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On a Weak Solution of a Fractional-order Temporal Equation

Iqbal M. Batiha[1,2,*], Zainouba Chebana[3], Taki-Eddine Oussaeif[3], Adel Ouannas[3], Iqbal H. Jebril[4]

[1]Department of Mathematics, Faculty of Science and Technology, Irbid National University, 2600 Irbid, Jordan

[2]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE

[3]Department of Mathematics and Computer Science, University of Larbi Ben M'hidi, Oum El Bouaghi, Algeria

[4]Mathematics Department, Al Zaytoonah University of Jordan, Queen Alia Airport St 594, Amman 11733, Jordan

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Abstract Several real-world phenomena emerging in engineering and science fields can be described successfully by developing certain models using fractional-order partial differential equations. The exact, analytical, semi-analytical or even numerical solutions for these models should be examined and investigated by distinguishing between their solvablities and non-solvabilities. In this paper, we aim to establish some sufficient conditions for exploring the existence and uniqueness of solution for a class of initial-boundary value problems with Dirichlet condition. The gained results from this research paper are established for the class of fractional-order partial differential equations by a method based on Lax Milgram theorem, which relies in its construction on properties of the symmetric part of the bilinear form. Lax Milgram theorem is deemed as a mathematical scheme that can be used to examine the existence and uniqueness of weak solutions for fractional-order partial differential equations. These equations are formulated here in view of the Caputo fractional-order derivative operator, which its inverse operator is the Riemann-Louville fractional-order integral one. The results of this paper will be supportive for mathematical analyzers and researchers when a fractional-order partial differential equation is handled in terms of finding its exact, analytical, semi-analytical or numerical solution.

Keywords Fractional Partial Differential Equation, Lax Milgram Theorem, Existence, Uniqueness.

1 Introduction

In 1695, the fractional-order partial derivatives began as a special correspondence between Leibniz and Lepital, when Leibniz described "paradoxes" and predicted that some "Useful results will be drawn one day". From this point of view, some new branches of mathematics have emerged, especially in the field of partial differential equations, not to mention the existence of the classics. In more recent times, fractional calculus have come and the research of its theories has developed more and more, especially in the last century [1, 2, 3, 4]. As a result, this topic has attracted the attention of many mathematical giants such as Riemann, Liouville, Abel, Laurent, Hardy, and Littlewood.

Fractional-order differential equations, which can be obtained by generalizing ordinary differential equations to an arbitrary order, play a crucial role in engineering, physics, and applied mathematics [7, 8, 9]. Several complex phenomena can be modeled with the help of using these equations [10, 11]. As a result, many applications of these equations can be found in the study of viscoelasticity, electrochemistry, signal processing, control theory, porous media, fluid mechanics, rheology, transport by diffusion, electrical networks, electromagnetic, probability distributions, and many other processes [5, 6, 12]. In fact, the existence and uniqueness results of the weak solutions were widely investigated for several fractional-order partial differential equations using Lax Milgram theorem, see [13, 14] and the references therein. Such scheme, which depends on certain energy inequalities, has been adopted by many authors, see for example [15, 16, 17].

Lax-Milgram theorem is deemed as a mathematical scheme that can be used to examine the existence and uniqueness of weak solutions for fractional-order partial differential equations. It is valid for the coercive linear operator on Hilbert space. In this work, we intend to apply this scheme for exploring the existence and uniqueness of a solution for a fractional-order partial differential equation with certain classical boundary conditions. We divide this paper into five sections so that Section 2 is devoted to remind readers of some basic tools and preliminary results essential to our work. In particular, we present some fundamental results and certain properties related to fractional-order operators and functional spaces. The third section aims to formulate our main problem, whereas the fourth one is devoted to the study of existence and uniqueness of a weak solution for a fractional-order diffusion problem with the help of using Lax-Milgram theorem, followed by Section 5 that declares the conclusions of this work.

2 Preliminaries and functional spaces

In this section, we will recall some basic facts and definitions associated to our work. To this aim, we let $\Gamma(\cdot)$ be the gamma function, and α be any positive real number lies in $(0, 1]$. The Caputo and Riemann-Liouville fractional-order derivative operators can be then defined respectively as follows.

- The left Caputo fractional-order derivative operator is defined by:

$$ {}_0^C D_t^\alpha u(x,t) := \frac{1}{\mathbf{\Gamma}(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} d\tau, \tag{1} $$

whereas the right Caputo fractional-order derivative operator is defined by:

$$ {}_t^C D_T^\alpha u(x,t) := \frac{-1}{\mathbf{\Gamma}(1-\alpha)} \int_t^T \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(\tau-t)^\alpha} d\tau. $$

- The left Riemann-Liouville fractional-order derivative operator is defined by:

$$ {}_0^R D_t^\alpha u(x,t) := \frac{1}{\mathbf{\Gamma}(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau, \tag{2} $$

whereas the right Riemann-Liouville fractional-order derivative operator is defined by:

$$ {}_t^R D_T^\alpha v(t) := \frac{-1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_t^T \frac{u(x,\tau)}{(t-\tau)^\alpha} d\tau. \tag{3} $$

Many authors think that the Caputo's version is more natural than Riemann-Liouville one as it can allow the handling of inhomogeneous initial conditions in an easier way. However, in view of the above two definitions, the Caputo operator and the Riemann-Liouville operator are linked to each other according to the following relationship:

$$ {}_0^R D_t^\alpha u(x,t) = {}_0^C D_t^\alpha u(x,t) + \frac{u(x,0)}{\mathbf{\Gamma}(1-\alpha) t^\alpha}. \tag{4} $$

Definition 1 *[13] For any real number $\sigma > 0$, we define the semi-norm:*

$$ |u|^2_{{}^l H^\sigma(I)} := \left\| {}_0^R D_t^\sigma u \right\|^2_{L_2(I)}, $$

and the norm:

$$ \|u\|_{{}^l H^\sigma(\Omega)} := \left(\|u\|^2_{L_2(I)} + |u|^2_{{}^l H_0^\sigma(I)} \right)^{\frac{1}{2}}. \tag{5} $$

In addition, we define ${}^l H_0^\sigma(I)$ as the closure of $C_0^\infty(I)$ with respect to the norm $\|\cdot\|_{{}^l H_0^\sigma(I)}$.

Definition 2 *For any real number $\sigma > 0$, we define the semi-norm:*

$$ |u|^2_{{}^r H_0^\sigma(I)} := \| {}_t^r D_T^\sigma u \|^2_{L_2(I)}, $$

and the norm:

$$ \|u\|_{{}^r H_0^\sigma(I)} := \left(\|u\|^2_{L_2(I)} + |u|^2_{{}^r H_0^\sigma(I)} \right)^{\frac{1}{2}}. \tag{6} $$

In addition, we define ${}^r H_0^\sigma(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{{}^r H_0^\sigma(\Omega)}$.

Definition 3 *For $\sigma \in \mathbb{R}_+$, $\sigma \neq n + \frac{1}{2}$, we define the semi-norm:*

$$ |u|_{{}^c H^\sigma(I)} = \left| \left({}^R D_t^\sigma u,\ {}_t^R D^\sigma u \right)_{L^2(I)} \right|^{1/2}, $$

and the norm:

$$ \|u\|_{{}^c H^\sigma(I)} = (\|u\|^2_{L^2(I)} + |u|^2_{{}^c H^\sigma(I)})^{1/2}. $$

n addition, we define ${}^c H^\sigma(I)$ as the closure of $C_0^\infty(I)$ with respect to the norm $\|\cdot\|_{{}^c H^\sigma(I)}$.

Lemma 1 *[13] For any real number $\sigma \in \mathbb{R}_+$ and $I = (0, T)$, if $u \in {}^l H^\alpha(I)$ and $v \in C_0^\infty(I)$, then we have:*

$$ ({}^R D_t^\sigma u(t), v(t))_{L^2(I)} = (u(t),\ {}_t^R D^\sigma v(t))_{L^2(I)}. $$

Lemma 2 *[13, 14] For $0 < \sigma < 2$, $\sigma \neq 1$ and $u \in H_0^{\frac{\sigma}{2}}(I)$, we have:*

$$ {}^R D_t^\sigma u(t) = {}^R D_t^{\frac{\sigma}{2}}\, {}^R D_t^{\frac{\sigma}{2}} u(t). $$

Lemma 3 *[13, 14] For $\sigma \in \mathbb{R}_+$ such that $\sigma \neq n + \frac{1}{2}$. Then, the semi-norms $|\cdot|_{{}^l H^\sigma(I)}$, $|\cdot|_{{}^r H^\sigma(I)}$ and $|\cdot|_{{}^c H^\sigma(I)}$ are equivalent, and we pose this assertion by:*

$$ |\cdot|_{{}^l H^\sigma(I)} \cong |\cdot|_{{}^r H^\sigma(I)} \cong |\cdot|_{{}^c H^\sigma(I)}. $$

Lemma 4 *[13] For any real number $\sigma > 0$, the space ${}^r H_0^\sigma(I)$ with respect to the norm (2.6) is complete.*

Definition 4 *The space $L^2(0, T, L_2(0,1)) := L_2(Q)$ is a space of functions that are square integrable in the Bochner sense according to the following scalar product:*

$$ (u, w)_{L_2(0,T,L^2(0,1))} = \int_0^T ((u,\cdot), (w,\cdot))_{L^2(0,1)}\, dt. \tag{7} $$

Since the space $L^2(0, T)$ is a Hilbert space, it can be shown that $L_2(0, T, L_2(0, 1))$ is a Hilbert space as well. In the same regard, we let $C^\infty(0, T)$ denote the space of infinitely differentiable functions on $(0, T)$, while the space $C_0^\infty(0, T)$ denote the space of infinitely differentiable functions with compact support in $(0, T)$.

3 Problem's formulation

In this part, we will pay our attention to formulate our main problem. To this aim, we consider the rectangular area $Q_T = (0,l) \times (0,T)$ such that $l, T < \infty$ and $0 < \alpha < 1$. Besides, we consider the following problem:

$$\begin{cases} {}^R D_t^\alpha u(x,t) - \frac{\partial}{\partial x}\left(a\,(x,t)\,\frac{\partial u(x,t)}{\partial x}\right) = f(x,t) \text{ in } Q_T, \\ u(x,0) = 0 \qquad \forall x \in (0,l), \\ u(0,t) = u(l,t) = 0 \qquad \forall t \in (0,T), \end{cases} \tag{8}$$

where the function a satisfies:

$$a_0 \leq a(x,t) \leq a_1, \quad a_0, a_1 \in \mathbb{R}_*^+, \forall (x,t) \in Q_T.$$

In order to move forward in our investigation, we propose a theoretical result that represents another variational formulation for our main problem. In this result, we will present the solution of such variational problem.

Proposition 1 *If $u \in B^{\frac{\alpha}{2}}(Q_T)$ is solution of problem (8), then u is a solution of the following variational problem:*

$$\left({}^R D_t^{\frac{\alpha}{2}} u, {}_t^R D^{\frac{\alpha}{2}} v\right)_{L^2(Q_T)} + \left(a\,(x,t)\,\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L^2(Q_T)} = \langle f, v\rangle\,,\ \forall v \in B^{\frac{\alpha}{2}}(Q_T),\ u \in B^{\frac{\alpha}{2}}(Q_T). \tag{9}$$

Proof 1 *Suppose u is a solution of (8), then $u \in B^{\frac{\alpha}{2}}(Q_T)$ and ${}^R D_t^\alpha u - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) = f$ over Q_T in $\left(C_0^\infty(Q_T)\right)'$. Therefore, we have:*

$$\left\langle {}^R D_t^\alpha u - \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right), v\right\rangle = \left\langle {}^R D_t^\alpha u, v\right\rangle - \left\langle \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right), v\right\rangle = \langle f, v\rangle\,,$$

for all $v \in C_0^\infty(Q_T)$. According to Lemma 1 and due to $\frac{\partial u}{\partial x} \in L^2(Q_T)$, we have:

$$\begin{aligned} &\left\langle {}^R D_t^\alpha u, v\right\rangle - \left\langle \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right), v\right\rangle \\ &= \left\langle {}^R D_t^\alpha u, v\right\rangle + \left\langle a\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right\rangle \\ &= \left({}^R D_t^{\frac{\alpha}{2}} u,\ {}_t^R D^{\frac{\alpha}{2}} v\right)_{L^2(Q_T)} + \left(a\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L^2(Q_T)} = \langle f, v\rangle \end{aligned}$$

4 Existence and uniqueness result

In what follow, we will endeavor to propose a new result that examines the existence and uniqueness of a weak solution for the fractional-order diffusion problem (8). For this purpose, it is sufficient to prove that problem (9) has a unique solution u, which has been previously proposed in Proposition 1.

Theorem 1 *For $u \in B^{\frac{\alpha}{2}}(Q_T)$ and $f \in \left(B^{\frac{\alpha}{2}}(Q_T)\right)'$, then problem (9) admits a unique solution satisfying the following priori estimate:*

$$\|u\|_{B^{\frac{\alpha}{2}}(Q_T)} \lesssim \|f\|_{\left(B^{\frac{\alpha}{2}}(Q_T)\right)'}.$$

Proof 2 *In order to prove this result, we apply the Lax Milgram theorem. To this aim, we have $V' = B^{\frac{\alpha}{2}}(Q_T)$ is a Hilbert space, and we let:*

$$B(u,v) = \left({}^R D_t^{\frac{\alpha}{2}} u, {}_t^R D^{\frac{\alpha}{2}} v\right)_{L^2(Q_T)} + \left(a\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L^2(Q_T)}.$$

According to the bilinearity of the scalar product and the linearity of the fractional-order derivative operator, we note that B is a bilinear form over $V' \times V'$. Now, let us check the continuity and coercivity of B. For the continuity of B, we confirm:

$$\begin{aligned} &|B(u,v)| \\ &= \left|\left({}^R D_t^{\frac{\alpha}{2}} u,\ {}_t^R D^{\frac{\alpha}{2}} v\right)_{L^2(Q_T)} + \left(a\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L^2(Q_T)}\right| \\ &\leq \left|\left({}^R D_t^{\frac{\alpha}{2}} u,\ {}_t^R D^{\frac{\alpha}{2}} v\right)_{L^2(Q_T)}\right| + \left|\left(a_1\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L^2(Q_T)}\right|, \end{aligned}$$

for all $(u,v) \in V' \times V'$. Now, by using the Cauchy-Schwarz inequality, we get:

$$\begin{aligned} |B(u,v)| &\leq \left\| {}^R D_t^{\frac{\alpha}{2}} u\right\|_{L^2(Q_T)} \left\| {}_t^R D^{\frac{\alpha}{2}} v\right\|_{L^2(Q_T)} \\ &+ a_1 \left\|\frac{\partial u}{\partial x}\right\|_{L^2(Q_T)} \left\|\frac{\partial v}{\partial x}\right\|_{L^2(Q_T)} \\ &\leq \left(\int_0^l |u|^2_{{}^l H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \left(\int_0^l |v|^2_{{}^r H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \\ &+ a_1 \left(\int_I \left\|\frac{\partial u}{\partial x}\right\|^2_{L^2(0,l)} dt\right)^{1/2} \left(\int_I \left\|\frac{\partial v}{\partial x}\right\|^2_{L^2(0,l)} dt\right)^{1/2} \\ &= \left(\int_0^l |u|^2_{{}^l H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \left(\int_0^l |v|^2_{{}^r H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \\ &+ a_1 \left(\int_I \|u\|^2_{H_0^1(0,l)}\,dt\right)^{1/2} \left(\int_I \|v\|^2_{H_0^1(0,l)}\,dt\right)^{1/2}. \end{aligned} \tag{10}$$

By Lemma 3, we can have:

$$\begin{aligned} &\left(\int_0^l |u|^2_{{}^l H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \left(\int_0^l |v|^2_{{}^r H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \\ &+ a_1 \left(\int_I \|u\|^2_{H_0^1(0,l)}\,dt\right)^{1/2} \left(\int_I \|v\|^2_{H_0^1(0,l)}\,dt\right)^{1/2} \\ &\lesssim \left(\int_0^l |u|^2_{H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \left(\int_0^l |v|^2_{H^{\frac{\alpha}{2}}(I)}\,dx\right)^{1/2} \\ &+ \|u\|_{L^2(I,H_0^1(0,l))} \|v\|_{L^2(I,H_0^1(0,l))} \\ &\lesssim \|u\|_{B^{\frac{\alpha}{2}}(Q_T)} \|v\|_{B^{\frac{\alpha}{2}}(Q_T)}. \end{aligned} \tag{11}$$

On the other hand, in order to deal with the coercivity of B, we let $u \in V'$ to have:

$$\begin{aligned} B(u,u) &= \left({}^{R}D_{t}^{\frac{\alpha}{2}} u,\ {}_{t}^{R}D^{\frac{\alpha}{2}} u \right)_{L^2(Q_T)} + \left(a\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \right)_{L^2(Q_T)} \\ &\geq \left({}^{R}D_{t}^{\frac{\alpha}{2}} u,\ {}_{t}^{R}D^{\frac{\alpha}{2}} u \right)_{L^2(Q_T)} + a_0 \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial x} \right)_{L^2(Q_T)} \\ &= \int_0^l \int_I {}^{R}D_{t}^{\frac{\alpha}{2}} u\,(x,t)\ {}_{t}^{R}D^{\frac{\alpha}{2}} u\,(x,t)\, dtdx \\ &+ \int_0^l \int_I \left(\frac{\partial u\,(x,t)}{\partial x} \right)^2 dtdx \\ &= \int_0^l \int_I {}^{R}D_{t}^{\frac{\alpha}{2}} u\,(x,t)\ {}_{t}^{R}D^{\frac{\alpha}{2}} u\,(x,t)\, dtdx \\ &+ \|u\|^2_{L^2(I,H_0^1(0,l))}. \end{aligned} \tag{12}$$

Consequently, with the help of using Lemmas 1, 2, 3 and 4, we obtain:

$$\begin{aligned} &\int_0^l \int_I {}^{R}D_{t}^{\frac{\alpha}{2}} u\,(x,t)\ {}_{t}^{R}D^{\frac{\alpha}{2}} u\,(x,t)\, dtdx + \|u\|^2_{L^2(I,H_0^1(0,l))} \\ &\quad \gtrsim \cos\left(\frac{\alpha}{2}\pi \right) \int_\Omega |u|^2_{H^{\frac{\alpha}{2}}(I)}\, dx + \|u\|^2_{L^2(I,H_0^1(0,l))}. \end{aligned}$$

Accordingly, since $0 < \alpha < 1$, we have:

$$0 < \cos\left(\frac{\alpha}{2}\pi \right) < 1.$$

This implies that $\exists \varepsilon < 1$ such that $\cos\left(\frac{\alpha}{2}\pi\right) \geq \varepsilon$. So (12) becomes:

$$\begin{aligned} &\cos\left(\frac{\alpha}{2}\pi \right) \int_\Omega |u|^2_{H^{\frac{\alpha}{2}}(I)}\, dx + \|u\|^2_{L^2(I,H_0^1(0,l))} \\ &\qquad \geq \varepsilon \int_\Omega |u|^2_{H^{\frac{\alpha}{2}}(I)}\, dx + \|u\|^2_{L^2(I,H_0^1(0,l))}. \end{aligned}$$

Therefore, we have:

$$B(u,u) \gtrsim \|u\|^2_{B^{\frac{\alpha}{2}}(Q_T)}. \tag{13}$$

Consequently, by the above inequality together with the Cauchy-Schwarz inequality, we have:

$$\|u\|^2_{B^{\frac{\alpha}{2}}(Q_T)} \lesssim B(u,u) = \langle f,u \rangle \leq \|u\|_{B^{\frac{\alpha}{2}}(Q_T)} \|f\|_{\left(B^{\frac{\alpha}{2}}(Q_T)\right)'}$$

Hence, we have:

$$\|u\|_{B^{\frac{\alpha}{2}}(Q_T)} \lesssim \|f\|_{\left(B^{\frac{\alpha}{2}}(Q_T)\right)'}.$$

5 Conclusion

Some sufficient conditions for examining the existence and uniqueness of solution for a class of initial-boundary value problems with Dirichlet condition have been established in this work. With the help of using Lax Milgram theorem, the gained results from this investigation have been established.

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