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Nano $\mathcal{I}$-connectedness and Strongly Nano $\mathcal{I}$-connectedness in Nano Topological Spaces

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Abstract This article's goals are to propose a brand-new category of space termed "nano-ideal topological spaces" and to look at how they relate to conventional topological spaces. To determine their relationships in these spaces, we create certain closed sets. These sets' fundamental characteristics and properties are provided. Additionally, we look into two theories of optimal connectivity in nano topological spaces. In particular, we obtain certain features of such spaces and define $\mathcal{I}$-connectedness and strongly $\mathcal{I}$-connectedness nano-topological spaces in terms of any ideal $\mathcal{I}$. This study aims to illustrate a novel kind of nano-topological space called nano-N^*-topological space, and we define the relationships between the various classes of open sets. We speak about how we might characterise them. Some of their characterizations are finally supported. The lower and upper approximations are used by the author to define nano topological space. As weak variants of Nano open sets, he also created Nano N^*-open sets, Nano semi-open sets, and Nano pre-open sets. Continuity, the fundamental notion of topology in nano topological space, was also introduced. Also, we introduce the notion of nano β-continuity between nano topological spaces and we investigate several properties of this type of near-nano continuity. Finally, we introduce two examples as applications in nano-topological spaces.

Keywords Nano Open, Nano Closed, Nano $\mathcal{I}$-connected, Strongly Nano $\mathcal{I}$-connected, υ-boundary Nano Ideal

AMS Subject Classification: 54A05, 54A10, 54B05.

1 Introduction

The authors of [5] introduced the concept of ideal on a set $\mathcal{P}$ as a non-empty collection of subsets of $\mathcal{P}$ that satisfies

(i) $H \in \mathcal{I} \,\&\, \mathcal{S} \subset H \Rightarrow \mathcal{S} \in \mathcal{I}$

(ii) $H, \mathcal{S} \in \mathcal{I} \Rightarrow H \cup \mathcal{S} \in \mathcal{I}$

Let $\mathcal{I}$ be an ideal on a topological space $(\mathcal{P}, \upsilon)$. If $\wp(\mathcal{P})$ is set of all subsets of $\mathcal{P}$, then the operator $(\cdot)^* : \wp(\mathcal{P}) \to \wp(\mathcal{P})$ is called local function [5] of H and $\mathcal{I}$ with respect to υ defined as follows:

$$H^*(\mathcal{I}, \upsilon) = \{x \in \mathcal{P} : U \cap H \notin \mathcal{I}, \text{ for every } U \in \upsilon(x)\}, \ \ H \subset \mathcal{P},$$

where $\upsilon(x) = \{U \in \upsilon, x \in U\}$.

The initiation of the ideal connectedness as $*$-connectedness is due to [1, 2]. So, the Author should be introduced this situation in the Introduction for the readers and the Literature.

Kuratowski's closure operator $cl^*(\cdot)$, which is known to be finer than υ, was defined by the authors of [5] as $cl^*(H) = H \cup H^*(\mathcal{I}, \upsilon)$. We shall simply write H^* for $H^*(\mathcal{I}, \upsilon)$ and υ^* for $\upsilon^*(\mathcal{I}, \upsilon)$ if there is no ambiguity. If $H \subset \mathcal{P}$, then $cl(H)$ and $int(H)$, respectively, stand for the closure and interior of H in $(\mathcal{P}, \upsilon)$. The interior and closure of H in $(\mathcal{P}, \upsilon^*)$ are denoted by $int^*(H)$ and $cl^*(H)$, respectively.

In general topology, the notation of connectedness and its features are extensively studied. My utilising equivalence relations on it, Lellis Thivagar. M. and Carmel Richard [8] defined nano topology in terms of approximations and the boundary region of a subset of the universe. In 2013, Lellis Thivagar

[7] investigated the characterization of a novel class of functions known as nano functions in nano topological space. Additionally, S. Krishna Prakash introduces nano-connectedness in nano topological space in [6] and examines its many features.

In [14] and [15], Hamlett and Jankovic have considered the local function in ideal topological space, and they have obtained a new topology. In this section, we shall introduce a similar type with a local function in nano topological space. Before starting the discussion, we shall consider the following concepts:

Consider (U, N) to be a nano topological space, with $N = \tau_R(X)$. A nano ideal topological space is denoted by $(U, N, \mathcal{I})$ and consists of a nano topological space (U, N) with an ideal I on U. Let (U, N) represent a nano topological space and $Q_n(q) = \{Q_n : q \in Q_n, Q_n \in N\}$ represent the family of nano open sets containing q.

2 Preliminaries

The following definitions should be kept in mind as they will be helpful in the sequel.

Definition 2.1. *[7] Let R represent an equivalence relation on U known as the indiscernibility relation, and U represent a non-empty finite collection of objects. After that, U is split up into distinct equivalence classes. It is argued that elements in the same equivalence class are indistinguishable from one another. The approximation space is stated to be the pair (U, R). If $\mathcal{P} \subseteq U$, then*

(i) The lower approximation of $\mathcal{P}$ with respect to R is indicated by $L_R(\mathcal{P})$, and it is the set of all objects that may be categorised as $\mathcal{P}$ with regard to R. Alternatively, $L_R(x) = \bigcup_{x\in U} \{R(x) : R(x) \subseteq \mathcal{P}\}$, where $R(x)$ signifies the equivalence class established by $x \in U$.

(ii) The set of all objects that might be categorised as belonging to $\mathcal{P}$ to R is known as the upper approximation of $\mathcal{P}$ to R, or $U_R(x)$. ie., $U_R(x) = \{R(x) : R(x) \cap \mathcal{P} \neq \emptyset\}$.

(iii) The set of all objects that cannot be categorised as either $\mathcal{P}$ or not $\mathcal{P}$ with regard to R is known as the boundary region of $\mathcal{P}$ with respect to R and is symbolised by the symbol $\mathcal{S}_R(\mathcal{P})$. ie., $\mathcal{S}_R(\mathcal{P}) = U_R(\mathcal{P}) - L_R(\mathcal{P})$.

Definition 2.2. *Let $(U, N, \mathcal{I})$ be a nano ideal topological space with an ideal $\mathcal{I}$ on U and $(\cdot)^*_n$ be a set operator from $P(U)$ to $P(U)$ ($P(U)$ is the set of all subsets of U). For a subset $A \subset U, A^*_n(\mathcal{I}, N) = \{q \in U : Q_n \cap A \notin \mathcal{I}$, for every $Q_n \in Q_n(q)\}$ is called the nano local function (briefly, n-local function) of A with respect to $\mathcal{I}$ and N. We will simply write A^*_n for $A^*_n(\mathcal{I}, N)$.*

Definition 2.3. *Let U be an equivalence relation on an universe U, and*

$$\upsilon_R(\mathcal{P}) = \{U, \emptyset, L_R(\mathcal{P}), U_R(\mathcal{P}), \mathcal{S}_R(\mathcal{P}),\ \ \mathcal{P} \subseteq U\}$$

satisfies the following axioms:

(i) $U, \emptyset \in \upsilon_R(\mathcal{P})$.

(ii) Any sub-collection of $\upsilon_R(\mathcal{P})$ contains the union of its elements.

(iii) Any finite sub-collection of $\upsilon_R(\mathcal{P})$ contains the intersection of its elements.

The nano topology on U with respect to $\mathcal{P}$ is denoted by $\upsilon_R(\mathcal{P})$. The nano topological space is the space $(U, \upsilon_R(\mathcal{P}))$. Nano open sets ([7]) are the name given to the components of $\upsilon_R(\mathcal{P})$.

Definition 2.4. *If $\mathcal{S} \subset \bigcup\{H_i : i \in I\}$ holds, a collection $\{H_i : i \in \mathcal{I}\}$ of nano-open sets in a nano topological space $\{U, \upsilon_R(\mathcal{P})\}$ is referred to as a nano-open cover of a subset $\mathcal{S}$ of U.*

Definition 2.5. *If there is a finite subset $\mathcal{I}_0$ of $\mathcal{I}$ such that $\mathcal{S} \subset \bigcup\{H_i : i \in \mathcal{I}_0\}$, then $\mathcal{S} \subset (U, \upsilon_R(\mathcal{P}))$ is said to be nano compact relative to $(U, \upsilon_R(\mathcal{P}))$.*

Definition 2.6. *If $\mathcal{I}$ is a nano ideal in $\mathcal{P}$ and $(U, \upsilon_R(\mathcal{P}))$ is a nano topological space, then $\mathcal{I}$ is referred to as a υ-boundary nano ideal if $\mathcal{I} \cap \upsilon_R(\mathcal{P}) = \{\emptyset\}$.*

3 Nano Connectedness Modulo a Nano Ideal

Definition 3.1. *If U cannot be written as a disjoint union of two non-nano ideal sets H and $\mathcal{S}$ of $(U, \upsilon_R(\mathcal{P}))$ such that $\overline{H} \cap \mathcal{S} = H \cap \overline{\mathcal{S}} = \emptyset$, then $(U, \upsilon_R(\mathcal{P}))$ is said to be nano connected over the nano ideal $\mathcal{I}$. H subset H of $\mathcal{P}$ ($H \notin \mathcal{I}$) is a non-nano ideal set in $\mathcal{P}$. If a subset of U is not nano $\mathcal{I}$-connected, then it is said to be nano $\mathcal{I}$-disconnected.*

Clearly every nano connected set is nano $\mathcal{I}$-connected, but the converse need not be true.

Example 3.1. *Let $U = \{l, m, n, o\}$, $\mathcal{P} = \{l, o\} \subset U$ and $U|R = \{\{l\}, \{m\}, \{n\}, \{o\}\}$ with nano topology $\upsilon_R(\mathcal{P}) = \{U, \emptyset, \{l, o\}\}$ and $\mathcal{I} = \{\{n\}, \{o\}\}$, then $(U, \upsilon_R(\mathcal{P}))$ is nano $\mathcal{I}$-connected but it is not nano connected.*

Theorem 3.1. *The following are equivalent for a nano topological space $(U, \upsilon_R(\mathcal{P}))$.*

(i) $(U, \upsilon_R(\mathcal{P}))$ is nano $\mathcal{I}$-connected.

(ii) $(U, \upsilon_R(\mathcal{P}))$ not be formulated as the union of two non-ideal open sets.

(iii) $(U, \upsilon_R(\mathcal{P}))$ not be formulated as the union of two disjoint non ideal closed sets.

Proof. **(i) ⇒ (ii)** Suppose (ii) is not true. Let H and $\mathcal{S}$ be nano open subsets of $(U, \upsilon_R(\mathcal{P}))$ such that $U = H \cup \mathcal{S}$, $H, \mathcal{S} \notin \mathcal{I}$ and $H \cap \mathcal{S} = \emptyset$. Then $H = \overline{H}$ and $\mathcal{S} = \overline{\mathcal{S}}$. So that $\overline{H} \cap \mathcal{S} = H \cap \overline{\mathcal{S}} = \emptyset$. This contradicts (i). Therefore (ii) is true.

(ii) ⇒ (iii) Suppose (iii) is false. Then $U = H \cup \mathcal{S}$, for some nano subsets $H, \mathcal{S}$ of $(U, \upsilon_R(\mathcal{P}))$, such that $H, \mathcal{S} \notin \mathcal{I}$, $H, \mathcal{S}$ are nano closed and $H \cap \mathcal{S} = \emptyset$. Then $U = H \cup \mathcal{S}$, where $H, \mathcal{S} \notin \mathcal{I}$, $H \cap \mathcal{S} = \emptyset$ and $H = \mathcal{P} - \mathcal{S}$, $\mathcal{S} = \mathcal{P} - H$ are nano open. This contradicts (ii). Therefore (iii) is true.

(iii) ⇒ (i) Suppose U is not $\mathcal{I}$-connected. Then $U = H \cup \mathcal{S}$, for some subsets $H, \mathcal{S}$ of $(U, \upsilon_R(\mathcal{P}))$ such that $H, \mathcal{S} \notin \mathcal{I}$, $\overline{H} \cap \mathcal{S} = H \cap \overline{\mathcal{S}} = \emptyset$. This implies that $\overline{H} \subseteq H$ and $\overline{\mathcal{S}} \subseteq \mathcal{S}$. Hence $U = H \cup \mathcal{S}$ where $H, \mathcal{S} \notin \mathcal{I}$, $H \cap \mathcal{S} = \emptyset$ and $H, \mathcal{S}$ are nano closed which is a contradiction to (iii). So $(U, \upsilon_R(\mathcal{P}))$ is nano $\mathcal{I}$-connected.

□

Theorem 3.2. *Let H_1 and H_2 be two nano $\mathcal{I}$-connected sets in $(U, \upsilon_R(\mathcal{P}))$ with $H_1 \cap H_2 \notin \mathcal{I}$. Then $H_1 \cup H_2$ is nano $\mathcal{I}$-connected.*

Proof. Suppose $H_1 \cup H_2$ is not nano $\mathcal{I}$-connected. Then $H_1 \cup H_2 = N \cup O$, where $N, O \notin \mathcal{I}$ and $(H_1 \cup H_2) \cap (\overline{N} \cap O) = \emptyset = N \cap \overline{O} \cap (H_1 \cup H_2)$, we have $H_1 \cap H_2 = (H_1 \cap H_2 \cap N) \cup (H_1 \cap H_2 \cap O) \notin \mathcal{I}$. So either $N \cap H_1 \cap H_2 \notin \mathcal{I}$ or $O \cap H_1 \cap H_2 \notin \mathcal{I}$. Suppose $N \cap H_1 \cap H_2 \notin \mathcal{I}$, then $N \cap H_1 \notin \mathcal{I}$ and $N \cap H_2 \notin \mathcal{I}$. Since $H_1 = (N \cap H_1) \cup (O \cap H_1)$ is nano $\mathcal{I}$-connected, either $N \cap H_1 \in \mathcal{I}$ or $O \cap H_1 \in \mathcal{I}$. As $N \cap H_1 \notin \mathcal{I}$, we have $O \cap H_1 \in \mathcal{I}$. Similarly, we have $O \cap H_2 \in \mathcal{I}$. So $O = (O \cap H_1) \cup (O \cap H_2) \in \mathcal{I} \Rightarrow O \in \mathcal{I}$ which is a contradiction. Hence $H_1 \cup H_2$ is nano $\mathcal{I}$-connected. □

Corollary 3.1. *The finite union of nano $\mathcal{I}$-connected sets $\{H_i\}$ for which $\bigcap_{i=1}^{n} H_i$ is a non-ideal set is also nano $\mathcal{I}$-connected.*

Theorem 3.3. *Let $\mathcal{I}$ be a nano ideal on $\mathcal{P}$ is defined as a nano topological space $(U, tauR(\mathcal{P}))$. If $H \subseteq U$ is nano $\mathcal{I}$-connected and $H \subseteq \mathcal{S} \subseteq cl^*(H)$ (closure of H in υ^*), then $\mathcal{S}$ is nano $\mathcal{I}$-connected.*

Proof. We are suppose to say that either $V \cap H \cap N \notin \mathcal{I}$ or $V \cap H \cap O \notin I$ and in such a case we have to consider two cases.

Suppose $\mathcal{S}$ is not $\mathcal{I}$-connected. Then $\mathcal{S} = N \cup O$, where $N, O \notin \mathcal{I}$ and $\mathcal{S} \cap \overline{N} \cap O = \emptyset = N \cap \overline{O} \cap \mathcal{S}$. Now $H = (H \cap N) \cup (H \cap O)$. Since H is nano $\mathcal{I}$-connected, either $H \cap N \in \mathcal{I}$ or $H \cap O \in \mathcal{I}$. Suppose $H \cap O \in \mathcal{I}$ and let $x \in O - H$. Then for every neighbourhood V of x, $V \cap H \notin \mathcal{I}$. As $V \cap H = (V \cap H \cap N) \cup (V \cap H \cap O) \notin \mathcal{I}$, we have $V \cap H \cap N \notin \mathcal{I}$.

We can not have two implies in the same statement so $\mathcal{I}$ suggest to rephrase this statement. If $V \cap N \neq \emptyset$ for any neighborhood of x, then $x \in \bar{N}$ but in the proof $x \notin \bar{N}$ which is a typo.

In particular $V \cap H \cap N \neq \emptyset$. Implies $V \cap N \neq \emptyset$ implies $x \notin \overline{N}$. Therefore $x \in O - H$ implies $x \in \overline{N}$ which is contradiction to $\mathcal{S} \cap \overline{N} \cap O = \emptyset$. Hence $O - H = \emptyset$. That is $O \subseteq H$. Therefore $O = O \cap H \in \mathcal{I}$, which is a contradiction. Then $\mathcal{S}$ is nano $\mathcal{I}$-connected. □

Remark 3.1. *Let $(U, \upsilon_R(\mathcal{P}))$ be a nano topological space with a nano ideal $\mathcal{I}$ on $\mathcal{P}$. Let U be nano $\mathcal{I}$-connected. If $\mathcal{J}$ is a nano ideal on U with $\mathcal{I} \subseteq \mathcal{J}$, then U is nano $\mathcal{J}$-connected.*

Theorem 3.4. *Let $f : (U, \upsilon_R(\mathcal{P})) \to (V, \upsilon_{R'}(\mathcal{Q}))$ be a continuous surjection mapping between two nano topological spaces $(U, \upsilon_R(\mathcal{P}))$ and $(V, \upsilon_{R'}(\mathcal{Q}))$. If $(\mathcal{P}, \upsilon)$ is nano $\mathcal{I}$-connected, then $(\mathcal{Q}, \upsilon)$ is nano $f(\mathcal{I})$-connected.*

Proof. Let $f : (U, \upsilon_R(\mathcal{P}), \mathcal{I}) \to (V, \upsilon_{R'}(\mathcal{Q}))$ be a continuous surjection and $\mathcal{P}$ is nano $\mathcal{I}$-connected. Assume that $\mathcal{Q}$ is not nano $f(\mathcal{I})$-connected, then $\mathcal{Q} = \mathcal{S} \cup N$, for some $\mathcal{S}, N \notin f(\mathcal{I})$. $\mathcal{S} \cap N = \emptyset$ and $\mathcal{S}, N$ are nano open. Since f is continuous, $f^{-1}(\mathcal{S}), f^{-1}(N)$ are nano open and $f^{-1}(\mathcal{S}) \cap f^{-1}(N) = f^{-1}(\mathcal{S} \cap N) = f^{-1}(\emptyset) = \emptyset$.

Also $f^{-1}(\mathcal{S}), f^{-1}(N) \notin \mathcal{I}$. [If $f^{-1}(\mathcal{S}) \in \mathcal{I}$, then $\mathcal{S} \in f(\mathcal{I})$]. Now $\mathcal{P} = f^{-1}(\mathcal{S}) \cup f^{-1}(N)$, where $f^{-1}(\mathcal{S}), f^{-1}(N)$ are nano open, $f^{-1}(\mathcal{S}) \cap f^{-1}(N) = \emptyset$ and $f^{-1}(\mathcal{S}), f^{-1}(N) \notin \mathcal{I}$. Hence $\mathcal{P}$ is not nano $\mathcal{I}$-connected, a contradiction to our assumption. Thus $\mathcal{Q}$ is nano $f(\mathcal{I})$-connected. □

Theorem 3.5. *Let $(U, \upsilon_R(\mathcal{P}))$ be a nano topological space with nano ideal $\mathcal{I}$ on $\mathcal{P}$. Let $H, \mathcal{S} \subseteq U$. If H is nano $\mathcal{I}$-connected and $\mathcal{S} \in \mathcal{I}$ then $H \cup \mathcal{S}$ is nano $\mathcal{I}$-connected.*

Proof. If $H \cup \mathcal{S}$ is not nano $\mathcal{I}$-connected, then there exist a nano open sets N, O of $\mathcal{P}$ such that $(H \cup \mathcal{S}) \cap N \notin \mathcal{I}$, $(H \cup \mathcal{S}) \cap O \notin \mathcal{I}$ and $(H \cup O) \cap (\overline{N} \cap O) = \emptyset = (H \cup \mathcal{S}) \cap (N \cap \overline{\mathcal{S}})$. As $\mathcal{S} \in \mathcal{I}$, we have $H \cap N \notin \mathcal{I}$ (as $\mathcal{S} \cap N \in \mathcal{I}$) and $H \cap O \notin \mathcal{I}$. As $H = (H \cap N) \cup (H \cap O)$, which is a contradiction to H is nano $\mathcal{I}$-connected. Hence $H \cup \mathcal{S}$ is nano $\mathcal{I}$-connected. □

Corollary 3.2. *Let $(U, \upsilon_R(\mathcal{P}))$ be a nano topological space with nano ideal $\mathcal{I}$ on $\mathcal{P}$. Let $H \subseteq \mathcal{P}$. If H is nano $\mathcal{I}$-connected and $\mathcal{P} - H \in \mathcal{I}$, then $\mathcal{P}$ is nano $\mathcal{I}$-connected.*

Theorem 3.6. *Let $(U, \upsilon_R(\mathcal{P}))$ be a nano topological space with respect to $\mathcal{P}$, where $\mathcal{P} \subseteq U$ and $(\mathcal{P}, \upsilon)$ is nano υ-boundary nano ideal $\mathcal{I}$ on $\mathcal{P}$. Then $\mathcal{P}$ is nano $\mathcal{I}$-connected if $\mathcal{P}$ is nano connected.*

Proof. It is enough to prove $\mathcal{P}$ is nano connected, if $\mathcal{P}$ is nano $\mathcal{I}$-connected.

Suppose $\mathcal{P}$ is not nano connected, $\mathcal{P} = H \cup \mathcal{S}$, where $H \neq \{\emptyset\}$, $\mathcal{S} \neq \{\emptyset\}$ and $\overline{H} \cap \mathcal{S} = H \cap \overline{\mathcal{S}} = \{\emptyset\}$. Since $\mathcal{I} \cap \upsilon = \{\emptyset\}$, we have $H, \mathcal{S} \notin \mathcal{I}$. So $\mathcal{P}$ is not nano $\mathcal{I}$-connected, gives a contradiction. Thus $\mathcal{P}$ is nano connected. □

Theorem 3.7. *Let $(U, \upsilon_R(\mathcal{P}))$ and $(V, \upsilon_R(\mathcal{Q}))$ are nano topological spaces and let $\mathcal{P}$ be a nano $\mathcal{I}_1$-connected and $\mathcal{Q}$ be a $\mathcal{I}_2$-connected. Assume that $\mathcal{I}_1 \cap \upsilon_R$ is nano closed under arbitrary unions. If $\mathcal{I}$ is a nano ideal such that $p_i^{-1}(\mathcal{I}_i) \subset \mathcal{I}$, $i = 1, 2$, then $\mathcal{P} \times \mathcal{Q}$ is nano $\mathcal{I}$-connected.*

Proof. If $\mathcal{P} \in \mathcal{I}_1$, then $\mathcal{P} \times \mathcal{Q}$ is not an ideal on $\mathcal{P} \times \mathcal{Q}$. So that $\mathcal{P} \notin \mathcal{I}_1$. Assume that $\mathcal{P} \times \mathcal{Q}$ is not nano $\mathcal{I}$-connected then $\mathcal{P} \times \mathcal{Q} = H \cup \mathcal{S}$, where $H, \mathcal{S} \notin \mathcal{I}$, $H \cap \mathcal{S} = \phi$ and $H, \mathcal{S}$ are nano open in $\mathcal{P} \times \mathcal{Q}$. To each $y \in \mathcal{Q}$, define $H_y = \{x \in \mathcal{P} : (x, y) \in H\}$ and $\mathcal{S}_y = \{x \in \mathcal{P} : (x, y) \in \mathcal{S}\}$. Let $N = \{y \in \mathcal{Q} : H_y \in \mathcal{I}_1\}$ and $O = \{y \in \mathcal{Q} : \mathcal{S}_y \in \mathcal{I}_1\}$. Then $\mathcal{P} = H_y \cup \mathcal{S}_y$. To each y, both H_y and $\mathcal{S}_y$ are open, $H_y \cap \mathcal{S}_y \neq \emptyset$. As $\mathcal{P}$ is nano $\mathcal{I}$-connected, either $H_y \in \mathcal{I}_1$ or $\mathcal{S}_y \in \mathcal{I}_1$ for any $y \in \mathcal{Q}$. Therefore $\mathcal{Q} = N \cup O$ and $N \cap O = \emptyset$. Now we claim that N is nano closed. Fix $y \in \overline{N}$. If $H_y \notin \mathcal{I}_1$, then $H_y = \emptyset$. Since H is open, to each $x \in H_y$, there exist a neighbourhood U_x of x and V_y of y such that $(x,y) \in U_x \times V_y \subset H$. As $y \in \overline{N}$, there is one $y' \in V_y \cap N$, so $U_x \times \{y'\} \subseteq H$ and hence $U_x \subset H_{y'}$ and $U_x \in \mathcal{I}_1$. Therefore $H_y \subseteq \bigcup_{x \in H_y} U_x \in \mathcal{I}_1$ and hence $y \in N$. Thus N is nano closed. Similarly O is nano closed. Since $\mathcal{Q}$ is nano $\mathcal{I}_2$-connected, we have $N \in \mathcal{I}_2$ or $O \in \mathcal{I}_2$.

Case (i) If $N \in \mathcal{I}_2$, then $\mathcal{P} \times N \notin \mathcal{I}$. Take $E = \cup\{\mathcal{S}_y : y \in O\} \in \mathcal{I}_1 \cap \upsilon$. So $E \times \mathcal{Q} \in \mathcal{I}$ and $(\mathcal{P} \times N) \cup (E \times \mathcal{Q}) \in \mathcal{I}$. Fix $(x,y) \in \mathcal{S}$. If $y \in N$, then $(x,y) \in \mathcal{P} \times N$. If $y \notin N$, then $y \in O$ and $x \in \mathcal{S}_y \subset E$. Therefore $(x,y) \in E \times \mathcal{Q}$. Hence $\mathcal{S} \subseteq (\mathcal{P} \times N) \cup (E \times \mathcal{Q})$. So $\mathcal{S} \in \mathcal{I}$. This contradicts the fact that $\mathcal{S} \notin \mathcal{I}$.

Case (ii) If $O \in \mathcal{I}_2$, then $\mathcal{P} \times O \in \mathcal{I}$ and as in Case (i), we obtain a contradiction. Thus $\mathcal{P} \times \mathcal{Q}$ is nano $\mathcal{I}$-connected. □

Corollary 3.3. *Let $(U_i, \upsilon_{R_i}(\mathcal{P}_i))$ be a nano topological space with nano ideal $\mathcal{I}_i$ on $\mathcal{P}_i$ respectively, for $i = 1,2,3,\ldots,n$. Let $\mathcal{P} = \prod_{i=1}^{n} \mathcal{P}_i$ and $\mathcal{I}$ be a nano ideal such that $p_i^{-1}(\mathcal{I}_i) \subset \mathcal{I}$, $i = 1,2,3,\ldots n$. If $\{\mathcal{I}_i \cap \upsilon_i, i = 1,2,3,\ldots n-1\}$ is nano closed under arbitrary unions and $\mathcal{P}_i$ is nano $\mathcal{I}$-connected, then $\mathcal{P}$ is nano $\mathcal{I}$-connected.*

Corollary 3.4. *Let $(U, \upsilon_R(\mathcal{P}))$ be a nano connected space and $(V, \upsilon_R(\mathcal{Q}))$ be a nano $\mathcal{I}_2$-connected. If $\mathcal{I}$ is an ideal containing $p_2^{-1}(\mathcal{I}_2)$, then $\mathcal{P} \times \mathcal{Q}$ is nano $\mathcal{I}$-connected.*

Proof. Consider $\mathcal{I}_1 = \{\emptyset\}$, then $\mathcal{I}_1 \cap \upsilon_1$ is closed under arbitrary unions, so by Theorem 3.7 we get $\mathcal{P} \times \mathcal{Q}$ is nano $\mathcal{I}$-connected. □

4 Strongly nano $\mathcal{I}$-connected sets

Definition 4.1. *Let $(U, \upsilon_R(\mathcal{P}))$ be a nano topological space and let $\mathcal{I}$ be a nano ideal on U. H subset H of U is said to be strongly nano $\mathcal{I}$-connected if there is a nano υ-connected subset $\mathcal{S}$ of U such that $H = \mathcal{S} \cup N$, where $N \in \mathcal{I}$.*

Remark 4.1. *Every nano connected set is strongly nano $\mathcal{I}$-connected, but converse need not be true.*

Example 4.1. *Let $U = \{l,m,n,o\}$, $\mathcal{P} = \{l,o\} \subset U$ and $U|R = \{\{l\},\{m\},\{n\},\{o\}\}$ with nano topology $\upsilon_R(\mathcal{P}) = \{U, \emptyset, \{l,o\}\}$. $\mathcal{I} = \{\{n\},\{o\}\}$. Take $\mathcal{S} = \upsilon_R(\mathcal{P})$ and $N = \{n\}$. Then $H = \mathcal{S} \cup N$ is strongly nano $\mathcal{I}$-connected but not connected.*

Theorem 4.1. *Let $(U, \upsilon_R(\mathcal{P}))$ be a topological space with a nano ideal $\mathcal{I}$ on $\mathcal{P}$. If $(\mathcal{P}, \upsilon)$ is strongly nano $\mathcal{I}$-connected, then it is nano $\mathcal{I}$-connected.*

Proof. Assume that $\mathcal{P}$ is strongly nano $\mathcal{I}$-connected, then $\mathcal{P} = \mathcal{S} \cup N$, where $\mathcal{S}$ is nano $\mathcal{I}$-connected and $N \in \mathcal{I}$. Suppose $\mathcal{P} = O_1 \cup O_2$, where O_1, O_2 are open and $O_1 \cap O_2 = \phi$. Then $\mathcal{S} = (O_1 \cap \mathcal{S}) \cup (O_2 \cap \mathcal{S})$ where $O_1 \cap \mathcal{S} = \emptyset$ or $O_2 \cap \mathcal{S} = \emptyset$. This implies $O_1 \subset \mathcal{P} - \mathcal{S}$ or $O_2 \subset \mathcal{P} - \mathcal{S} \Rightarrow O_1 \subset N$ or $O_2 \subset N$, implies $O_1 \in \mathcal{I}$ or $O_2 \in \mathcal{I}$. Hence $\mathcal{P}$ is nano $\mathcal{I}$-connected. □

Remark 4.2. *Let $(U, \upsilon_R(\mathcal{P}))$ be a nano topological space with a nano ideal $\mathcal{I}$ on U. Let $H, \mathcal{S} \subseteq U$. If H is strongly nano $\mathcal{I}$-connected and $\mathcal{S} \in \mathcal{I}$, then $H \cup \mathcal{S}$ is strongly nano $\mathcal{I}$-connected.*

Theorem 4.2. *If $f : (U, \upsilon_R(\mathcal{P})) \to (V, \upsilon_{R'}(\mathcal{Q}))$ is a nano continuous surjection and $\mathcal{P}$ is strongly nano $\mathcal{I}$-connected, then $\mathcal{Q}$ is strongly nano $f(\mathcal{I})$-connected.*

Proof. Let $f : (U, \upsilon_R(\mathcal{P})) \to (V, \upsilon_{R'}(\mathcal{Q}))$ be a nano continuous surjection and let $(\mathcal{P}, \upsilon)$ is strongly nano $\mathcal{I}$-connected. Then $\mathcal{P} = \mathcal{S} \cup N$, where $\mathcal{S}$ be nano connected and $N \in \mathcal{I}$. Therefore $\mathcal{Q} = f(\mathcal{P}) = f(\mathcal{S} \cup N) = f(\mathcal{S}) \cup f(N)$ where $f(\mathcal{S})$ is nano connected and $f(N) \in f(\mathcal{I})$. Then $\mathcal{Q}$ is strongly $f(\mathcal{I})$-connected. □

Theorem 4.3. *If $\overline{H} \in \mathcal{I}$, for all $H \in \mathcal{I}$, then whenever H is strongly nano $\mathcal{I}$-connected then $\mathcal{S}$ is also strongly nano $\mathcal{I}$-connected, for all $\mathcal{S}$ with $H \subseteq \mathcal{S} \subseteq \overline{H}$. In particular $\overline{H}$ is strongly nano $\mathcal{I}$-connected, for any $H \in \mathcal{I}$.*

Proof. Suppose H is strongly nano $\mathcal{I}$-connected. Then $H = N \cup O$, where N is nano connected and $O \in \mathcal{I}$. Since $H \subseteq \mathcal{S} \subseteq \overline{H}$ and $H = N \cup O \subseteq \mathcal{S}$, we have $\mathcal{S} = (\overline{N} \cap \mathcal{S}) \cup (\overline{O} \cap \mathcal{S})$, where $\overline{N} \cap \mathcal{S}$ is nano connected as $N \subseteq \overline{N} \cap \mathcal{S} \subseteq \overline{N}$ and $\overline{O} \cap \mathcal{S} \in \mathcal{I}$. Hence $\mathcal{S}$ is strongly nano $\mathcal{I}$-connected. As a particular case when H is strongly nano $\mathcal{I}$-connected, $\overline{H}$ is strongly nano $\mathcal{I}$-connected, for all $\overline{I} \in \mathcal{I}$. □

Theorem 4.4. *Let H_i $(i = 1,2,3,\ldots n)$ be a strongly nano $\mathcal{I}$-connected such that $\bigcap_{i=1}^{n} H_i \notin \mathcal{I}$, then $\bigcup_{i=1}^{n} H_i$ is strongly nano $\mathcal{I}$-connected.*

Proof. Since each H_i $(i = 1,2,3,\ldots n)$ is a strongly nano $\mathcal{I}$-connected, we have $H_i = \mathcal{S}_i \cup N_i$, where $\mathcal{S}_i$ is nano connected and $N_i \in \mathcal{I}$. As $\cap H_i \notin \mathcal{I}$, we get $H_i \notin \mathcal{I}$ and $\mathcal{S}_i \notin \mathcal{I}$, for any i. Let $F_j = (\cap H_j) \cap N_j$. Then $F_j \in \mathcal{I}$, for any j. Therefore $\bigcup_{j=1}^{n} F_j \in \mathcal{I}$, put $E = (\cap H_i) - (\cup F_j)$. Then $E \notin \mathcal{I}$, because $\cap H_i \notin \mathcal{I}$ and $\cup F_j \in \mathcal{I}$.

Now $E \subset \mathcal{S}_j$ for any j and hence $E \subset \cap \mathcal{S}_j \notin \mathcal{I}$. In particular $\cap \mathcal{S}_j \neq \emptyset$. Hence $\bigcup_{i=1}^{n} \mathcal{S}_i$ is nano connected and hence $\bigcup_{i=1}^{n} H_i = \left(\bigcup_{i=1}^{n} \mathcal{S}_i\right) \cup N$, where $N \subseteq \bigcup_{i=1}^{n}(H_i - \mathcal{S}_i) \subseteq \bigcup_{i=1}^{n} N_i \in \mathcal{I}$. Thus $\bigcup_{i=1}^{n} H_i$ is strongly nano $\mathcal{I}$-connected. □

Theorem 4.5. *Let $(U, \upsilon_R(\mathcal{P}))$ and $(V, \sigma_R(\mathcal{Q}))$ be two nano topological spaces and let $(\mathcal{P}, \upsilon)$ be a strongly nano $\mathcal{I}_1$-connected and $(\mathcal{Q}, \sigma)$ be a strongly nano $\mathcal{I}_2$-connected. If $\mathcal{I}$ is nano ideal on $\mathcal{P} \times \mathcal{Q}$ such that the projection $\wp_i^{-1}(\mathcal{I}_i) \subset \mathcal{I}$, $i = 1,2$, then $\mathcal{P} \times \mathcal{Q}$ is strongly nano $\mathcal{I}$-connected.*

Proof. Suppose $\mathcal{P}$ is strongly nano $\mathcal{I}_1$-connected and $\mathcal{Q}$ is strongly nano $\mathcal{I}_2$-connected then $\mathcal{P} = H \cup N_1$ and $\mathcal{Q} = \mathcal{S} \cup N_2$, where $H, \mathcal{S}$ are nano connected subsets of $\mathcal{P}$ and $\mathcal{Q}$ respectively and $N_1 \in \mathcal{I}, N_2 \in \mathcal{I}_2$. Therefore $\mathcal{P} \times \mathcal{Q} = (H \times \mathcal{S}) \cup$

$[(N_1 \times \mathcal{Q}) \cup (\mathcal{P} \times N_2)]$. Since $H \times \mathcal{S}$ is nano connected with $\upsilon \times \sigma$ and $N_1 \times \mathcal{Q}, \mathcal{P} \times N_2 \in \mathcal{I} \Rightarrow (N_1 \times \mathcal{Q}) \cup (\mathcal{P} \times N_2) \in \mathcal{I}$. Thus $\mathcal{P} \times \mathcal{Q}$ is strongly connected. □

5 Conclusion

This study aims to illustrate a novel kind of nano-topological space called nano-N^*-topological space. The relationships between the various classes of open sets are defined. We spoke about how we might characterise them. Some of their characterizations are finally supported. The lower and upper approximations are used by the author to define nano topological space. As weak variants of Nano open sets, he also created Nano N^*-open sets, Nano semi-open sets, and Nano pre-open sets. Continuity, the fundamental notion of topology in nano topological space, was also introduced.

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