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Characterization of a Class of Generalised Core-satellite Graphs Using Average Degree

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Abstract Network equilibrium models are significantly distinct in supply chain networks, traffic networks, and e-waste flow networks. The idea of network equilibrium is strongly perceived while determining the tuner sets of a graph (network).

Tuner sets are subsets of vertices of the graph G whose degrees are lower than the average degree of G, $d(G)$ that can compensate or balance the presence of vertices whose degrees are greater than $d(G)$. Generalised core-satellite graph $\Theta\left(c, \bar{S}, \bar{\eta}\right) = K_c \bigtriangledown (\bar{\eta} K_{\bar{S}})$ comprises $\bar{\eta}$ copies of $K_{\bar{S}}$ (the satellites) meeting in K_c (the core) and it belongs to the family of graphs of diameter two. It has a central core of vertices connected to a few satellites, where all satellite cliques need not be identical and can be of different sizes. Properties like hierarchical structure of large real-world networks, are competently modeled using core-satellite graphs [1, 2, 5]. This family of graphs exhibits the properties similar to scale-free network as they possess anomalous vertex connectivity, where a small fraction of vertices (the core) are densely connected. Since these graphs possess such a structural property, interesting results are obtained for these graphs when tuner sets are determined. In this paper, we have considered the graph $G = (\eta K_q + \gamma K_p) \bigtriangledown K_1$, with $p > q$, a subclass of the generalized core-satellite graph which is a join of η copies of the clique K_q and γ copies of the clique K_p with the core K_1. We have obtained the tuner set for this subclass and established the relation between the Top $T(G)$ and the cardinality of the tuner set $|\Psi|$ through necessary and sufficient conditions. We analyze and characterize these graphs and obtain some interesting results while simultaneously examining the existence of tuner sets.

Keywords Network Equilibrium, Generalised Core-satellite Graphs, Graphs of Diameter Two, Average Degree, Tuner Set

1 Introduction

Biological networks, transportation systems, electric power grids, wireless communication networks, the Internet, neural networks and protein networks are few examples of the complex networks that exist in nature and in human society. Recent research focuses on developing the theory and modeling of complex networks especially the small-world network models and scale-free network models. In particular, in scale-free network, the vertices exhibit anomalous vertex connectivity, where a small fraction of vertices are very densely connected [7].

Generalized core-satellite graph $\Theta\left(c, \bar{S}, \bar{\eta}\right) = K_c \bigtriangledown (\bar{\eta} K_{\bar{S}})$ comprises $\bar{\eta}$ copies of $K_{\bar{S}}$ (the satellites) meeting in K_c (the core). It has a central core of vertices connected to a few satellites, where all satellite cliques need not be identical and can be of different sizes. They exhibit the scale-free vertex degree distribution. The family of graphs which we have analysed in this paper are a subclass of the generalised core-satellite graphs which possess atleast one universal vertex. Generalised core-satellite graphs are subclass of Quasi-threshold graphs, belonging to the wide family of graphs of diameter two [4]. These graph are more realistic and flexible to model hierarchical structural properties of the complex networks [1, 2, 5].

Rodrigues et al. [6] observe the association between the vertex degree distribution and the average degree of the graph

$d(G)$. Maximiliano Pinto Damas et al. [3] introduced the definitions of Top $T(G)$, tuner set, balanced and non-balanced graphs using the average vertex degree $d(G)$. According to Maximiliano et al. [3], the idea of network equilibrium is found in the definition of tuner set. Tuner sets are the subsets of vertices of G whose degrees are lower than $d(G)$ that compensate or balance the presence of vertices with degrees greater than $d(G)$.

In this paper, we characterize the family of graphs involving two distinct cliques γK_p and $\eta\ K_q$ with $p > q$, where γ copies of the clique K_p and η copies of the clique K_q joined K_1. We have established the relation between the Top $T(G)$ and the cardinality of the tuner set $|\Psi|$ through necessary and sufficient conditions. We recognize that the vertices in the tuner set are determined by the the Top $T(G)$ and the non-empty upper vertex set $U_G = \{v \in V(G) : d(v) > T(G)\}$.

1.1 BASIC NOTATIONS AND DEFINITIONS

Consider the graph $G = (V, E)$. Let $d(v_i)$ denote the degree of vertex $v_i \in V(G)$. Let δ and Δ be the minimum and maximum degree of G.

Definition 1.1. *The* ***average degree*** *of G is* $d(G) = \frac{\Sigma d(v_i)}{n}$, *for* $1 \leq i \leq n$ *where* $0 \leq d(G) \leq (n-1)$, *where* $d(G)$ *need not be an integer.*

Definition 1.2. *[4] The* ***Top*** *of graph G,* $T(G)$ *is defined as* $\lceil d(G) \rceil$.

Definition 1.3. *[4] The* ***balanced vertex set*** *is defined as*

$$B_G = \{v \in V(G) : d(v) = T(G)\},$$

The ***upper vertex set*** *is defined as*

$$U_G = \{v \in V(G) : d(v) > T(G)\} \text{ and}$$

$$L_G = \{v \in V(G) : d(v) < T(G)\}$$

is the ***lower vertex set****, where* $|B_G| + |U_G| + |L_G| = n$.

Definition 1.4. *[4] Let* L_G, U_G *be the lower vertex set and the upper vertex set of G, respectively, such that* $U_G \neq \varnothing$. *There exists a tuner set* $\Psi \subseteq L_G$ *for which the following equality holds:*

$$T(G) = \frac{\Sigma_{t \in \Psi \subseteq L(G)} d(t) + \Sigma_{u \in U_G} d(u)}{|\Psi| + |U_G|} \quad (1)$$

U_G *determines* ***tuner set*** Ψ *of the graph G. In general* $\Psi \subseteq L_G$. *When* $\Psi = L_G$ *then the tuner set is said to be a full tuner set of G. When* $k = |\ \Psi\ | < |L_G|$, Ψ *is called a proper tuner set of G.*

Example 1.1. Here we illustrate this definition of tuner set Ψ with an example of a Quasi-threshold graph (Figure 1).

In this graph G given in Figure 1, the Top, upper vertex set, balanced vertex set and lower vertex set are,

$$T(G) = \lceil d(G) \rceil = 3$$

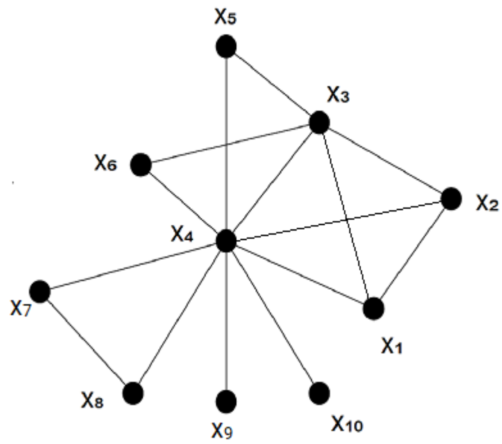

Figure 1. Quasi-threshold graph

$$U_G = \{v : d(v) > T(G)\} = \{x_4, x_3\}$$
$$B_G = \{v : d(v) = T(G)\} = \{x_1, x_2\}$$
$$L_G = \{v : d(v) < T(G)\}$$
$$= \{x_5, x_6, x_7, x_8, x_9, x_{10}\}$$

We construct the tuner set making use of the fact that $T(G) = 3$ for this graph. To construct the tuner set, we need to appropriately choose the vertices from the lower vertex set L_G. For the given graph, the sum of the degrees of the vertices of U_G i.e $\Sigma_{u \in U_G} d(u) = 14$ and cardinality of the the upper vertex set $|\ U_G\ |= 2$ are known, whereas $\Sigma_{t \in \Psi \subseteq (G)} d(t)$ and $|\Psi|$ are to be determined. We note that only by choosing all the vertices of L_G we obtain $T(G) = 3$. i.e,

$$T(G) = \frac{\{\Sigma(d(x_i) + d(x_4) + d(x_3)\}}{(|\Psi| + |U_G|)} = \frac{(10 + 14)}{(6 + 2)} = 3.$$

where $i = 5, 6, 7, 8, 9, 10$.

The tuner set for the above graph is $\Psi(G) = L_G$.

In section 2, we use this concept to the subclass of generalised core satellite graph defined as $(\eta K_q + \gamma K_p) \bigtriangledown K_1, p > q$, which is a join of η copies of the clique K_q and γ copies of the clique K_p with K_1 denoted as O, the universal vertex. If $U_G = \varnothing$ then G is said to be a balanced graph. Otherwise, G is said to be non-balanced graph [4]. Clearly, we can observe that graphs which we have considered in this paper are non-balanced.

2 Main Results

The following theorem which gives the necessary and sufficient condition for $T(G) = p$.

Theorem 2.1. *Let G be the graph* $(\gamma K_p + \eta K_q) \bigtriangledown K_1$, *where* $p > q$ *and* $|\ \eta - \gamma\ |= 1$. *Then* $T(G) = p$ *if and only if* $|\ \Psi\ |= \frac{(\Delta - p)}{(p-q)}$.

Proof. Assuming $T(G) = p$, we have to prove that $\mid \Psi \mid = \frac{(\Delta - p)}{(p-q)}$. The condition $\mid \eta - \gamma \mid = 1$ implies $(\eta - \gamma) = 1$ or $(\gamma - \eta) = 1$.
We have

$$B_G = \{d(v) = T(G) = p\}$$

Therefore

$$|B_G| = \gamma p$$

$$U_G = \{v : d(v) > T(G)\} = \{O\}$$

where O is the universal vertex of the graph G, such that $d(O) = (n-1) = \Delta > p$ and $|U_G| = 1$.

$$L_G = \{v \in V(G) : d(v) < p\}$$

Hence

$$|L_G| = \eta q$$

Since we have $\Sigma_{u \in U_G} d(u) = (n-1)$, we get

$$T(G) = p = \frac{\Sigma_{t \in \Psi \subseteq L_G} d(t) + \Sigma_{u \in U_G} d(u)}{|\Psi| + |U_G|}$$

$$= \frac{\Sigma_{t \in \Psi} d(t) + (n-1)}{|\Psi| + 1}.$$

As Ψ is a subset of $L_G = \{v \in V(G) : d(v) = q < T(G)\}$, we have

$$\Sigma_{t \in \Psi \subseteq L_G} d(t) = |\Psi| q$$

Therefore

$$p = \frac{\Sigma_{t \in \Psi} d(t) + (n-1)}{|\Psi| + 1} = \frac{|\Psi| q + (n-1)}{|\Psi| + 1}$$

On simplifying, we obtain

$$|\Psi| = \frac{(\Delta - p)}{(p-q)}$$

Now consider $|\Psi| = \frac{(\Delta - p)}{(p-q)}$, we prove that $T(G) = p$.
Consider

$$T(G) = \frac{\Sigma_{t \in \Psi \subseteq L_G} d(t) + \Sigma_{u \in U_G} d(u)}{|\Psi| + |U_G|}.$$

Substituting $\Sigma_{u \in U_G} d(G) = (n-1) = \Delta$ and $\Sigma_{t \in \Psi \subseteq L_G} d(t) = |\Psi| q$

$$T(G) = \frac{|\Psi| q + \Delta}{|\Psi| + 1}.$$

Using $|\Psi|$ in the above expression, we have

$$T(G) = \frac{(\Delta - p) q + \Delta (p - q)}{(\Delta - p) + (p - q)}.$$

On simplifying, we get $T(G) = p$ □

Corollary 2.1. *Let* $G = (\eta K_q + \gamma K_{q+1}) \bigtriangledown K_1$, *where* $|\eta - \gamma| = 1$. *Then* $T(G) = (q+1)$ *if and only if*

$$|\Psi| = (\Delta - (q+1)) = \Delta - T(G).$$

Proof. Let $T(G) = (q+1)$. We observe

$$B_G = \{v : d(v) = (q+1)\}$$

$$= \{v : v \in V(\gamma K_{(q+1)})\}$$

is a nonempty set. We have

$$L_G = \{v : d(v) < T(G) = (q+1)\}$$

$$= \{v : v \in V(\eta K_q)\}$$

Therefore,

$$T(G) = (q+1) = \frac{\Sigma_{t \in \Psi \subseteq L_G} d(t) + \Sigma_{u \in U_G} d(u)}{|\Psi| + |U_G|}.$$

$$(q+1) = \frac{|\Psi| q + \Delta}{|\Psi| + 1}.$$

Rewriting, the above expression, we obtain

$$|\Psi| = (\Delta - (q+1)) = (\Delta - T(G))$$

To prove the converse, assume $|\Psi| = (\Delta - T(G))$.

On using the expression for $|\Psi|$ in

$$T(G) = \frac{|\Psi| q + \Delta}{|\Psi| + 1}$$

We obtain

$$T(G) = (q+1)$$

□

Example 2.1. The graph $G = (2K_5 + 3K_3) \bigtriangledown K_1$ illustrated in Figure 2 is an example for Theorem 2.1.

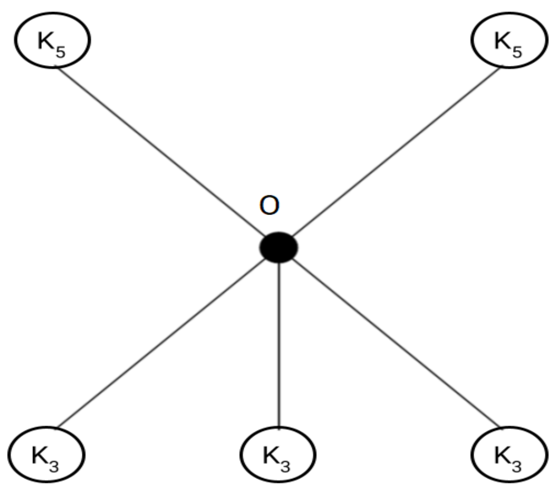

Figure 2. $(2K_5 + 3K_3) \bigtriangledown K_1$

For this graph the average degree $d(G) = 4.8$ and $T(G) = \lceil d(G) \rceil = 5$. The balanced vertex set is $B_G = \{v : d(v) = T(G) = 5\}$. Hence $B_G = \{v : v \in 2K_5\}$.

Let the universal vertex of the graph G be labelled as O, $d(O) = \Delta = (n-1) = 19$. So $U_G = \{v : d(v) > 5\} = \{O\}$.

The lower vertex set is

$$L_G = \{v : d(v) < 5\} = \{v : v \in 3K_3\}$$

We have from Theorem 2.1

$$|\Psi| = \frac{(\Delta - p)}{(p-q)} = \frac{(19-5)}{(5-3)} = \frac{14}{2} = 7$$

.
Here Ψ comprises 7 vertices which are taken from L_G. In this example $\Psi \subset L_G$. We confirm the value of $T(G)$ by substituting the value of $|\Psi|$, q and Δ in $T(G)$

$$T(G) = \frac{(|\Psi|q + \Delta)}{(|\Psi| + 1)} = 5.$$

Theorem 2.2. *Let $G = (\gamma K_p + \eta K_q) \bigtriangledown K_1$, where $p > (q+1)$, and $(\gamma - \eta) \geq 2$. Then $T(G) = (p+1)$ if and only if*

$$|\Psi| = \frac{(\Delta - (p+1) + m_1(p-q))}{(p-q+1)} \tag{2}$$

where $|\Psi|$ = $m_1 + m_2$, where m_1 and m_2 are positive integers ≥ 1.

Proof. Let $T(G) = (p+1) > p > q$, therefore $B_G = \varnothing$.

$$L_G = \{v : d(v) < (p+1)\}$$
$$= \{v : v \in V(\gamma K_p \cup \eta K_q)\}$$

We choose the vertices of the tuner set from L_G in the ratio $m_1 : m_2$, where m_1, m_2 are some positive integer ≥ 1. The m_1 vertices are chosen from γK_p and m_2 vertices are chosen from ηK_q.
Therefore,

$$\Sigma_{t \in \Psi \subseteq L_G} d(t) = (m_1 p + m_2 q)$$

Also

$$\Sigma_{u \in U_G} d(u) = \Delta = (n-1)$$

Having $T(G) = (p+1)$, the above expression yields

$$(p+1) = \frac{\Sigma_{t \in \Psi \subseteq L_G} d(t) + \Sigma_{u \in U_G} d(u)}{|\Psi| + |U_G|}$$

$$= \frac{((m_1 p + m_2 q) + \Delta)}{(m_1 + m_2 + 1)}.$$

$$m_2 = \frac{\Delta - (m_1 + m_2 + p + 1)}{(p-q)}$$

$$m_2 = \frac{(\Delta - (|\Psi| + p + 1))}{(p-q)}$$

As

$$m_1 = |\Psi| - m_2$$

we get

$$m_1 = \frac{(|\Psi|(p-q+1) - \Delta + (p+1))}{(p-q)}.$$

This is rewritten as

$$|\Psi| = \frac{(\Delta - (p+1) + m_1(p-q))}{(p-q+1)}.$$

Assuming $|\Psi|$ as given in (2), we now show that $T(G) = (p+1)$. Using

$$\Sigma_{t \in \Psi \subseteq L_G} d(t) = (m_1 p + m_2 q)$$

and

$$\Sigma_{u \in U_G} d(u) = \Delta = (n-1)$$

in (1). We have

$$T(G) = \frac{((m_1 p + m_2 q) + \Delta)}{(|\Psi| + 1)}$$

$$= \frac{(|\Psi|(p^2 - pq + p) - \Delta(p-q)}{(p-q)(|\Psi|+1)}$$

$$+ \frac{p(p+1) - q(p+1) + \Delta(p-q))}{(p-q)(|\Psi|+1)}$$

On simplifying, we obtain

$$T(G) = \frac{(|\Psi|+1)(p+1)(p-q)}{(p-q)(|\Psi|+1)} = (p+1)$$

□

Corollary 2.2. *Let $G = (\eta K_q + \gamma K_{q+1}) \bigtriangledown K_1$, where $\gamma \geq 2$ for any positive η. Then $T(G) = (q+2)$ if and only if*

$$|\Psi| = \frac{(\Delta - (q+2) + m_1)}{2}$$
$$= (\Delta - (q+2) - m_2).$$

where $|\Psi| = m_1 + m_2$.

Proof. Let $T(G) = (q+2)$, where $(q+2) > (q+1) > q$. Therefore $B_G = \varnothing$. The lower vertex set L_G comprises of the vertices of degrees q and $(q+1)$.

$$L_G = \{v : d(v) < T(G) = (q+2)\}$$
$$= \{v : v \in V(\eta K_q \cup \gamma K_{q+1})\}$$

On choosing the vertices from L_G such that m_1 vertices from γK_{q+1} and m_2 from ηK_q, we get

$$\Sigma_{t \in \Psi \subseteq L_G} d(t) = (m_1(q+1) + m_2 q)$$

Therefore

$$T(G) = (q+2) = \frac{\Sigma_{t \in \Psi \subseteq L_G} d(t) + \Sigma_{u \in U_G} d(u)}{|\Psi| + |U_G|}.$$

$$= \frac{(m_1(q+1) + m_2 q + \Delta)}{|\Psi| + 1}.$$

On simplifying the above expression, we have

$$|\Psi| = (\Delta - (q+2) - m_2)$$

As $|\Psi| = m_1 + m_2$.

$$m_1 = (2|\Psi| + (q+2) - \Delta)$$

$$m_2 = (\Delta - (q+2) - |\Psi|)$$

To prove the converse, assume

$$|\Psi| = (\Delta - (q+2) + m_2)$$

We need to prove $T(G) = (q+2)$.

$$T(G) = \frac{m_1(q+1) + m_2 q + \Delta}{|\Psi| + 1}$$

Substituting for m_1, m_2 in terms of $|\Psi|$, Δ and q, we obtain

$$T(G) = \frac{(q+2)(-q + |\Psi| + q + 1)}{|\Psi| + 1} = (q+2).$$

□

Example 2.2. Theorem 2.2 is illustrated using this graph $G = (K_4 + 3K_6) \bigtriangledown K_1$ given in Figure 3.

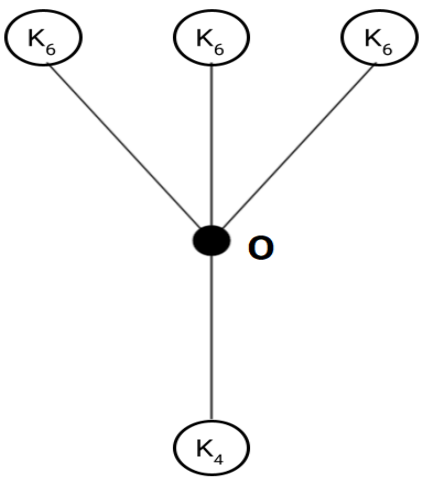

Figure 3. $(K_4 + 3K_6) \bigtriangledown K_1$ with $B_G = \phi$.

For this graph the average degree $d(G)$ = 6.3, $\lceil d(G) \rceil$ = 7. Since the Top

$$T(G) = (p+1) = 7$$

The balanced vertex set is

$$B_G = \{v : d(v) = T(G) = 7\} = \varnothing.$$

The upper vertex set is

$$U_G = \{v : d(v) > 7\} = \{O\}$$

where $d(O) = \Delta = (n-1) = 23$.

The lower vertex set is

$$L_G = \{v : d(v) < 7\} = \{v : v \in V(K_4 \cup 3K_6)\}$$

In this example, by choosing 6 vertices from $3K_6$ and 3 vertices from K_4, i.e m_1 = 6 and m_2 = 3.

$$|\Psi| = m_1 + m_2 = 9$$

This combination of choosing 6 vertices from $3K_6$ and 3 vertices from K_4 can be verified by substituting the value of Δ, p, q, m_1 and m_2 in $|\Psi|$.

$$|\Psi| = \frac{(\Delta - (p+1) + m_1(p-q))}{(p-q+1)} = 9$$

The value of $T(G)$ can be confirmed by substituting the value p, q, Δ, m_1 and m_2 in $T(G)$.

$$T(G) = \frac{((m_1 p + m_2 q) + \Delta)}{(m_1 + m_2 + 1)}$$

$$= \frac{(6 \times 6 + 3 \times 4 + 22)}{(9+1)} = 7 = (p+1).$$

Hence verified.

The following theorem, characterizes the class of graphs G that have no tuner set.

Theorem 2.3. *There exists no tuner set for the graphs* $(\eta K_q + (\eta+1)K_{q+4} \bigtriangledown K_1$*, where* q *is an odd integer and* $\eta \neq 4l$ *for some integer* l.

Proof. We observe

$$n = (\eta+1)(q+4) + \eta q + 1$$

$$d(G) = \frac{((\eta+1)(q+4)^2) + \eta q^2 + (\eta+1)(q+4) + \eta q)}{((q+4) + 2\eta q + 4\eta + 1))}$$

$$= \frac{(q+4)((q+4) + 2\eta q + 4\eta)}{((q+4) + 2\eta q + 4\eta + 1)}$$

Taking

$$\frac{((q+4) + 2\eta q + 4\eta)}{((q+4) + 2\eta q + 4\eta + 1)} = \epsilon$$

where $0 < \epsilon < 1$, we have

$$d(G) = (q+4)\epsilon < (q+4)$$

Therefore $\lceil d(G) \rceil = T(G) = (q+4)$.
Hence $T(G) = (q+4)$.
For this class of graphs, we have

$$B_G = \{v : v \in V(\eta+1)K_{(q+4))}\} \neq \varnothing$$

The lower vertex set L_G comprises of vertices whose degree is less than $(q+4)$.

$$L_G = \{v : d(v) < T(G) = (q+4)\}$$

$$= \{v : v \in V(\eta K_q)\}$$

Therefore

$$T(G) = (q+4) = \frac{|\Psi|q + (n-1)}{|\Psi| + 1} \tag{3}$$

From (3), we have

$$|\Psi| = \frac{(n-q-5)}{4}$$

Substituting for n in the above expression, we obtain

$$|\Psi| = \frac{\eta(2q+4)}{4}$$

Since q takes positive odd integer values, we have $q = (2k+1)$, for $k > 0$.

Therefore

$$|\Psi| = \frac{\eta(4k+6)}{4}$$

The above expression is not divisible by 4. Hence $|\Psi|$ is a non-integer. Therefore, this class of graphs do not have a tuner set as $\eta \neq 4l$ for some integer l. □

3 Conclusions

In this paper, we identify and discuss the existence of the tuner set for a subclass of the generalised core satellite graph. We have established the relation between the Top $T(G)$ and the cardinality of the tuner set $|\Psi|$ through necessary and sufficient conditions.

REFERENCES

[1] Ernesto Estrada, The Structure of Complex Networks. Theory and Applications, Oxford University Press, 2011.

[2] Ernesto Estrada, Michele Benzi, Core–satellite graphs: Clustering, assortativity and spectral properties, Linear Algebra and its Applications 517(2017)30–52.

[3] Maximiliano Pinto Damas, Lilian Markenzon, Nair Maria Maia de Abreu, New Concepts and results on the Average degree of a graph, Applicable Analysis and Discrete Mathematics,1 (2007), 284-292.

[4] Nair Abreu et al., Integer Laplacian eigenvalues of chordal graphs, Linear Algebra and its Applications 614(2021),68-81, DOI:10.1016/j.laa.2019.12.030

[5] E. Ravasz, Albert-Laszlo Barabasi, Hierarchical organization in complex networks, Phys. Rev. E 67 (2003) 026112, DOI: 10.1103/PhysRevE.67.026112.

[6] R. M. N. D. Rodrigues, N. M. M. Abreu, L. Markenzon: Maxregularity and Maximal Outerplanar Graphs. Electronic Notes in Discrete Mathematics, 3 (1999),171–175.

[7] Wenjun Xiao, Longxin Lin, Guanrong Chenc, Vertex-degree sequences in complex networks: New characteristics and applications,Physica A 437 (2015) 437–441.