4.1.

Definition 4.2 *Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$ *and* $(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\in}}\ \mathcal{BSS}(\Omega)$. *Then the collection*

$$\widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)} = \{(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\cap}}\ (\Lambda_i, \Theta_i, \varrho) : (\Lambda_i, \Theta_i, \varrho) \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}},\ i \in \mathcal{I}\}$$

is said to be a bipolar soft generalized subspace on $(\Lambda, \Theta, \varrho)$.

Theorem 4.2 *In Definition 4.2, we will show that* $\widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)}$ *is a bipolar soft generalized topological subspace* $\mathcal{BSGTSS}$ *on* $(\Lambda, \Theta, \varrho)$.

Proof. Since $(\Phi, \Omega, \varrho) \ \widetilde{\widetilde{\cap}}\ (\Lambda, \Theta, \varrho) = (\Phi, \Omega, \varrho) \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)}$ and $\widetilde{\widetilde{\cup}}\,((\Lambda_i, \Theta_i, \varrho) \ \widetilde{\widetilde{\cap}}\ (\Lambda, \Theta, \varrho)) = (\widetilde{\widetilde{\cup}}\,(\Lambda_i, \Theta_i, \varrho)) \ \widetilde{\widetilde{\cap}}\ (\Lambda, \Theta, \varrho)$ for $\widetilde{\widetilde{\mathfrak{g}}} = \{(\Lambda_i, \Theta_i, \varrho) : i \in \mathcal{I}\}$. Thus, the bipolar soft union of any member of $\mathcal{BSS}s$ in $\widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)}$ belongs to $\widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)}$. Hence, $\widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)}$ is a $\mathcal{BSGT}$ over $(\Lambda, \Theta, \varrho)$.

Remark 4.1 *We say that* $(\Omega_{(\Lambda,\Theta,\varrho)}, \widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)}, \varrho, \neg\varrho)$ *a bipolar soft generalized topological subspace of* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *and we denote it by* $\mathcal{BSGTSS}$.

Example 4.1 *Consider* $\widetilde{\widetilde{\mathfrak{g}}}$ *in Example 3.3. If* $(\Lambda, \Theta, \varrho) = \{(\varsigma_1, \{\pi_1\}, \{\pi_3\}), (\varsigma_2, \{\pi_1, \pi_2\}, \{\pi_3\})\}$ *Then,*

$$\widetilde{\widetilde{B}} = \{(\Phi, \widetilde{\widetilde{\Omega}}, \varrho), (\Lambda_1, \Theta_1, \varrho), (\Lambda_2, \Theta_2, \varrho)\}$$

is a $\mathcal{BSGB}$ *for* $\widetilde{\widetilde{\mathfrak{g}}}$.
If we take $(\Lambda, \Theta, \varrho) = \{(\varsigma_1, \{\pi_1, \pi_2, \pi_3\}, \{\pi_4\}), (\varsigma_2, \{\pi_3, \pi_4\}, \{\pi_1, \pi_2\})\} \ \widetilde{\widetilde{\in}}\ \mathcal{BSS}(\Omega)$. *Then*

$$\widetilde{\widetilde{\mathfrak{g}}}_{(\Lambda,\Theta,\varrho)} = \Big\{(\Phi, \widetilde{\widetilde{\Omega}}, \varrho), \{(\varsigma_1, \{\pi_3\}, \{\pi_1, \pi_4\}), (\varsigma_2, \{\pi_3\}, \{\pi_1, \pi_2\})\}, \{(\varsigma_1, \phi, \{\pi_2, \pi_3, \pi_4\}), (\varsigma_2, \phi, \{\pi_1, \pi_2, \pi_3\})\}, \{(\varsigma_1, \{\pi_3\}, \{\pi_4\}), (\varsigma_2, \{\pi_3\}, \{\pi_1, \pi_2\})\}\Big\}$$

is a $\mathcal{BSGTS}s$ *of* $\widetilde{\widetilde{\mathfrak{g}}}$.

Definition 4.3 *Let* $(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\in}}\ \mathcal{BSS}(\Omega)$ *and* Γ *be a non-empty subset of* Ω. *Then the bipolar sub-soft set of* $(\Lambda, \Theta, \varrho)$ *over* Γ *denoted by* $({}^{\Gamma}\Lambda, {}^{\Gamma}\Theta, \varrho)$ *is defined as*

${}^{\Gamma}\Lambda(\varsigma) = \Gamma \ \widetilde{\widetilde{\cap}}\ \Lambda(\varsigma)$ *and* ${}^{\Gamma}\Theta(\neg\varsigma) = \Gamma \ \widetilde{\widetilde{\cap}}\ \Theta(\neg\varsigma)$, *for each* $\varsigma \in \varrho$.

Theorem 4.3 *Let* $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \varrho, \neg\varrho)$ *be a* $\mathcal{BSGTS}$ *and* $\widetilde{\widetilde{\mathfrak{B}}} \ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\mathfrak{g}}}$. *Then*

1. *The family* $\widetilde{\widetilde{\mathfrak{B}}}$ *is a* $\mathcal{BSGB}$ *of* $\widetilde{\widetilde{\mathfrak{g}}}$ *if and only if there exists* $\widetilde{\widetilde{\mathfrak{B}}}_{\pi^{\varsigma}_{\upsilon}} \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}$ *such that* $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}_{\pi^{\varsigma}_{\upsilon}} \ \widetilde{\widetilde{\subseteq}}\ (\Lambda, \Theta, \varrho)$ *for every* $(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}}$ *and every* $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ (\Lambda, \Theta, \varrho)$.
2. *If the family* $\widetilde{\widetilde{\mathfrak{B}}} = \{\mathfrak{B}_i\}_{i\in\mathfrak{I}}$ *is a* $\mathcal{BSGB}$ *of* $\widetilde{\widetilde{\mathfrak{g}}}$, *then there exists* $\mathfrak{B}_i \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}$ *such that* $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ \mathfrak{B}_{i_3} \ \widetilde{\widetilde{\subseteq}}\ \mathfrak{B}_{i_1} \ \widetilde{\widetilde{\cap}}\ \mathfrak{B}_{i_2}$ *for every* $\mathfrak{B}_{i_1}, \mathfrak{B}_{i_2} \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{B}}}$ *and every* $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ \mathfrak{B}_{i_1} \ \widetilde{\widetilde{\cap}}\ \mathfrak{B}_{i_2}$.

Proof.

1. Let the family $\widetilde{\widetilde{\mathfrak{B}}}$ be a $\mathcal{BSGB}$ of $\widetilde{\widetilde{\mathfrak{g}}}$, $(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}}$ and $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ (\Lambda, \Theta, \varrho)$. Since $\widetilde{\widetilde{\mathfrak{B}}}$ is a $\mathcal{BSGB}$ of $\widetilde{\widetilde{\mathfrak{g}}}$, then

   $$\exists\ \widetilde{\widetilde{\mathfrak{B}}}' \ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\mathfrak{B}}} : (\Lambda, \Theta, \varrho) = \widetilde{\widetilde{\bigcup}}_{\mathfrak{B}\in\widetilde{\widetilde{\mathfrak{B}}}'} \mathfrak{B}.$$

   From $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ (\Lambda, \Theta, \varrho) = \widetilde{\widetilde{\bigcup}}_{\mathfrak{B}\in\widetilde{\widetilde{\mathfrak{B}}}'} \mathfrak{B}$, $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ \mathfrak{B}_{\pi^{\varsigma}_{\upsilon}} \ \widetilde{\widetilde{\subseteq}}\ (\Lambda, \Theta, \varrho)$ is satisfied for at least one $\mathfrak{B} \in \widetilde{\widetilde{\mathfrak{B}}}'$.
   Conversely, assume that there exists $\mathfrak{B}_{\pi^{\varsigma}_{\upsilon}} \in \widetilde{\widetilde{\mathfrak{B}}}$ such that $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ \mathfrak{B}_{\pi^{\varsigma}_{\upsilon}} \ \widetilde{\widetilde{\subseteq}}\ (\Lambda, \Theta, \varrho)$ for every $(\Lambda, \Theta, \varrho) \ \widetilde{\widetilde{\in}}\ \widetilde{\widetilde{\mathfrak{g}}}$ and every $\pi^{\varsigma}_{\upsilon} \ \widetilde{\widetilde{\in}}\ (\Lambda, \Theta, \varrho)$. Then

   $$(\Lambda, \Theta, \varrho) = \widetilde{\widetilde{\bigcup}}_{\pi^{\varsigma}_{\upsilon}\widetilde{\widetilde{\in}}(\Lambda,\Theta,\varrho)} \{\pi^{\varsigma}_{\upsilon}\} \ \widetilde{\widetilde{\subseteq}}\ \widetilde{\widetilde{\bigcup}}_{\pi^{\varsigma}_{\upsilon}\widetilde{\widetilde{\in}}(\Lambda,\Theta,\varrho)} \mathfrak{B}_{\pi^{\varsigma}_{\upsilon}} \ \widetilde{\widetilde{\subseteq}}\ (\Lambda, \Theta, \varrho)$$

   Therefore,

   $$(\Lambda, \Theta, \varrho) = \widetilde{\widetilde{\bigcup}}_{\pi^{\varsigma}_{\upsilon}\widetilde{\widetilde{\in}}(\Lambda,\Theta,\varrho)} \mathfrak{B}_{\pi^{\varsigma}_{\upsilon}}.$$

2. It is obtained in a similar way.

5 Conclusions

In this paper, we continued to present the main idea with the context of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-limit point, $\mathcal{BSGB}$ and $\mathcal{BSGSS}$ in $\mathcal{BSGTS}$ which are defined over an initial universe with a fixed set of parameters. The main purpose of this study was to define the concepts of $\mathcal{BSP}s$ in $\mathcal{BSGTS}$. Furthermore, we presented some of its basic properties with the help of some counterexamples. In future work, these results may be extended to new results of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-mappings, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation axioms and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-compactness in $\mathcal{BSGTS}s$.

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