Mathematics and Statistics 10(5): 1121-1126, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100523

Anti-hesitant Fuzzy Subalgebras, Ideals and Deductive Systems of Hilbert Algebras

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Received June 5, 2022; Revised September 13, 2022; Accepted September 21, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] Aiyared Iampan, S. Yamunadevi, P. Maragatha Meenakshi, N. Rajesh, "Anti-hesitant Fuzzy Subalgebras, Ideals and Deductive Systems of Hilbert Algebras," Mathematics and Statistics, Vol.10, No.5, pp. 1121-1126, 2022. DOI: 10.13189/ms.2022.100523*

***(b)**: Aiyared Iampan, S. Yamunadevi, P. Maragatha Meenakshi, N. Rajesh, (2022). Anti-hesitant Fuzzy Subalgebras, Ideals and Deductive Systems of Hilbert Algebras. Mathematics and Statistics, 10(5), 1121-1126. DOI: 10.13189/ms.2022.100523*

Abstract The Hilbert algebra, one of several algebraic structures, was first described by Diego in 1966 [7] and has since been extensively studied by other mathematicians. Torra [18] was the first to suggest the idea of hesitant fuzzy sets (HFSs) in 2010, which is a generalization of the fuzzy sets defined by Zadeh [20] in 1965 as a function from a reference set to a power set of the unit interval. The significance of the ideas of hesitant fuzzy subalgebras, ideals, and filters in the study of the different logical algebras aroused our interest in applying these concepts to Hilbert algebras. In this paper, the concepts of HFSs to subalgebras (SAs), ideals (IDs), and deductive systems (DSs) of Hilbert algebras are introduced in terms of anti-types. We call them anti-hesitant fuzzy subalgebras (AHFSAs), anti-hesitant fuzzy ideals (AHFIDs), and anti-hesitant fuzzy deductive systems (AHFDSs). The relationships between AHFSAs, AHFIDs, and AHFDSs and their lower and strong level subsets are provided. As a result of the study, we found their generalization as follows: every AHFID of a Hilbert algebra Ω is an AHFSA and an AHFDS of Ω. We also study and find the conditions for the complement of an HFS to be an AHFSA, an AHFID, and an AHFDS. In addition, the relationships between the complements of AHFSAs, AHFIDs, and AHFDSs and their upper and strong level subsets are also provided.

Keywords Hilbert Algebra, Anti-hesitant Fuzzy Subalgebra, Anti-hesitant Fuzzy Ideal and Anti-hesitant Fuzzy Deductive System

1 Introduction

Zadeh introduced the idea of fuzzy sets (FSs) in [20]. Numerous academics have studied the notion of FSs, which have numerous applications in everyday life. Numerous investigations were undertaken into the generalizations of FSs. In [1, 2, 5], it is explained how FSs can be integrated with some uncertainty-reduction strategies like soft sets and rough sets. The concept of HFSs, a function from a reference set to a power set of the unit interval, was introduced in 2009–2010 by Torra and Narukawa [18, 19]. The idea of FSs has been expanded to include HFSs. Numerous uses for the HFS ideas created by Torra and others can be found outside of mathematics and in it. Following Torra and Narukawa's [18, 19] introduction of the concept of HFSs, numerous studies on its generalizations and applications to numerous logical algebras have been conducted. Examples of related articles can be found at [3, 4, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 22].

In this paper, the concepts of HFSs to SAs, IDs, and DSs of Hilbert algebras are introduced in terms of anti-types. The relationships between AHFSAs, AHFIDDs, and AHFDSs and their lower and strong level subsets are provided. In addition, the relationships between the complements of AHFSAs, AH-

FIDDs, and AHFDSs and their upper and strong level subsets are also provided.

2 Preliminaries

Let's briefly review the idea of a Hilbert algebra as it was introduced by Diego [7] before we get started.

Definition 2.1. [7] A *Hilbert algebra* is a triplet with the formula $\Omega = (\Omega, \star, \omega)$, where $\Omega \neq \emptyset$, $\star$ is a binary operation, and ω is a fixed element of Ω that satisfies the conditions:

(1) $(\forall \iota, \varepsilon \in \Omega)(\iota \star (\varepsilon \star \iota) = \omega)$,

(2) $(\forall \iota, \varepsilon, \eta \in \Omega)((\iota \star (\varepsilon \star \eta)) \star ((\iota \star \varepsilon) \star (\iota \star \eta)) = \omega)$,

(3) $(\forall \iota, \varepsilon \in \Omega)(\iota \star \varepsilon = \omega, \varepsilon \star \iota = \omega \Rightarrow \iota = \varepsilon)$.

Now let $\Omega = (\Omega, \star, \omega)$ be a Hilbert algebra.

Lemma 2.2. *[8] In Ω, we have*

(1) $(\forall \iota \in \Omega)(\iota \star \iota = \omega)$,

(2) $(\forall \iota \in \Omega)(\omega \star \iota = \iota)$,

(3) $(\forall \iota \in \Omega)(\iota \star \omega = \omega)$,

(4) $(\forall \iota, \varepsilon, \eta \in \Omega)(\iota \star (\varepsilon \star \eta) = \varepsilon \star (\iota \star \eta))$.

In Ω, the binary relation $\leq$ is given by

$$(\forall \iota, \varepsilon \in \Omega)(\iota \leq \varepsilon \Leftrightarrow \iota \star \varepsilon = \omega),$$

which is a partial order on Ω.

Definition 2.3. [21] $\emptyset \neq \Sigma \subseteq \Omega$ is called a *SA* of Ω if $\iota \star \varepsilon \in \Sigma$ $\forall \iota, \varepsilon \in \Sigma$.

Definition 2.4. [6] $\emptyset \neq \Sigma \subseteq \Omega$ is called an *ID* of Ω if the statements below are accurate:

(1) $\omega \in \Sigma$,

(2) $(\forall \iota, \varepsilon \in \Omega)(\varepsilon \in \Sigma \Rightarrow \iota \star \varepsilon \in \Sigma)$,

(3) $(\forall \iota, \varepsilon_1, \varepsilon_2 \in \Omega)(\varepsilon_1, \varepsilon_2 \in \Sigma \Rightarrow (\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota \in \Sigma)$.

Definition 2.5. [9] $\emptyset \neq \Sigma \subseteq \Omega$ is called a *DS* of Ω if

(1) $\omega \in \Sigma$,

(2) $(\forall \iota, \varepsilon \in \Omega)(\iota, \iota \star \varepsilon \in \Sigma \Rightarrow y \in \Sigma)$.

In $\Omega \neq \emptyset$, a *fuzzy set* (FS) is defined as a function $f : \Omega \to [0, 1] \subseteq \mathbb{R}$.

Definition 2.6. [18] Let Ω serve as the reference set. A *hesitant fuzzy set* (HFS) on Ω is described in terms of a function H that, when applied to Ω returns a subset of $[0, 1]$; specifically, $\mathrm{H} : \Omega \to 2^{[0,1]}$, where $2^{[0,1]}$ means the power set of $[0, 1]$.

For convenience, we set H instead of an HFS on Ω.

Definition 2.7. [18] The HFS $\overline{\mathrm{H}}$ is defined by $\overline{\mathrm{H}}(\iota) = [0, 1] \setminus \mathrm{H}(\iota)$ $\forall \iota \in \Omega$.

3 AHFSAs, AHFIDs, and AHFDSs

We discuss the ideas of AHFSAs, AHFIDs, and AHFDSs of Hilbert algebras in this part and look at various related characteristics.

Definition 3.1. If an HFS H on Ω meets the following condition, it is referred to as a *AHFSA* of Ω:

$$(\forall \iota, \varepsilon \in \Omega)(\mathrm{H}(\iota \star \varepsilon) \subseteq \mathrm{H}(\iota) \cup \mathrm{H}(\varepsilon)). \tag{3.1}$$

Example 3.2. Let $\Omega = \{\omega, \iota, \varepsilon, \eta, 0\}$ be a Hilbert algebra as shown in the table below that contains ω.

$\star$	ω	ι	ε	η	0
ω	ω	ι	ε	η	0
ι	ω	ω	ε	η	0
ε	ω	ι	ω	η	η
η	ω	ω	ε	ω	ε
0	ω	ω	ω	ω	ω

Define an HFS H on Ω as follows:

$$\mathrm{H}(\omega) = \emptyset, \mathrm{H}(\iota) = \{0.1, 0.2\}, \mathrm{H}(\varepsilon) = \{0.2\},$$
$$\mathrm{H}(\eta) = \{0.2\}, \mathrm{H}(0) = \{0.2\}.$$

Then H is an AHFSA of Ω.

Proposition 3.3. *If H is an AHFSA of Ω, then*

$$(\forall \iota \in \Omega)(\mathrm{H}(\omega) \subseteq \mathrm{H}(\iota)).$$

Proof. For any $\iota \in \Omega$, we obtain $\mathrm{H}(\omega) = \mathrm{H}(\iota \star \iota) \subseteq \mathrm{H}(\iota) \cup \mathrm{H}(\iota) = \mathrm{H}(\iota)$. □

Proposition 3.4. *If H is an AHFSA of Ω, then*

$$(\forall \iota \in \Omega)(\mathrm{H}(\omega \star \iota) \subseteq \mathrm{H}(\iota)).$$

Proof. For any $\iota \in \Omega$, we obtain $\mathrm{H}(\omega \star \iota) \subseteq \mathrm{H}(\omega) \cup \mathrm{H}(\iota) = \mathrm{H}(\iota \star \iota) \cup \mathrm{H}(\iota) \subseteq \mathrm{H}(\iota) \cup \mathrm{H}(\iota) \cup \mathrm{H}(\iota) = \mathrm{H}(\iota)$. □

Definition 3.5. If an HFS H on Ω meets the following conditions, it is referred to as a *AHFID* of Ω:

$$(\forall \iota \in \Omega)(\mathrm{H}(\omega) \subseteq \mathrm{H}(\iota)), \tag{3.2}$$
$$(\forall \iota, \varepsilon \in \Omega)(\mathrm{H}(\iota \star \varepsilon) \subseteq \mathrm{H}(\varepsilon)), \tag{3.3}$$
$$(\forall \iota, \varepsilon_1, \varepsilon_2 \in \Omega)(\mathrm{H}((y_1 \star (\varepsilon_2 \star \iota)) \star \iota) \subseteq \mathrm{H}(\varepsilon_1) \cup \mathrm{H}(\varepsilon_2)). \tag{3.4}$$

Example 3.6. Let $\Omega = \{\omega, \iota, \varepsilon, \eta\}$ be a Hilbert algebra as shown in the table below that contains ω.

$\star$	ω	ι	ε	η
ω	ω	ι	ε	η
ι	ω	ω	ε	ε
ε	ω	ι	ω	ι
η	ω	ω	ω	ω

Define an HFS H on Ω as follows:

$$\mathrm{H}(\omega) = \emptyset, \mathrm{H}(\iota) = \{0.1, 0.2\}, \mathrm{H}(\varepsilon) = \{0.1\},$$
$$\mathrm{H}(\eta) = \{0.1, 0.2\}.$$

Then H is an AHFID of Ω.

Proposition 3.7. *If* H *is an AHFID of* Ω*, then*

$$(\forall \iota, \varepsilon \in \Omega)(\mathrm{H}((\varepsilon \star \iota) \star \iota) \subseteq \mathrm{H}(\varepsilon)).$$

Proof. Putting $\varepsilon_1 = \varepsilon$ and $\varepsilon_2 = \omega$ in (3.4), we have $\mathrm{H}((\varepsilon \star \iota) \star \iota) \subseteq \mathrm{H}(\varepsilon) \cup \mathrm{H}(\omega) = \mathrm{H}(\varepsilon)\ \forall \iota, \varepsilon \in \Omega$. □

Lemma 3.8. *If* H *is an AHFID of* Ω*, then*

$$(\forall \iota, \varepsilon \in \Omega)(\iota \leq \varepsilon \Rightarrow \mathrm{H}(\iota) \supseteq \mathrm{H}(\varepsilon)).$$

Proof. Let $\iota, \varepsilon \in \Omega$ in which $\iota \leq \varepsilon$. Then $\iota \star \varepsilon = \omega$ and so $\mathrm{H}(\varepsilon) = \mathrm{H}(\omega \star \varepsilon) = \mathrm{H}(((\iota \star \varepsilon) \star (\iota \star \varepsilon)) \star \varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon) \cup \mathrm{H}(\iota) = \mathrm{H}(\omega) \cup \mathrm{H}(\iota) = \mathrm{H}(\iota)$. □

Theorem 3.9. *Every AHFID of* Ω *is an AHFSA of* Ω*.*

Proof. Let H be an AHFID of Ω. Let $\iota, \varepsilon \in \Omega$. Because $\varepsilon \leq \iota \star \varepsilon$ and by Lemma 3.8, $\mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon)$. It follows from (3.3) that $\mathrm{H}(\iota \star \varepsilon) \subseteq \mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon) \cup \mathrm{H}(\iota) \subseteq \mathrm{H}(\iota) \cup \mathrm{H}(\varepsilon)$. Summarize H is an AHFSA of Ω. □

Definition 3.10. If an HFS H on Ω meets the following conditions, it is referred to as a *AHFDS* of Ω:

$$(\forall \iota \in \Omega)(\mathrm{H}(\omega) \subseteq \mathrm{H}(\iota)), \tag{3.5}$$

$$(\forall \iota, \varepsilon \in \Omega)(\mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon) \cup \mathrm{H}(\iota)). \tag{3.6}$$

Example 3.11. From Example 3.6, we get H is an AHFDS of Ω.

Proposition 3.12. *Every AHFID of* Ω *is an AHFDS of* Ω*.*

Proof. Let H be an AHFID of Ω. Let $\iota, \varepsilon \in \Omega$. If $\varepsilon_1 = \iota \star \varepsilon$ and $\varepsilon_2 = \iota$, then by (1), (2) of Lemma 2.2 and (3.4), we have $\mathrm{H}(\varepsilon) = \mathrm{H}(\omega \star \varepsilon) = \mathrm{H}(((\iota \star \varepsilon) \star (\iota \star \varepsilon)) \star \varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon) \cup \mathrm{H}(\iota)$. Summarize H is an AHFDS of Ω. □

Lemma 3.13. *If* H *is an AHFDS of* Ω*, then*

$$(\forall \iota, \varepsilon, \eta \in \Omega)(\eta \leq \iota \star \varepsilon \Rightarrow \mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota) \cup \mathrm{H}(\eta)).$$

Proof. Let $\iota, \varepsilon, \eta \in \Omega$ in which $\eta \leq \iota \star \varepsilon$. Then $\eta \star (\iota \star \varepsilon) = \omega$ and so $\mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon) \cup \mathrm{H}(\iota) \subseteq \mathrm{H}(\eta \star (\iota \star \varepsilon)) \cup \mathrm{H}(\eta) \cup \mathrm{H}(\iota) = \mathrm{H}(\omega) \cup \mathrm{H}(\eta) \cup \mathrm{H}(\iota) = \mathrm{H}(\iota) \cup \mathrm{H}(\eta)$. □

Lemma 3.14. *If* H *is an AHFDS of* Ω*, then*

$$(\forall \iota, \varepsilon \in \Omega)(\iota \leq \varepsilon \Rightarrow \mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota)).$$

Proof. Let $\iota, \varepsilon \in \Omega$ in which $\iota \leq \varepsilon$. Then $\iota \star \varepsilon = \omega$ and so $\mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon) \cup \mathrm{H}(\iota) = \mathrm{H}(\omega) \cup \mathrm{H}(\iota) = \mathrm{H}(\iota)$. □

4 Level subsets of an HFS

In this section, we provide the relationship between AHFSAs, AHFIDs and AHFDSs and their level subsets.

For any $\pi \in 2^{[0,1]}$, the sets $S_{\supseteq}(\mathrm{H}, \pi) = \{\iota \in \Omega \mid \mathrm{H}(\iota) \supseteq \pi\}$, $S_{\supset}(\mathrm{H}, \pi) = \{\iota \in \Omega \mid \mathrm{H}(\iota) \supset \pi\}$, $S_{\subseteq}(\mathrm{H}, \pi) = \{\iota \in \Omega \mid \mathrm{H}(\iota) \subseteq \pi\}$, $S_{\subset}(\mathrm{H}, \pi) = \{\iota \in \Omega \mid \mathrm{H}(\iota) \subset \pi\}$, and $S(\mathrm{H}, \pi) = \{\iota \in \Omega \mid \mathrm{H}(\iota) = \pi\}$.

Theorem 4.1. *An HFS* H *on* Ω *is an AHFSA of* Ω *if and only if* $\forall \pi \in 2^{[0,1]}$*,* $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ *is a SA of* Ω*.*

Proof. Let's say H is an AHFSA of Ω. Let $\pi \in 2^{[0,1]}$ in which $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ and let $\iota \in S_{\subseteq}(\mathrm{H}, \pi)$. Then $\mathrm{H}(\iota) \subseteq \pi$. Because H is an AHFSA of Ω, $\mathrm{H}(\omega) \subseteq \mathrm{H}(\iota) \subseteq \pi$. Thus $\omega \in S_{\subseteq}(\mathrm{H}, \pi)$. Let $\iota, \varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. Then $\mathrm{H}(\iota) \subseteq \pi$ and $\mathrm{H}(\varepsilon) \subseteq \pi$. Because H is an AHFSA of Ω, $\mathrm{H}(\iota \star \varepsilon) \subseteq \mathrm{H}(\iota) \cup \mathrm{H}(\varepsilon) \subseteq \pi$ and hence $\iota \star \varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. Summarize $S_{\subseteq}(\mathrm{H}, \pi)$ is a SA of Ω.

On the other hand, assume $\forall \pi \in 2^{[0,1]}$, $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ is a SA of Ω. Let $\iota, \varepsilon \in \Omega$. Put $\pi = \mathrm{H}(\iota) \cup \mathrm{H}(\varepsilon) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \subseteq \pi$ and $\mathrm{H}(\varepsilon) \subseteq \pi$. So $\iota, \varepsilon \in S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$. By assuming, $S_{\subseteq}(\mathrm{H}, \pi)$ is a SA of Ω and hence $\iota \star \varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. So, $\mathrm{H}(\iota \star \varepsilon) \subseteq \pi = \mathrm{H}(\iota) \cup \mathrm{H}(\varepsilon)$. Summarize H is an AHFSA of Ω. □

Theorem 4.2. *An HFS* H *on* Ω *is an AHFID of* Ω *if and only if* $\forall \pi \in 2^{[0,1]}$*,* $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ *is an ID of* Ω*.*

Proof. Let's say H is an AHFID of Ω. Let $\pi \in 2^{[0,1]}$ in which $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ and let $\iota \in S_{\subseteq}(\mathrm{H}, \pi)$. Then $\mathrm{H}(\iota) \subseteq \pi$. As H is an AHFID of Ω, $\mathrm{H}(\omega) \subseteq \mathrm{H}(\iota) \subseteq \pi$. Thus $\omega \in S_{\subseteq}(\mathrm{H}, \pi)$. Next, let $\iota, \varepsilon \in \Omega$ in which $\varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. Then $\mathrm{H}(\varepsilon) \subseteq \pi$. Because H is an AHFID of Ω, $\mathrm{H}(\iota \star \varepsilon) \subseteq \mathrm{H}(\varepsilon) \subseteq \pi$. So, $\iota \star \varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. Let $\iota, \varepsilon_1, \varepsilon_2 \in \Omega$ in which $\varepsilon_1, \varepsilon_2 \in S_{\subseteq}(\mathrm{H}, \pi)$. Then $\mathrm{H}(\varepsilon_1) \subseteq \pi$ and $\mathrm{H}(\varepsilon_2) \subseteq \pi$. Because H is an AHFID of Ω, $\mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota)) \subseteq \mathrm{H}(\varepsilon_1) \cup \mathrm{H}(\varepsilon_2) \subseteq \pi$. So, $(\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota \in S_{\subseteq}(\mathrm{H}, \pi)$. Summarize $S_{\subseteq}(\mathrm{H}, \pi)$ is an ID of Ω.

On the other hand, assume $\forall \pi \in 2^{[0,1]}$, $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ is an ID of Ω. Let $\iota \in \Omega$. Then $\mathrm{H}(\iota) \in 2^{[0,1]}$. Put $\pi = \mathrm{H}(\iota) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \subseteq \pi$. So $\iota \in S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$. By assuming, we obtain $S_{\subseteq}(\mathrm{H}, \pi)$ is an ID of Ω and hence $\omega \in S_{\subseteq}(\mathrm{H}, \pi)$. So $\mathrm{H}(\omega) \subseteq \pi = \mathrm{H}(\iota)$. Next, let $\iota, \varepsilon \in \Omega$. Then $\mathrm{H}(\iota), \mathrm{H}(\varepsilon) \in 2^{[0,1]}$. Put $\pi = \mathrm{H}(\varepsilon) \in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon) \subseteq \pi$, so $\iota \in \Omega$ and $\varepsilon \in S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$. By assuming, we obtain $S_{\subseteq}(\mathrm{H}, \pi)$ is an ID of Ω and then $\iota \star \varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. Thus $\mathrm{H}(\iota \star \varepsilon) \subseteq \pi = \mathrm{H}(\varepsilon)$. Let $\iota, \varepsilon_1, \varepsilon_2 \in \Omega$. Then $\mathrm{H}(\iota), \mathrm{H}(\varepsilon_1), \mathrm{H}(\varepsilon_2) \in 2^{[0,1]}$. Put $\pi = \mathrm{H}(\varepsilon_1) \cup \mathrm{H}(\varepsilon_2) \in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon_1) \subseteq \pi$ and $\mathrm{H}(\varepsilon_2) \subseteq \pi$, so $\iota \in \Omega$ and $\varepsilon_1, \varepsilon_2 \in S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$. By assuming, we obtain $S_{\subseteq}(\mathrm{H}, \pi)$ is an ID of Ω and then $(\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota \in S_{\subseteq}(\mathrm{H}, \pi)$. Thus $\mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \subseteq \pi = \mathrm{H}(\varepsilon_1) \cup \mathrm{H}(\varepsilon_2)$. Summarize H is an AHFID of Ω. □

Theorem 4.3. *An HFS* H *on* Ω *is an AHFDS of* Ω *if and only if* $\forall \pi \in 2^{[0,1]}$*,* $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ *is a DS of* Ω*.*

Proof. Let's say H is an AHFDS of Ω. Let $\pi \in 2^{[0,1]}$ in which $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ and let $\iota \in S_{\subseteq}(\mathrm{H}, \pi)$. Then $\mathrm{H}(\iota) \subseteq \pi$. Because H is an AHFDS of Ω, $\mathrm{H}(\omega) \subseteq \mathrm{H}(\iota) \subseteq \pi$. Thus $\omega \in S_{\subseteq}(\mathrm{H}, \pi)$. Next, let $\iota, \varepsilon \in \Omega$ in which $\iota, \iota \star \varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. Then $\mathrm{H}(\iota) \subseteq \pi$ and $\mathrm{H}(\iota \star \varepsilon) \subseteq \pi$. Because H is an AHFDS of Ω, $\mathrm{H}(\varepsilon) \subseteq \mathrm{H}(\iota \star \varepsilon) \cup \mathrm{H}(\iota) \subseteq \pi$. So, $\varepsilon \in S_{\subseteq}(\mathrm{H}, \pi)$. Summarize $S_{\subseteq}(\mathrm{H}, \pi)$ is a DS of Ω.

On the other hand, assume $\forall \pi \in 2^{[0,1]}$, $S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$ is a DS of Ω. Let $\iota \in \Omega$. Then $\mathrm{H}(\iota) \in 2^{[0,1]}$. Put $\pi = \mathrm{H}(\iota) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \subseteq \pi$. So $\iota \in S_{\subseteq}(\mathrm{H}, \pi) \neq \emptyset$. By assuming, we obtain $S_{\subseteq}(\mathrm{H}, \pi)$ is a DS of Ω and hence $\omega \in S_{\subseteq}(\mathrm{H}, \pi)$. So, $\mathrm{H}(\omega) \subseteq \pi = \mathrm{H}(\iota)$. Let $\iota, \varepsilon \in \Omega$. Then $\mathrm{H}(\iota), \mathrm{H}(\iota \star \varepsilon) \in 2^{[0,1]}$. Put $\pi = \mathrm{H}(\iota) \cup \mathrm{H}(\iota \star \varepsilon) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \subseteq \pi$ and

$\mathrm{H}(\iota\star\varepsilon)\subseteq\pi$, so $\iota,\iota\star\varepsilon\in S_{\subseteq}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\subseteq}(\mathrm{H},\pi)$ is a DS of Ω and then $\varepsilon\in S_{\subseteq}(\mathrm{H},\pi)$. Thus $\mathrm{H}(\varepsilon)\subseteq\pi=\mathrm{H}(\iota)\cup\mathrm{H}(\iota\star\varepsilon)$. Summarize H is an AHFDS of Ω. □

Theorem 4.4. *The statements below are accurate.*

(1) If H *is an AHFSA of* Ω*, then* $\forall\pi\in 2^{[0,1]}$, $S_{\subset}(\mathrm{H},\pi)\neq\emptyset$ *is a SA of* Ω *and* $S(\mathrm{H},\pi)=\emptyset$*.*

(2) If H *is an AHFSA of* Ω*, then* $\forall\pi\in 2^{[0,1]}$, $S(H,\pi)\neq\emptyset$ *is a SA of* Ω *and* $S_{\subset}(H,\pi)=\emptyset$*.*

(3) If $\mathrm{H}(\Omega)$ *is comparable and* $\forall\pi\in 2^{[0,1]}$, $S_{\subset}(\mathrm{H},\pi)\neq\emptyset$ *is a SA of* Ω*, then* H *is an AHFSA of* Ω*.*

Proof. (1) and (2) are straightforward by Theorem 4.1.

(3) Let's say $\mathrm{H}(\Omega)$ is comparable and $\forall\pi\in 2^{[0,1]}$, $S_{\subset}(\mathrm{H},\pi)\neq\emptyset$ is a SA of Ω. Suppose there are $\iota,\varepsilon\in\Omega$ such that $\mathrm{H}(\iota\star\varepsilon)\nsubseteq\mathrm{H}(\iota)\cup\mathrm{H}(\varepsilon)$. Because $\mathrm{H}(\Omega)$ is comparable, $\mathrm{H}(\iota\star\varepsilon)\supset\mathrm{H}(\iota)\cup\mathrm{H}(\varepsilon)$. Then $\mathrm{H}(\iota\star\varepsilon)\in 2^{[0,1]}$. Put $\pi=\mathrm{H}(\iota\star\varepsilon)\in 2^{[0,1]}$. Then $\mathrm{H}(\iota)\subset\pi$ and $\mathrm{H}(\varepsilon)\subset\pi$. So $\iota,\varepsilon\in S_{\subset}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\subset}(\mathrm{H},\pi)$ is a SA of Ω and hence $\iota\star\varepsilon\in S_{\subset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\iota\star\varepsilon)\subset\pi=\mathrm{H}(\iota\star\varepsilon)$, which is impossible. Summarize $\mathrm{H}(\iota\star\varepsilon)\subseteq\mathrm{H}(\iota)\cup\mathrm{H}(\varepsilon)$ $\forall\iota,\varepsilon\in\Omega$. In conclusion, H is an AHFSA of Ω. □

Theorem 4.5. *The statements below are accurate.*

(1) If H *is an AHFID of* Ω*, then* $\forall\pi\in 2^{[0,1]}$, $S_{\subset}(\mathrm{H},\pi)\neq\emptyset$ *is an ID of* Ω *and* $S(\mathrm{H},\pi)=\emptyset$*.*

(2) If H *is an AHFID of* Ω*, then* $\forall\pi\in 2^{[0,1]}$, $S(H,\pi)\neq\emptyset$ *is an ID of* Ω *and* $S_{\subset}(H,\pi)=\emptyset$*.*

(3) If $\mathrm{H}(\Omega)$ *is comparable and* $\forall\pi\in 2^{[0,1]}$, $\emptyset\neq S_{\subset}(\mathrm{H},\pi)\subseteq\Omega$ *is an ID of* Ω*, then* H *is an AHFID of* Ω*.*

Proof. (1) and (2) are straightforward by Theorem 4.2.

(3) Let's say $\mathrm{H}(\Omega)$ is comparable and $\forall\pi\in 2^{[0,1]}$, $S_{\subset}(\mathrm{H},\pi)\neq\emptyset$ is an ID of Ω. Suppose there is $\iota\in\Omega$ such that $\mathrm{H}(\omega)\nsubseteq\mathrm{H}(\iota)$. Because $\mathrm{H}(\Omega)$ is comparable, $\mathrm{H}(\omega)\supset\mathrm{H}(\iota)$. Put $\pi=\mathrm{H}(\omega)\in 2^{[0,1]}$. Then $\mathrm{H}(\iota)\subset\mathrm{H}(\omega)=\pi$. So $\iota\in S_{\subset}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\subset}(\mathrm{H},\pi)$ is an ID of Ω and hence $\omega\in S_{\subset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\omega)\subset\pi=\mathrm{H}(\omega)$, which is impossible. Summarize $\mathrm{H}(\omega)\subseteq\mathrm{H}(\iota)$ $\forall\iota\in\Omega$. Suppose there are $\iota,\varepsilon\in\Omega$ such that $\mathrm{H}(\iota\star\varepsilon)\nsubseteq\mathrm{H}(\varepsilon)$. Because $\mathrm{H}(\Omega)$ is comparable, $\mathrm{H}(\iota\star\varepsilon)\supset\mathrm{H}(\varepsilon)$. Put $\pi=\mathrm{H}(\iota\star\varepsilon)\in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon)\subset\pi$. So $\varepsilon\in S_{\subset}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\subset}(\mathrm{H},\pi)$ is an ID of Ω and hence $\iota\star\varepsilon\in S_{\subset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\iota\star\varepsilon)\subset\pi=\mathrm{H}(\iota\star\varepsilon)$, which is impossible. Summarize $\mathrm{H}(\iota\star\varepsilon)\subseteq\mathrm{H}(\varepsilon)$ $\forall\iota,\varepsilon\in\Omega$. Suppose there are $\iota,\varepsilon_1,\varepsilon_2\in\Omega$ such that $\mathrm{H}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota)\nsubseteq\mathrm{H}(\varepsilon_1)\cup\mathrm{H}(\varepsilon_2)$. Because $\mathrm{H}(\Omega)$ is comparable, $\mathrm{H}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota)\supset\mathrm{H}(\varepsilon_1)\cup\mathrm{H}(\varepsilon_2)$. Put $\pi=\mathrm{H}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota)\in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon_1)\subset\pi$ and $\mathrm{H}(\varepsilon_2)\subset\pi$. So $\varepsilon_1,\varepsilon_2\in S_{\subset}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\subset}(\mathrm{H},\pi)$ is an ID of Ω and hence $(\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota\in S_{\subset}(\mathrm{H},\pi)$. So, $\mathrm{H}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota)\subset\pi=\mathrm{H}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota)$, which is impossible. Summarize $\mathrm{H}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota)\subseteq\mathrm{H}(\varepsilon_1)\cup\mathrm{H}(\varepsilon_2)$ $\forall\iota,\varepsilon_1,\varepsilon_2\in\Omega$. In conclusion, H is an AHFID of Ω. □

Theorem 4.6. *The statements below are accurate.*

(1) If H *is an AHFDS of* Ω*, then* $\forall\pi\in 2^{[0,1]}$, $S_{\subset}(\mathrm{H},\pi)\neq\emptyset$ *is a DS of* Ω *and* $S(\mathrm{H},\pi)=\emptyset$*.*

(2) If H *is an AHFDS of* Ω*, then* $\forall\pi\in 2^{[0,1]}$, $S(H,\pi)\neq\emptyset$ *is a DS of* Ω *and* $S_{\subset}(H,\pi)=\emptyset$*.*

(3) If $\mathrm{H}(\Omega)$ *is comparable and* $\forall\pi\in 2^{[0,1]}$, $\emptyset\neq S_{\subset}(\mathrm{H},\pi)\subseteq\Omega$ *is a DS of* Ω*, then* H *is an AHFDS of* Ω*.*

Proof. (1) and (2) are straightforward by Theorem 4.3.

(3) Let's say $\mathrm{H}(\Omega)$ is comparable and $\forall\pi\in 2^{[0,1]}$, $S_{\subset}(\mathrm{H},\pi)\neq\emptyset$ is a DS of Ω. Suppose there is $\iota\in\Omega$ such that $\mathrm{H}(\omega)\nsubseteq\mathrm{H}(\iota)$. Because $\mathrm{H}(\Omega)$ is comparable, $\mathrm{H}(\omega)\supset\mathrm{H}(\iota)$. Put $\pi=\mathrm{H}(\omega)\in 2^{[0,1]}$. Then $\mathrm{H}(\iota)\subset\mathrm{H}(\omega)=\pi$. So $\iota\in S_{\subset}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\subset}(\mathrm{H},\pi)$ is a DS of Ω and hence $\omega\in S_{\subset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\omega)\subset\pi=\mathrm{H}(\omega)$, which is impossible. Summarize $\mathrm{H}(\omega)\subseteq\mathrm{H}(\iota)$ $\forall\iota\in\Omega$. Suppose there are $\iota,\varepsilon\in\Omega$ such that $\mathrm{H}(\varepsilon)\nsubseteq\mathrm{H}(\iota\star\varepsilon)\cup\mathrm{H}(\iota)$. Because $\mathrm{H}(\Omega)$ is comparable, $\mathrm{H}(\varepsilon)\supset\mathrm{H}(\iota\star\varepsilon)\cup\mathrm{H}(\iota)$. Put $\pi=\mathrm{H}(\varepsilon)\in 2^{[0,1]}$. Then $\mathrm{H}(\iota\star\varepsilon)\subset\pi$ and $\mathrm{H}(\iota)\subset\pi$. So $\iota\star\varepsilon,\iota\in S_{\subset}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\subset}(\mathrm{H},\pi)$ is a DS of Ω and hence $\varepsilon\in S_{\subset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\varepsilon)\subset\pi=\mathrm{H}(\varepsilon)$, which is impossible. Summarize $\mathrm{H}(\varepsilon)\subseteq\mathrm{H}(\iota\star\varepsilon)\cup\mathrm{H}(\iota)$ $\forall\iota,\varepsilon\in\Omega$. In conclusion, H is an AHFDS of Ω. □

Theorem 4.7. $\overline{\mathrm{H}}$ *is an AHFSA of* Ω *if and only if* $\forall\pi\in 2^{[0,1]}$, $S_{\supseteq}(\mathrm{H},\pi)\neq\emptyset$ *is a SA of* Ω*.*

Proof. Let's say $\overline{\mathrm{H}}$ is an AHFSA of Ω. Let $\pi\in 2^{[0,1]}$ in which $S_{\supseteq}(\mathrm{H},\pi)\neq\emptyset$ and let $\iota,\varepsilon\in S_{\supseteq}(\mathrm{H},\pi)$. Then $\mathrm{H}(\iota)\supseteq\pi$ and $\mathrm{H}(\varepsilon)\supseteq\pi$. Because $\overline{\mathrm{H}}$ is an AHFSA of Ω, $\overline{\mathrm{H}}(\iota\star\varepsilon)\subseteq\overline{\mathrm{H}}(\iota)\cup\overline{\mathrm{H}}(\varepsilon)$ and so $[0,1]\setminus\mathrm{H}(\iota\star\varepsilon)\subseteq([0,1]\setminus\mathrm{H}(\iota))\cup([0,1]\setminus\mathrm{H}(\varepsilon))=[0,1]\setminus(\mathrm{H}(\iota)\cap\mathrm{H}(\varepsilon))$. Thus $\mathrm{H}(\iota\star\varepsilon)\supseteq\mathrm{H}(\iota)\cap\mathrm{H}(\varepsilon)\supseteq\pi$. So, $\iota\star\varepsilon\in S_{\supseteq}(\mathrm{H},\pi)$. Summarize $S_{\supseteq}(\mathrm{H},\pi)$ is a SA of Ω.

On the other hand, assume $\forall\pi\in 2^{[0,1]}$, $S_{\supseteq}(\mathrm{H},\pi)\neq\emptyset$ is a SA of Ω. Let $\iota,\varepsilon\in\Omega$. Put $\pi=\mathrm{H}(\iota)\cap\mathrm{H}(\varepsilon)\in 2^{[0,1]}$. Then $\mathrm{H}(\iota)\supseteq\pi$ and $\mathrm{H}(\varepsilon)\supseteq\pi$. So $\iota,\varepsilon\in S_{\supseteq}(\mathrm{H},\pi)\neq\emptyset$. By assuming, we obtain $S_{\supseteq}(\mathrm{H},\pi)$ is a SA of Ω and hence $\iota\star\varepsilon\in S_{\supseteq}(\mathrm{H},\pi)$. So, $\mathrm{H}(\iota\star\varepsilon)\supseteq\pi=\mathrm{H}(\iota)\cap\mathrm{H}(\varepsilon)$. Thus $\overline{\mathrm{H}}(\iota\star\varepsilon)=[0,1]\setminus\mathrm{H}(\iota\star\varepsilon)\subseteq[0,1]\setminus(\mathrm{H}(\iota)\cap\mathrm{H}(\varepsilon))=([0,1]\setminus\mathrm{H}(\iota))\cup([0,1]\setminus\mathrm{H}(\varepsilon))=\overline{\mathrm{H}}(\iota)\cup\overline{\mathrm{H}}(\varepsilon)$. Summarize $\overline{\mathrm{H}}$ is an AHFSA of Ω. □

Theorem 4.8. $\overline{\mathrm{H}}$ *is an AHFID of* Ω *if and only if* $\forall\pi\in 2^{[0,1]}$, $S_{\supseteq}(\mathrm{H},\pi)\neq\emptyset$ *is an ID of* Ω*.*

Proof. Let's say $\overline{\mathrm{H}}$ is an AHFID of Ω. Let $\pi\in 2^{[0,1]}$ in which $S_{\supseteq}(\mathrm{H},\pi)\neq\emptyset$ and let $\iota\in S_{\supseteq}(\mathrm{H},\pi)$. Then $\mathrm{H}(\iota)\supseteq\pi$. Because $\overline{\mathrm{H}}$ is an AHFID of Ω, $\overline{\mathrm{H}}(\omega)\subseteq\overline{\mathrm{H}}(\iota)$. Thus $[0,1]\setminus\mathrm{H}(\omega)\subseteq[0,1]\setminus\mathrm{H}(\iota)$. So, $\mathrm{H}(\omega)\supseteq\mathrm{H}(\iota)\supseteq\pi$. Summarize $\omega\in S_{\supseteq}(\mathrm{H},\pi)$. Next, let $\iota,\varepsilon\in\Omega$ in which $\varepsilon\in S_{\supseteq}(\mathrm{H},\pi)$. Then $\mathrm{H}(\varepsilon)\supseteq\pi$. As $\overline{\mathrm{H}}$ is an AHFID of Ω, $\overline{\mathrm{H}}(\iota\star\varepsilon)\subseteq\overline{\mathrm{H}}(\varepsilon)$ and so $[0,1]\setminus\mathrm{H}(\iota\star\varepsilon)\subseteq[0,1]\setminus\mathrm{H}(\varepsilon)$. Thus $\mathrm{H}(\iota\star\varepsilon)\supseteq\mathrm{H}(\varepsilon)\supseteq\pi$. So, $\iota\star\varepsilon\in S_{\supseteq}(\mathrm{H},\pi)$. Also, let $\iota,\varepsilon_1,\varepsilon_2\in\Omega$ in which $\varepsilon_1,\varepsilon_2\in S_{\supseteq}(\mathrm{H},\pi)$. Then $\mathrm{H}(\varepsilon_1)\supseteq\pi$ and $\mathrm{H}(\varepsilon_2)\supseteq\pi$. Because $\overline{\mathrm{H}}$ is an AHFID of Ω, $\overline{\mathrm{H}}((\varepsilon_1\star(\varepsilon_2\star\iota)\star\iota))\subseteq\overline{\mathrm{H}}(\varepsilon_1)\cup\overline{\mathrm{H}}(\varepsilon_2)$ and so $[0,1]\setminus\mathrm{H}(((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota))\subseteq([0,1]\setminus\mathrm{H}(\varepsilon_1))\cup([0,1]\setminus\mathrm{H}(\varepsilon_2))=[0,1]\setminus(\mathrm{H}(\varepsilon_1)\cap\mathrm{H}(\varepsilon_2))$. Thus $\mathrm{H}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota)\supseteq\mathrm{H}(\iota)\cap\mathrm{H}(\varepsilon)\supseteq\pi$. So, $(\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota\in S_{\supseteq}(\mathrm{H},\pi)$. Summarize $S_{\supseteq}(\mathrm{H},\pi)$ is an ID of Ω.

On the other hand, assume $\forall\pi \in 2^{[0,1]}$, $S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$ is an ID of Ω. Let $\iota \in \Omega$. Put $\pi = \mathrm{H}(\iota) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \supseteq \pi$, so $\iota \in S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supseteq}(\mathrm{H},\pi)$ is an ID of Ω. Thus $\omega \in S_{\supseteq}(\mathrm{H},\pi)$. So, $\mathrm{H}(\omega) \supseteq \pi = \mathrm{H}(\iota)$. Summarize $\overline{\mathrm{H}}(\omega) = [0,1] \setminus \mathrm{H}(\omega) \subseteq [0,1] \setminus \mathrm{H}(\iota) = \overline{\mathrm{H}}(\iota)$. Next, Let $\iota,\varepsilon \in \Omega$. Put $\pi = \mathrm{H}(\varepsilon) \in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon) \supseteq \pi$. So $\varepsilon \in S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supseteq}(\mathrm{H},\pi)$ is an ID of Ω and hence $\iota \star \varepsilon \in S_{\supseteq}(\mathrm{H},\pi)$. So, $\mathrm{H}(\iota \star \varepsilon) \supseteq \pi = \mathrm{H}(\varepsilon)$. Thus $\overline{\mathrm{H}}(\iota\star\varepsilon) = [0,1]\setminus\mathrm{H}(\iota\star\varepsilon) \subseteq [0,1]\setminus\mathrm{H}(\varepsilon) = \overline{\mathrm{H}}(\varepsilon)$. Also, let $\iota,\varepsilon_1,\varepsilon_2 \in \Omega$. Put $\pi = \mathrm{H}(\varepsilon_1) \cap \mathrm{H}(\varepsilon_2) \in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon_1) \supseteq \pi$ and $\mathrm{H}(\varepsilon_2) \supseteq \pi$. So $\varepsilon_1,\varepsilon_2 \in S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supseteq}(\mathrm{H},\pi)$ is an ID of Ω and hence $(\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota \in S_{\supseteq}(\mathrm{H},\pi)$. So, $\mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \supseteq \pi = \mathrm{H}(\varepsilon_1) \cap \mathrm{H}(\varepsilon_2)$. Thus $\overline{\mathrm{H}}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) = [0,1] \setminus \mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \subseteq [0,1] \setminus (\mathrm{H}(\varepsilon_1) \cap \mathrm{H}(\varepsilon_2)) = ([0,1] \setminus \mathrm{H}(\varepsilon_1)) \cup ([0,1] \setminus \mathrm{H}(\varepsilon_2)) = \overline{\mathrm{H}}(\varepsilon_1) \cup \overline{\mathrm{H}}(\varepsilon_2)$. Summarize $\overline{\mathrm{H}}$ is an AHFID of Ω. □

Theorem 4.9. *$\overline{\mathrm{H}}$ is an AHFDS of Ω if and only if $\forall\pi \in 2^{[0,1]}$, $S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$ is a DS of Ω.*

Proof. Let's say $\overline{\mathrm{H}}$ is an AHFDS of Ω. Let $\pi \in 2^{[0,1]}$ in which $S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$ and let $\iota \in S_{\supseteq}(\mathrm{H},\pi)$. Then $\mathrm{H}(\iota) \supseteq \pi$. Because $\overline{\mathrm{H}}$ is an AHFDS of Ω, $\overline{\mathrm{H}}(\omega) \subseteq \overline{\mathrm{H}}(\iota)$. Thus $[0,1] \setminus \mathrm{H}(\omega) \subseteq [0,1] \setminus \mathrm{H}(\iota)$. So, $\mathrm{H}(\omega) \supseteq \mathrm{H}(\iota) \supseteq \pi$. Summarize $\omega \in S_{\supseteq}(\mathrm{H},\pi)$. Next, let $\iota,\varepsilon \in \Omega$ in which $\iota \star \varepsilon, \iota \in S_{\supseteq}(\mathrm{H},\pi)$. Then $\mathrm{H}(\iota \star \varepsilon) \supseteq \pi$ and $\mathrm{H}(\iota) \supseteq \pi$. Because $\overline{\mathrm{H}}$ is an AHFDS of Ω, $\overline{\mathrm{H}}(\varepsilon) \subseteq \overline{\mathrm{H}}(\iota \star \varepsilon) \cup \overline{\mathrm{H}}(\varepsilon)$ and so $[0,1] \setminus \mathrm{H}(\varepsilon) \subseteq ([0,1] \setminus \mathrm{H}(\iota \star \varepsilon)) \cup ([0,1] \setminus \mathrm{H}(\iota)) = [0,1] \setminus (\mathrm{H}(\iota \star \varepsilon) \cap \mathrm{H}(\iota))$. Thus $\mathrm{H}(\varepsilon) \supseteq \mathrm{H}(\iota \star \varepsilon) \cap \mathrm{H}(\iota) \supseteq \pi$. So, $\varepsilon \in S_{\supseteq}(\mathrm{H},\pi)$. Summarize $S_{\supseteq}(\mathrm{H},\pi)$ is a DS of Ω.

On the other hand, assume $\forall\pi \in 2^{[0,1]}$, $S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$ is a DS of Ω. Let $\iota \in \Omega$. Put $\pi = \mathrm{H}(\iota) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \supseteq \pi$, so $\iota \in S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supseteq}(\mathrm{H},\pi)$ is a DS of Ω. Thus $\omega \in S_{\supseteq}(\mathrm{H},\pi)$. So, $\mathrm{H}(\omega) \supseteq \pi = \mathrm{H}(\iota)$. Summarize $\overline{\mathrm{H}}(\omega) = [0,1] \setminus \mathrm{H}(\omega) \subseteq [0,1] \setminus \mathrm{H}(\iota) = \overline{\mathrm{H}}(\iota)$. Next, Let $\iota,\varepsilon \in \Omega$. Put $\pi = \mathrm{H}(\iota \star \varepsilon) \cap \mathrm{H}(\iota) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota \star \varepsilon) \supseteq \pi$ and $\mathrm{H}(\iota) \supseteq \pi$. So $\iota \star \varepsilon, \iota \in S_{\supseteq}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supseteq}(\mathrm{H},\pi)$ is a DS of Ω and hence $\varepsilon \in S_{\supseteq}(\mathrm{H},\pi)$. So, $\mathrm{H}(\varepsilon) \supseteq \pi = \mathrm{H}(\iota \star \varepsilon) \cap \mathrm{H}(\iota)$. Thus $\overline{\mathrm{H}}(\varepsilon) = [0,1] \setminus \mathrm{H}(\varepsilon) \subseteq [0,1] \setminus (\mathrm{H}(\iota\star\varepsilon) \cap \mathrm{H}(\iota)) = ([0,1] \setminus \mathrm{H}(\iota\star\varepsilon)) \cup ([0,1] \setminus \mathrm{H}(\iota)) = \overline{\mathrm{H}}(\iota \star \varepsilon) \cup \overline{\mathrm{H}}(\iota)$. Summarize $\overline{\mathrm{H}}$ is an AHFDS of Ω. □

Theorem 4.10. *The statements below are accurate.*

(1) If $\overline{\mathrm{H}}$ is an AHFSA of Ω, then $\forall\pi \in 2^{[0,1]}$, $S_{\supset}(\mathrm{H},\pi) \neq \emptyset$ is a SA of Ω and $S(\mathrm{H},\pi) = \emptyset$.

(2) If $\overline{\mathrm{H}}$ is an AHFSA of Ω, then $\forall\pi \in 2^{[0,1]}$, $S(\mathrm{H},\pi) \neq \emptyset$ is a SA of Ω and $S_{\supset}(\mathrm{H},\pi) = \emptyset$.

(3) If $\mathrm{H}(\Omega)$ is comparable and $\forall\pi \in 2^{[0,1]}$, $\emptyset \neq S_{\supset}(\mathrm{H},\pi) \subseteq \Omega$ is a SA of Ω, then $\overline{\mathrm{H}}$ is an AHFSA of Ω.

Proof. (1) and (2) are straightforward by Theorem 4.7.

(3) Let's say $\mathrm{H}(\Omega)$ is comparable and $\forall\pi \in 2^{[0,1]}$, $S_{\supset}(\mathrm{H},\pi) \neq \emptyset$ is a SA of Ω. Suppose there are $\iota,\varepsilon \in \Omega$ such that $\overline{\mathrm{H}}(\iota \star \varepsilon) \nsubseteq \overline{\mathrm{H}}(\iota) \cup \overline{\mathrm{H}}(\varepsilon)$. Because $\mathrm{H}(\Omega)$ is comparable, $\overline{\mathrm{H}}(\iota \star \varepsilon) \supset \overline{\mathrm{H}}(\iota) \cup \overline{\mathrm{H}}(\varepsilon)$ and so $[0,1] \setminus \mathrm{H}(\iota \star \varepsilon) \supset ([0,1] \setminus \mathrm{H}(\iota)) \cup ([0,1] \setminus \mathrm{H}(\varepsilon)) = [0,1] \setminus (\mathrm{H}(\iota) \cap \mathrm{H}(\varepsilon))$. Then $\mathrm{H}(\iota \star \varepsilon) \subset \mathrm{H}(\iota) \cap \mathrm{H}(\varepsilon)$. Put $\pi = \mathrm{H}(\iota \star \varepsilon) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \supset \pi$ and $\mathrm{H}(\varepsilon) \supset \pi$. So $\iota,\varepsilon \in S_{\supset}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supset}(\mathrm{H},\pi)$ is a SA of Ω and hence $\iota \star \varepsilon \in S_{\supset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\iota \star \varepsilon) \supset \pi = \mathrm{H}(\iota \star \varepsilon)$, which is impossible. Summarize $\overline{\mathrm{H}}(\iota \star \varepsilon) \subseteq \overline{\mathrm{H}}(\iota) \cup \overline{\mathrm{H}}(\varepsilon)\ \forall\iota,\varepsilon \in \Omega$. In conclusion, $\overline{\mathrm{H}}$ is an AHFSA of Ω. □

Theorem 4.11. *The statements below are accurate.*

(1) If $\overline{\mathrm{H}}$ is an AHFID of Ω, then $\forall\pi \in 2^{[0,1]}$, $S_{\supset}(\mathrm{H},\pi) \neq \emptyset$ is an ID of Ω and $S(\mathrm{H},\pi) = \emptyset$.

(2) If $\overline{\mathrm{H}}$ is an AHFID of Ω, then $\forall\pi \in 2^{[0,1]}$, $S(\mathrm{H},\pi) \neq \emptyset$ is an ID of Ω and $S_{\supset}(\mathrm{H},\pi) = \emptyset$.

(3) If $\mathrm{H}(\Omega)$ is comparable and $\forall\pi \in 2^{[0,1]}$, $\emptyset \neq S_{\supset}(\mathrm{H},\pi) \subseteq \Omega$ is an ID of Ω, then $\overline{\mathrm{H}}$ is an AHFID of Ω.

Proof. (1) and (2) are straightforward by Theorem 4.8.

(3) Let's say $\mathrm{H}(\Omega)$ is comparable and $\forall\pi \in 2^{[0,1]}$, $S_{\supset}(\mathrm{H},\pi) \neq \emptyset$ is an ID of Ω. Suppose there is $\iota \in \Omega$ such that $\overline{\mathrm{H}}(\omega) \nsubseteq \overline{\mathrm{H}}(\iota)$. Because $\mathrm{H}(\Omega)$ is comparable, $\overline{\mathrm{H}}(\omega) \supset \overline{\mathrm{H}}(\iota)$. Then $[0,1] \setminus \mathrm{H}(\omega) \supset [0,1] \setminus \mathrm{H}(\iota)$. Put $\pi = \mathrm{H}(\omega) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \supset \mathrm{H}(\omega) = \pi$. So $\iota \in S_{\supset}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supset}(\mathrm{H},\pi)$ is an ID of Ω and hence $\omega \in S_{\supset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\omega) \subset \pi = \mathrm{H}(\omega)$, which is impossible. Summarize $\overline{\mathrm{H}}(\omega) \subseteq \overline{\mathrm{H}}(\iota)\ \forall\iota \in \Omega$. Suppose there are $\iota,\varepsilon \in \Omega$ such that $\overline{\mathrm{H}}(\iota\star\varepsilon) \nsubseteq \overline{\mathrm{H}}(\varepsilon)$. Because $\mathrm{H}(\Omega)$ is comparable, $\overline{\mathrm{H}}(\iota\star\varepsilon) \supset \overline{\mathrm{H}}(\varepsilon)$ and so $[0,1]\setminus\mathrm{H}(\iota\star\varepsilon) \supset [0,1]\setminus\mathrm{H}(\varepsilon)$. Then $\mathrm{H}(\iota\star\varepsilon) \subset \mathrm{H}(\varepsilon)$. Put $\pi = \mathrm{H}(\iota \star \varepsilon) \in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon) \supset \pi$. So $\varepsilon \in S_{\supset}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supset}(\mathrm{H},\pi)$ is an ID of Ω and hence $\iota\star\varepsilon \in S_{\supset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\iota\star\varepsilon) \subset \pi = \mathrm{H}(\iota\star\varepsilon)$, which is impossible. Summarize $\overline{\mathrm{H}}(\iota\star\varepsilon) \subseteq \overline{\mathrm{H}}(\varepsilon)\ \forall\iota,\varepsilon \in \Omega$. Suppose there are $\iota,\varepsilon_1,\varepsilon_2 \in \Omega$ such that $\overline{\mathrm{H}}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota) \nsubseteq \overline{\mathrm{H}}(\varepsilon_1)\cup\overline{\mathrm{H}}(\varepsilon_2)$. Because $\mathrm{H}(\Omega)$ is comparable, $\overline{\mathrm{H}}((\varepsilon_1\star(\varepsilon_2\star\iota))\star\iota) \supset \overline{\mathrm{H}}(\varepsilon_1)\cup\overline{\mathrm{H}}(\varepsilon_2)$ and so $[0,1] \setminus \mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \supset ([0,1] \setminus \mathrm{H}(\varepsilon_1)) \cup ([0,1] \setminus \mathrm{H}(\varepsilon_2)) = [0,1] \setminus (\mathrm{H}(\varepsilon_1) \cap \mathrm{H}(\varepsilon_2)$. Then $\mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \subset \mathrm{H}(\varepsilon_1) \cap \mathrm{H}(\varepsilon_2)$. Put $\pi = \mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \in 2^{[0,1]}$. Then $\mathrm{H}(\varepsilon_1) \supset \pi$ and $\mathrm{H}(\varepsilon_2) \supset \pi$. So $\varepsilon_1,\varepsilon_2 \in S_{\supset}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supset}(\mathrm{H},\pi)$ is an ID of Ω and hence $(\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota \in S_{\supset}(\mathrm{H},\pi)$. So, $\mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \subset \pi = \mathrm{H}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota)$, which is impossible. Summarize $\overline{\mathrm{H}}((\varepsilon_1 \star (\varepsilon_2 \star \iota)) \star \iota) \subseteq \overline{\mathrm{H}}(\varepsilon_1) \cup \overline{\mathrm{H}}(\varepsilon_2)\ \forall\iota,\varepsilon_1,\varepsilon_2 \in \Omega$. In conclusion, $\overline{\mathrm{H}}$ is an AHFID of Ω. □

Theorem 4.12. *The statements below are accurate.*

(1) If H is an AHFDS of Ω, then $\forall\pi \in 2^{[0,1]}$, $S_{\supset}(\mathrm{H},\pi) \neq \emptyset$ is a DS of Ω and $S(\mathrm{H},\pi) = \emptyset$.

(2) If $\overline{\mathrm{H}}$ is an AHFDS of Ω, then $\forall\pi \in 2^{[0,1]}$, $S(\mathrm{H},\pi) \neq \emptyset$ is a DS of Ω and $S_{\supset}(\mathrm{H},\pi) = \emptyset$.

(3) If $\mathrm{H}(\Omega)$ is comparable and $\forall\pi \in 2^{[0,1]}$, $\emptyset \neq S_{\supset}(\mathrm{H},\pi) \subseteq \Omega$ is a DS of Ω, then $\overline{\mathrm{H}}$ is an AHFDS of Ω.

Proof. (1) and (2) are straightforward by Theorem 4.9.

(3) Let's say $\mathrm{H}(\Omega)$ is comparable and $\forall\pi \in 2^{[0,1]}$, $S_{\supset}(\mathrm{H},\pi) \neq \emptyset$ is a DS of Ω. Suppose there is $\iota \in \Omega$ such that $\overline{\mathrm{H}}(\omega) \nsubseteq \overline{\mathrm{H}}(\iota)$. Because $\mathrm{H}(\Omega)$ is comparable, $\overline{\mathrm{H}}(\omega) \supset \overline{\mathrm{H}}(\iota)$. Then $[0,1] \setminus \mathrm{H}(\omega) \supset [0,1] \setminus \mathrm{H}(\iota)$. Put $\pi = \mathrm{H}(\omega) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota) \supset \mathrm{H}(\omega) = \pi$. So $\iota \in S_{\supset}(\mathrm{H},\pi) \neq \emptyset$. By assuming,

we obtain $S_{\supset}(\mathrm{H},\pi)$ is a DS of Ω and hence $\omega \in S_{\supset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\omega) \subset \pi = \mathrm{H}(\omega)$, which is impossible. Summarize $\overline{\mathrm{H}}(\omega) \subseteq \overline{\mathrm{H}}(\iota)\ \forall\iota \in \Omega$. Suppose there are $\iota,\varepsilon \in \Omega$ such that $\overline{\mathrm{H}}(\varepsilon) \nsubseteq \overline{\mathrm{H}}(\iota\star\varepsilon) \cup \overline{\mathrm{H}}(\iota)$. Because $\mathrm{H}(\Omega)$ is comparable, $\overline{\mathrm{H}}(\varepsilon) \supset \overline{\mathrm{H}}(\iota\star\varepsilon) \cup \overline{\mathrm{H}}(\iota)$ and so $[0,1] \setminus \mathrm{H}(\varepsilon) \supset ([0,1] \setminus \mathrm{H}(\iota\star\varepsilon)) \cup ([0,1] \setminus \mathrm{H}(\iota)) = [0,1] \setminus (\mathrm{H}(\iota\star\varepsilon) \cap \mathrm{H}(\iota))$. Then $\mathrm{H}(\varepsilon) \subset \mathrm{H}(\iota\star\varepsilon) \cap \mathrm{H}(\iota)$. Put $\pi = \mathrm{H}(\varepsilon) \in 2^{[0,1]}$. Then $\mathrm{H}(\iota\star\varepsilon) \supset \pi$ and $\mathrm{H}(\iota) \supset \pi$. So $\iota\star\varepsilon, \iota \in S_{\supset}(\mathrm{H},\pi) \neq \emptyset$. By assuming, we obtain $S_{\supset}(\mathrm{H},\pi)$ is a DS of Ω and hence $\varepsilon \in S_{\supset}(\mathrm{H},\pi)$. So, $\mathrm{H}(\varepsilon) \subset \pi = \mathrm{H}(\varepsilon)$, which is impossible. Summarize $\overline{\mathrm{H}}(\varepsilon) \subseteq \overline{\mathrm{H}}(\iota\star\varepsilon) \cup \overline{\mathrm{H}}(\iota)\ \forall\iota,\varepsilon \in \Omega$. In conclusion, $\overline{\mathrm{H}}$ is an AHFDS of Ω. □

5 Conclusion

In the current study, we have explained HFSs to SAs, IDs, and DSs of Hilbert algebras in terms of anti-types referred to as an AHFSA, an AHFID, and an AHFDS. It supplies the link between their level subsets, AHFSAs, AHFIDs, and AHFDSs.

Acknowledgements

The income budget for 2022 at the School of Science, University of Phayao, Thailand, provided funding for this study.

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