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Finite Domination Type for Monoid Presentations

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Abstract In [5], Squier, Otto and Kobayashi explored a homotopical property for monoids called finite derivation type (FDT) and proved that FDT is a necessary condition that a finitely presented monoid must satisfy if it is to have a finite canonical presentation. In the latter development in [2], Kobayashi proved that the property bi-FP_1 is equivalent with what is called in [2] finite domination type. It was indicated in the end of [2] that there are bi-FP_1 monoids which are not even finitely generated, and as a consequence are not of FDT. It was this indication that inspired us to look for the possibility of defining a property of monoids which encapsulates both, FDT and finite domination type. This is realized in the current paper by extending the notion of finite domination from monoids to rewriting systems, and to achieve this, we are based on the approach of Isbell in [1], who defined the notion of the dominion of a subcategory $\mathcal{C}$ of a category $\mathcal{D}$ and characterized that dominion in terms of zigzags in $\mathcal{D}$ over $\mathcal{C}$. The reason we followed this approach is that to every rewriting system $(\mathbf{x}, \mathbf{r})$ which gives a monoid M, there is always a category $\mathcal{D}(\mathbf{x}, \mathbf{r})$ associated to it which contains three types of information at the same time: (i) all the possible ways in which the elements of M are written in terms of words with letters from $\mathbf{x}$, (ii) all the possible ways one can transform a word with letters from $\mathbf{x}$ into another one representing the same element of M by using rewriting rules from $\mathbf{r}$. Each of such way gives is in fact a path in the reduction graph of $(\mathbf{x}, \mathbf{r})$. The last information (iii) encoded in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ is that $\mathcal{D}(\mathbf{x}, \mathbf{r})$ contains all the possible ways that two parallel paths of the reduction graph are linked to each other by a series of compositions of whiskerings of other parallel paths. This category $\mathcal{D}(\mathbf{x}, \mathbf{r})$ turns out to have the advantage that it can "measure" the extent to which a set U of parallel paths is sufficient to express any pair of parallel paths by composing whiskers from U. The gadget used to measure this, is the Isbell dominion of the whisker category $W(U)$ generated by U over $\mathcal{D}(\mathbf{x}, \mathbf{r})$. We then define the monoid M given by $(\mathbf{x}, \mathbf{r})$ to be of finite domination type (FDOT) if both $\mathbf{x}$ and $\mathbf{r}$ are finite and there is a finite set U of morphisms such that $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ is exactly $\mathcal{D}(\mathbf{x}, \mathbf{r})$. The first main result of our paper is that likewise FDT, FDOT is an invariant of the monoid presentation, and the second one is that that FDT implies FDOT, while remains open whether the converse is true or not. The importance of FDOT stands in the fact that not only it generalizes FDT, but the way it is defined has a lot in common with bi-FP_1, giving thus hope that FDOT is the right tool to put FDT and bi-FP_1 into the same framework.

Keywords Monoid, Presentation, Category, Homology, Dominion

1 Introduction and basic definitions

Monoids given by a finite convergent rewriting systems have attracted for years a considerable attention because they have a solvable word problem. In an attempt to find necessary conditions for a finitely presented monoid in order that it is given by a convergent presentation, Squier, Otto and Kobayashi discovered in [5] the property FDT. In topological terms, a monoid M is FDT whenever it is given by a finite presentation $(\mathbf{x}, \mathbf{r})$ and its reduction graph $\Gamma(\mathbf{x}, \mathbf{r})$ has a finite homotopy base. It is important to mention that Pride proved in [4] that for groups FDT is equivalent to the homological property bi-FP_3 indicating that FDT has also a homological nature aside with the topological one. More recently, Kobayashi introduced in [2] another property for monoids called finite domination type. In categorical terms, a monoid M is of finite domination type, if there is a finitely generated submonoid S, such that $Dom_M(S) = M$. It was proved in [2] that this property is in fact equivalent with the homological property bi-FP_1, and likewise with the group case, bi-FP_1 and finite generation coincide for inverse monoids. Thus we have so far two different conditions, FDT for groups is the same as bi-FP_3, and finite domination type is the same as bi-FP_1 for all monoids. The main purpose of the present paper is to generalize FDT in a way that it mimics the definition of Kobayashi's finite domination type with the hope that in prospects it will be proved to be equivalent with a homological invariant of the aforementioned types. Our new condition here is called finite domination type for monoid presentation (FDOT) and is proved to be an invariant of the presentation, which is our first main result. The second result is that FDT monoids are also FDOT proving that FDOT generalizes FDT. It remains open though the converse of the above holds true as well.

The rest of the paper is organized as follows. The remaining of this section contains a number of known definitions and constructions which are used latter to make the definition of our FDOT easy to understand. Section 2 contains the definition of FDOT and our two main results, the invariance of FDOT and its relationship with FDT. Section 3 contains conclusions and a number of suggestions for future study.

We begin with the definition of a certain category arising from a monoid presentation in the form it appears in [3] but with notation compatible with those used in [7]. First we say that $(\mathbf{x}, \mathbf{r})$ is a monoid presentation of a monoid M, if $\mathbf{x}$ is a nonempty set, $\mathbf{r}$ is a set of distinct pairs (R_+, R_-) of words from the free monoid $\mathbf{x}^*$, and $M \cong \mathbf{x}^*/\mathbf{r}^\sharp$ where $\mathbf{r}^\sharp$ is the congruence on $\mathbf{x}^*$ generated by $\mathbf{r}$. Associated with a monoid presentation $(\mathbf{x}, \mathbf{r})$, it is constructed a category $M(\mathbf{x}, \mathbf{r})$ as follows:

- An object is a word of the free monoid $\mathbf{x}^*$.
- A positive atomic derivation $R_+ \xrightarrow{A} R_-$ is given by a pair $(R_+, R_-) \in \mathbf{r}$.
- A negative atomic derivation $R_- \xrightarrow{A^{-1}} R_+$ is associated with every positive atomic derivation $R_+ \xrightarrow{A} R_-$.

  An atomic derivation $R_\varepsilon \xrightarrow{A^\varepsilon} R_{-\varepsilon}$ where $\varepsilon = \pm 1$ is then a positive or a negative atomic derivation depending on the sign of ε.
- An elementary derivation $x \xrightarrow{E} y$ is given by two words $U, V \in \mathbf{x}^*$ and an atomic derivation $R_\varepsilon \xrightarrow{A^\varepsilon} R_{-\varepsilon}$ such that $x = UR_{+\varepsilon}V$ and $y = UR_{-\varepsilon}V$. If $U = V = 1$, we identify E with the atomic derivation A^ε.
- A derivation $x \xrightarrow{F} y$ is given by a sequence

$$x = x_0 \xrightarrow{E_1} x_1 \xrightarrow{E_2} \cdots \xrightarrow{E_n} x_n = y$$

  of elementary derivations. If $n = 1$, we identify F with the elementary derivation E_1, and if $n = 0$, we get the identity derivation id_x.

The composition of two derivations is defined in the obvious way. Also there is a bi-action of $\mathbf{x}^*$ on the set of derivations defined by setting for every derivation $z \xrightarrow{F} z'$ and every $x, y \in \mathbf{x}^*$, xFy to be the derivation $xzy \xrightarrow{xFy} xz'y$. Further we factor out disjoint derivations, which means that for every $U, V, W \in \mathbf{x}^*$ and two atomic derivations $R_\varepsilon \xrightarrow{A^\varepsilon} R_{-\varepsilon}$ and $R'_{\varepsilon'} \xrightarrow{A'^{\varepsilon'}} R'_{-\varepsilon'}$, we identify $UR_{-\varepsilon}VA'W \circ UAVR'_{\varepsilon'}W$ and $UAVR'_{-\varepsilon'}W \circ UR_\varepsilon VA'W$ as in the diagram below

$$UR_\varepsilon VR'_{\varepsilon'}W \xrightarrow{UA^\varepsilon VR'_{\varepsilon'}W} UR_{-\varepsilon}VR'_{\varepsilon'}W \xrightarrow{UR_{-\varepsilon}VA'^{\varepsilon'}W} UR_{-\varepsilon}VR'_{-\varepsilon'}W$$

$$UR_\varepsilon VR'_{\varepsilon'}W \xrightarrow{UR_\varepsilon VA'^{\varepsilon'}W} UR_\varepsilon VR'_{-\varepsilon'}W \xrightarrow{UA^\varepsilon VR'_{-\varepsilon'}W} UR_{-\varepsilon}VR'_{-\varepsilon'}W$$

More generally, if $x \xrightarrow{F} UR_\varepsilon VR'_{\varepsilon'}W$ and $UR_{-\varepsilon}VR'_{-\varepsilon'}W \xrightarrow{G} y$ are arbitrary derivations, we identify the derivations $G \circ UR_{-\varepsilon}VA'^{\varepsilon'}W \circ UA^\varepsilon VR'_{\varepsilon'}W \circ F$ and $G \circ UA^\varepsilon VR'_{-\varepsilon'}W \circ UR_\varepsilon VA'^{\varepsilon'}W \circ F$. From this it follows easily that for every two derivations $x \xrightarrow{F} x'$ and $y \xrightarrow{G} y'$, $x'G \circ Fy$ and $Fy' \circ xG$ are equivalent by permutation of disjoint derivations. From this follows that the multiplication of equivalence classes of derivations is well defined. In particular we see that the bi-action of $\mathbf{x}^*$ on derivations extends to a bi-action on the set of equivalence classes of derivations. We observe in addition that equivalent derivations have the same length, therefore we can talk of the length of an equivalence class of derivations which is the length of any of derivations in the class. From now and on the equivalence class of a derivation F will be denoted by $\mathbb{F}$ which we keep calling a derivation.

All these data in fact define a monoidal category $M(\mathbf{x}, \mathbf{r})$ which is called in [3] the free monoidal category associated with a presentation $(\mathbf{x}, \mathbf{r})$. Let now $M(\mathbf{x}, \mathbf{r})$ and $M(\mathbf{x}', \mathbf{r}')$ be the free monoidal categories associated with presentations $(\mathbf{x}, \mathbf{r})$ and $(\mathbf{x}', \mathbf{r}')$ of monoids M and M' respectively, and let $\Phi : M(\mathbf{x}, \mathbf{r}) \to M(\mathbf{x}', \mathbf{r}')$ be a 2-functor, that is a functor that preserves the multiplication structure. This clearly induces a morphism $\varphi : M \to M'$ by the rule $\varphi(\bar{U}) = \overline{\Phi(U)}$ for every $U \in \mathbf{x}^*$.

Conversely, if $\varphi : M \to M'$ is a monoid morphism, we can define a 2-functor $\Phi : M(\mathbf{x}, \mathbf{r}) \to M(\mathbf{x}', \mathbf{r}')$ in the following way. For every generator $a \in \mathbf{x}$ we choose a word $x_a \in \mathbf{x}'^*$ such that $\overline{x_a} = \varphi(\bar{a})$. This map extends to a morphism $\xi : \mathbf{x}^* \to \mathbf{x}'^*$ with the property $\varphi(\bar{U}) = \overline{\xi(U)}$ for all $U \in \mathbf{x}^*$. Now for each atomic derivation $R_\varepsilon \xrightarrow{\mathbb{A}^\varepsilon} R_{-\varepsilon}$ in $M(\mathbf{x}, \mathbf{r})$

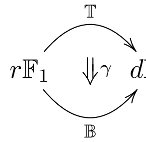
we have that $\overline{\xi(R_\varepsilon)} = \varphi(\overline{R_\varepsilon}) = \varphi(\overline{R_{-\varepsilon}}) = \overline{\xi(R_{-\varepsilon})}$, hence we can choose a derivation $\xi(R_\varepsilon) \xrightarrow{\mathbb{F}_{\mathbb{A}^\varepsilon}} \xi(R_{-\varepsilon})$. The map $\mathbb{A}^\varepsilon \mapsto \mathbb{F}^\varepsilon_{\mathbb{A}}$ extends to a 2-functor $\Phi : M(\mathbf{x}, \mathbf{r}) \to M(\mathbf{x}', \mathbf{r}')$ such that $\Phi(U) = \xi(U)$ for every $U \in \mathbf{x}^*$.

If $\Psi : M(\mathbf{x}, \mathbf{r}) \to M(\mathbf{x}', \mathbf{r}')$ is another 2-functor arising from $\varphi : M \to M'$, then for every $U \in \mathbf{x}^*$, $\overline{\Phi(U)} = \overline{\Psi(U)}$. We can choose for every $a \in \mathbf{x}$ a derivation $\Phi(a) \xrightarrow{\mathbb{H}_a} \Psi(a)$. An inductive argument on the number of letters that occurs in $x \in \mathbf{x}^*$ shows that this extends to a derivation $\Phi(x) \xrightarrow{\mathbb{H}_x} \Psi(x)$ such that for every $x, y \in \mathbf{x}^*$, $\mathbb{H}_{xy} = \mathbb{H}_x\mathbb{H}_y$.

2 The property FDOT

Having defined the category $M(\mathbf{x}, \mathbf{r})$ associated with a presentation $(\mathbf{x}, \mathbf{r})$ of a monoid M, we define a new category $\mathcal{D}(\mathbf{x}, \mathbf{r})$. First we consider a graph $D(\mathbf{x}, \mathbf{r})$ with vertices of the morphisms of $M(\mathbf{x}, \mathbf{r})$, which as explained before, are equivalence classes of derivations. To define arrows, we consider the set of all parallel morphisms in $M(\mathbf{x}, \mathbf{r})$, that is morphisms with the same domain and codomain. Then for two vertices $\mathbb{F}, \mathbb{G} \in D(\mathbf{x}, \mathbf{r})$, there is an edge $\alpha : \mathbb{F} \to \mathbb{G}$ if $\mathbb{F}$ and $\mathbb{G}$ are parallel morphisms of $M(\mathbf{x}, \mathbf{r})$, that is with the same domain $d\mathbb{F} = d\mathbb{G}$ and the same range $r\mathbb{F} = r\mathbb{G}$, $\mathbb{F}$ and $\mathbb{G}$ decompose as $\mathbb{F} = \mathbb{F}_2 \circ \mathbb{T} \circ \mathbb{F}_1$, $\mathbb{G} = \mathbb{F}_2 \circ \mathbb{B} \circ \mathbb{F}_1$ where $\mathbb{T}$ and $\mathbb{B}$ are also parallel morphisms, and in that case α is by definition the quadruple $(\mathbb{F}_1, \mathbb{T}, \mathbb{B}, \mathbb{F}_2)$. If it happens that $\mathbb{F}_1$ and $\mathbb{F}_2$ are identity morphisms, then the quadruple $(id_{d\mathbb{T}}, \mathbb{T}, \mathbb{B}, id_{r\mathbb{T}})$ is written simply as the pair $(\mathbb{T}, \mathbb{B})$. The bi-action of $\mathbf{x}^*$ on $M(\mathbf{x}, \mathbf{r})$ extends to a bi-action on $D(\mathbf{x}, \mathbf{r})$, on vertices of $D(\mathbf{x}, \mathbf{r})$ in the obvious fashion, and on the set of edges of $D(\mathbf{x}, \mathbf{r})$ by setting for all $x, y \in \mathbf{x}^*$ and every edge $\alpha = (\mathbb{F}_1, \mathbb{T}, \mathbb{B}, \mathbb{F}_2)$, $x\alpha y = (x\mathbb{F}_1 y, x\mathbb{T}y, x\mathbb{B}y, x\mathbb{F}_2 y)$. Sometimes we call $x\alpha y$ a translate of α by x and y. In the case when $\mathbb{F}_1$ and $\mathbb{F}_2$ are identity morphisms and $\alpha = (\mathbb{T}, \mathbb{B})$, then $x\alpha y = (x\mathbb{T}y, x\mathbb{B}y)$. Further, the free category generated by $D(\mathbf{x}, \mathbf{r})$ is denoted by $\mathcal{D}(\mathbf{x}, \mathbf{r})$. The bi-action defined so far now extends to morphisms of $\mathcal{D}(\mathbf{x}, \mathbf{r})$ in a natural way because each such morphism is written uniquely as a concatenation of edges from $D(\mathbf{x}, \mathbf{r})$.

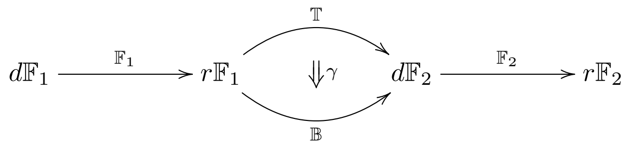
It is clear that there is always at least a morphism between two parallel morphisms in $M(\mathbf{x}, \mathbf{r})$ as long as we have added a morphism in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ connecting any two parallel morphisms in $M(\mathbf{x}, \mathbf{r})$. But the existence of all such morphisms in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ leaves the possibility open that there could be multiple morphisms of $\mathcal{D}(\mathbf{x}, \mathbf{r})$ in general connecting two parallel morphisms in $M(\mathbf{x}, \mathbf{r})$. To make this precise, assume that $\mathbb{F}$ and $\mathbb{G}$ decompose as $\mathbb{F} = \mathbb{F}_2 \circ \mathbb{T} \circ \mathbb{F}_1$, $\mathbb{G} = \mathbb{F}_2 \circ \mathbb{B} \circ \mathbb{F}_1$ with $\mathbb{T}$ and $\mathbb{B}$ being parallel morphisms, and let



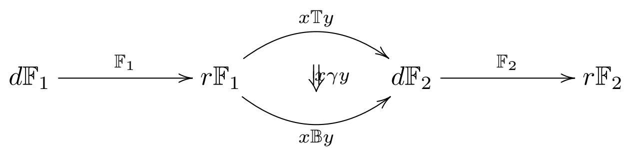
be a morphism connecting $\mathbb{T}$ with $\mathbb{B}$. A morphism that connects $\mathbb{F}$ with $\mathbb{G}$ now is

$$\delta : \mathbb{F}_2 \circ \mathbb{T} \circ \mathbb{F}_1 \Rightarrow \mathbb{F}_2 \circ \mathbb{B} \circ \mathbb{F}_1$$

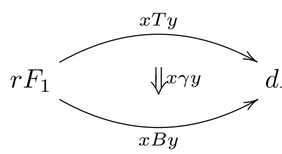
depicted as below



The property FDOT, as we will see in a while, has to do with the possibility of using only finitely many morphisms of $\mathcal{D}(\mathbf{x}, \mathbf{r})$ to connect any two objects in that category. If $\gamma : T \Rightarrow B$ is a morphism in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ of length one, that is γ is arising from an edge in $D(\mathbf{x}, \mathbf{r})$, $x, y \in \mathbf{x}^*$ any two words, and F_1, F_2 are two morphisms of $M(\mathbf{x}, \mathbf{r})$ such that $xTy \circ F_1$ and $F_2 \circ xTy$ exist, then the morphism



is called the whiskering of



by F_1 and F_2 and is written for short by $F_1 \circ x\gamma y \circ F_2$. The notion of whiskering and the related notions of derivation schemes and 2-categories can be found in [6]. For a set of morphisms U of length one in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ we form a subcategory denoted by $W(U)$ in the following fashion. First we define two sets of morphisms. The first set is the set of all whiskers of edges in $\mathbf{x}^*U\mathbf{x}^*$, where

$$\mathbf{x}^*U\mathbf{x}^* = \{x\gamma y : x, y \in \mathbf{x}^*, \gamma \in U\},$$

and the second one is the set of edges

$$\{(F^{-1} \circ F, id_x), (F \circ F^{-1}, id_y) : \forall \text{ derivation } x \xrightarrow{F} y\}.$$

Now we denote by $W(U)$ the subcategory of $\mathcal{D}(\mathbf{x}, \mathbf{r})$ generated by the above two sets. Before we give the definition of finite domination type for finitely presented monoids, we recall from [1] the following definition and a useful theorem. Let $\mathcal{C}$ be a subcategory of a small category $\mathcal{D}$. The dominion $Dom_{\mathcal{C}}\mathcal{D}$ of $\mathcal{C}$ in $\mathcal{D}$ is defined to be the set of all morphisms γ in $\mathcal{D}$ with the property that for every small category $\mathcal{E}$ and every pair of functors $\varphi, \psi : \mathcal{D} \to \mathcal{E}$ such that coincide in $\mathcal{C}$, then they should coincide in γ. It is proved in theorem 2.1 there that $\gamma \in Dom_{\mathcal{C}}\mathcal{D}$ if and only if we have the following factorizations

$$\begin{array}{rclcrcl} \gamma & = & c_1u_1 & , & u_1 & = & v_1d_1 \\ c_{i-1}v_{i-1} & = & c_iu_i & , & u_id_{i-1} & = & v_id_i \\ c_{m-1}v_{m-1} & = & u_m & , & u_md_{m-1} & = & \gamma \end{array}$$

where $(i = 2, ..., m-1)$ and $u_1, ..., u_m, v_1, ..., v_{m-1} \in \mathcal{C}$. These equations are known with the name zigzag of length m in $\mathcal{D}$ over $\mathcal{C}$ with value γ.

Definition 1. *We say that the monoid presentation $(\mathbf{x}, \mathbf{r})$ has finite domination type (FDOT) if there is a finite set of edges U in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ such that for any two parallel morphisms F and G in $M(\mathbf{x}, \mathbf{r})$, there is γ in the dominion of $W(U)$ which connects F with G.*

We will prove that FDOT is an intrinsic property of the monoid, in fact we prove that if it holds true for a particular finite presentation of a monoid, then it holds true for any other finite presentation of that monoid. Before we prove that, we need some preparatory work.

Lemma 1. *Let $\gamma \in \mathcal{D}(\mathbf{x}, \mathbf{r})$ be a morphism that belongs to $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ for some U. Then all its possible whiskers $F_1 \circ \gamma \circ F_2$ belong to $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ as well.*

Proof. We will prove the lemma for left whiskers $F \circ \gamma$. The general proof is analogous. From the zig-zag Isbell's theorem we have the following factorizations

$$\begin{array}{rclcrcl} \gamma & = & c_1 u_1 & , & u_1 & = & v_1 d_1 \\ c_{i-1} v_{i-1} & = & c_i u_i & , & u_i d_{i-1} & = & v_i d_i \\ c_{m-1} v_{m-1} & = & u_m & , & u_m d_{m-1} & = & \gamma \end{array}$$

where $(i = 2, ..., m-1)$ and $u_1, ..., u_m, v_1, ..., v_{m-1} \in W(U)$. Whiskering now on the left by F we obtain

$$\begin{array}{rcll} F \circ \gamma & = & (F \circ c_1)(F \circ u_1) & , \\ (F \circ c_{i-1})(F \circ v_{i-1}) & = & (F \circ c_i)(F \circ u_i) & , \\ (F \circ c_{m-1})(F \circ v_{m-1}) & = & F \circ u_m & , \end{array}$$

and

$$\begin{array}{rcl} F \circ u_1 & = & (F \circ v_1)(F \circ d_1) \\ (F \circ u_i)(F \circ d_{i-1}) & = & (F \circ v_i)(F \circ d_i) \\ (F \circ u_m)(F \circ d_{m-1}) & = & F \circ \gamma \end{array}$$

where from the definition of $W(U)$ all $F \circ u_1, ..., F \circ u_m, F \circ v_1, ..., F \circ v_{m-1} \in W(U)$. The zig-zag theorem implies that $F \circ \gamma \in Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ □

Lemma 2. *If for each atomic derivation $r \xrightarrow{A} s$ in $M(\mathbf{x}, \mathbf{r})$, there is a $\gamma \in \mathcal{D}(\mathbf{x}, \mathbf{r})$ connecting $H_s \circ \Phi(A)$ with $\Psi(A) \circ H_r$ that belongs to $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ for some U, then there is some δ connecting $H_y \circ \Phi(F)$ with $\Psi(F) \circ H_x$ that belongs to $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ for every derivation $x \xrightarrow{F} y$.*

Proof. We proceed by induction on the length of F. For derivations of length one the property is already true. Let split F of an ordinary length as $x \xrightarrow{A} x' \xrightarrow{F'} y$ where A is an atomic derivation. From hypothesis, there is a some $\delta' \in Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ connecting $H_y \circ \Phi(F')$ with $\Psi(F') \circ H_{x'}$. Lemma 1 implies that $\Phi(A) \circ \delta'$ is a morphism that connects $H_y \circ \Phi(F) = H_y \circ \Phi(F') \circ \Phi(A)$ with $\Psi(F') \circ H_{x'} \circ \Phi(A)$ and it belongs to $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$. Also from the hypothesis we have that there is some $\gamma \in Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ connecting $H_{x'} \circ \Phi(A)$ with $\Psi(A) \circ H_x$. Again Lemma 1 implies that $\gamma \circ \Psi(F')$ is in $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ and it connects $\Psi(F') \circ H_{x'} \circ \Phi(A)$ with $\Psi(F') \circ \Psi(A) \circ H_x = \Psi(F) \circ H_x$. As the end of $\Phi(A) \circ \delta'$ matches with the initial of $\gamma \circ \Psi(F')$, then the two morphisms can be composed to give one that again belongs to $Dom_{\mathcal{D}(\mathbf{x},\mathbf{r})}(W(U))$ and that connects $H_y \circ \Phi(F)$ with $\Psi(F) \circ H_x$. □

Before we prove our main theorem we will make a definition. Assume we are given two monoid presentation $(\mathbf{x}, \mathbf{r})$ and $(\mathbf{x}', \mathbf{r}')$. Let $M(\mathbf{x}, \mathbf{r})$ and $M(\mathbf{x}', \mathbf{r}')$ be the corresponding categories and $\Phi : M(\mathbf{x}, \mathbf{r}) \to M(\mathbf{x}', \mathbf{r}')$ a 2-morphism. This Φ induces a functor $\tilde{\Phi} : \mathcal{D}(\mathbf{x}, \mathbf{r}) \to \mathcal{D}(\mathbf{x}', \mathbf{r}')$. On objects $\tilde{\Phi}$ is defined by $\tilde{\Phi}(F) = \Phi(F)$ for every $F \in M(\mathbf{x}, \mathbf{r})$. To define $\tilde{\Phi}$ on morphisms, it is sufficient to define it on edges of the generating graph $D(\mathbf{x}, \mathbf{r})$. If (F_1, T, B, F_2) is such an edge, then by definition

$$\tilde{\Phi}(F_1, T, B, F_2) = (\Phi(F_1), \Phi(T), \Phi(B), \Phi(F_2)).$$

The freeness of $\mathcal{D}(\mathbf{x}, \mathbf{r})$ on $D(\mathbf{x}, \mathbf{r})$ allows to extend $\tilde{\Phi}$ on ordinary morphisms. With the notations just established we have the following.

Lemma 3. *If U is a set of edges in $D(\mathbf{x}, \mathbf{r})$, then $\tilde{\Phi}(W(U)) \subseteq W(\tilde{\Phi}(U))$.*

Proof. We will make the proof for those elements of $W(U)$ that are whiskers of an edge from U. So let $u = (F_1, T, B, F_2) \in U$, $xuy = (xF_1y, xTy, xBy, xF_2y)$ its translate by x and y, and $G_1 \circ xuy \circ G_2 = (G_1 \circ xF_1y, xTy, xBy, xF_2y \circ G_2)$ any whisker of xuy by G_1 and G_2. We see that

$$\tilde{\Phi}(G_1 \circ xuy \circ G_2) = \\ = (\tilde{\Phi}(G_1) \circ x\tilde{\Phi}(F_1)y, x\tilde{\Phi}(T)y, x\tilde{\Phi}(B)y, x\tilde{\Phi}(F_2)y \circ \tilde{\Phi}(G_2)),$$

where the right hand side belongs to $W(\tilde{\Phi}(U))$. □

Theorem 1. *Let $(\mathbf{x}, \mathbf{r})$ and $(\mathbf{x}', \mathbf{r}')$ be finite presentations of isomorphic monoids M and M'. Then $(\mathbf{x}, \mathbf{r})$ is of finite domination type if and only if $(\mathbf{x}', \mathbf{r}')$ is of finite domination type.*

Proof. By hypothesis there are two monoid morphisms $\varphi : M \to M'$ and $\varphi' : M' \to M$ such that $\varphi' \circ \varphi = id_M$ and $\varphi \circ \varphi' = id_{M'}$. As we explained earlier, φ is induced by a 2-morphism $\Phi : M(\mathbf{x}, \mathbf{r}) \to M(\mathbf{x}', \mathbf{r}')$ and φ' by a 2-morphism $\Phi' : M(\mathbf{x}', \mathbf{r}') \to M(\mathbf{x}, \mathbf{r})$. On the other hand, the identity morphism id_M is induced by the identity 2-morphism in $M(\mathbf{x}, \mathbf{r})$ and also by $\Phi' \circ \Phi$. From above there is a derivation $x \xrightarrow{H_x} \Phi'(\Phi(x))$ for every $x \in \mathbf{x}^*$. Assume now that $(\mathbf{x}', \mathbf{r}')$ is of finite domination type and let U' be the finite set of edges in $\mathcal{D}(\mathbf{x}', \mathbf{r}')$ satisfying the definition of FDOT for $(\mathbf{x}', \mathbf{r}')$. Now we define a set U of morphisms in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ as follows. For every $\gamma' : F \to G$ in U', as F and G are parallel morphisms, their images $\Phi'(F)$ and $\Phi'(G)$ are parallel as well. Consider the morphism $\tilde{\Phi}'(\gamma)$ from $\mathcal{D}(\mathbf{x}, \mathbf{r})$

where $\tilde{\Phi}'$ is the functor induced by Φ' as explained above, and let $\tilde{\Phi}'(U') = \{\tilde{\Phi}'(\gamma) : \gamma \in U'\}$. Further for every atomic derivation $r \xrightarrow{A} s$ we let δ_A be the morphism $(H_s \circ A, \Phi'(\Phi(A)) \circ H_r)$ in $\mathcal{D}(\mathbf{x}, \mathbf{r})$. Now define

$$U = \tilde{\Phi}'(U') \cup \{\delta_A : A \text{ is an atomic derivation in } M(\mathbf{x}, \mathbf{r})\}.$$

This is a finite set of edges in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ since U' is finite and $\mathbf{r}$ is finite. We will prove that U "generates" $W(U)$ such that for any two parallel morphisms F and G in $M(\mathbf{x}, \mathbf{r})$ as in the diagram $x \underset{G}{\overset{F}{\rightrightarrows}} y$, there is $\gamma \in Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$ that connects F with G. Since F and G are parallel, $\Phi(F)$ and $\Phi(G)$ are parallel in $M(\mathbf{x}', \mathbf{r}')$, therefore there is $\gamma' \in Dom_{W(U')}\mathcal{D}(\mathbf{x}', \mathbf{r}')$ which connects $\Phi(F)$ with $\Phi(G)$. From the zig-zag theorem of Isbell we have the following zig-zag

$$\begin{array}{rclcrcl} \gamma' & = & c_1 u_1 & , & u_1 & = & v_1 d_1 \\ c_{i-1} v_{i-1} & = & c_i u_i & , & u_i d_{i-1} & = & v_i d_i \\ c_{m-1} v_{m-1} & = & u_m & , & u_m d_{m-1} & = & \gamma' \end{array}$$

where $i = 2, ..., m-1$ and $u_1, ..., u_m, v_1, ..., v_{m-1} \in W(U')$. Applying $\tilde{\Phi}'$ on the zig-zag we obtain

$$\begin{array}{rcll} \Phi'(\gamma') & = & \Phi'(c_1)\Phi'(u_1) & , \\ \Phi'(c_{i-1})\Phi'(v_{i-1}) & = & \Phi'(c_i)\Phi'(u_i) & , \\ \Phi'(c_{m-1})\Phi'(v_{m-1}) & = & \Phi'(u_m) & , \end{array}$$

and

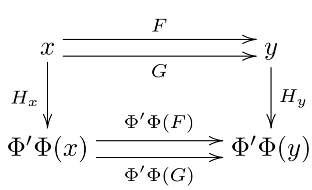
$$\begin{array}{rcll} \Phi'(u_1) & = & \Phi'(v_1)\Phi'(d_1) & \\ \Phi'(u_i)\Phi'(d_{i-1}) & = & \Phi'(v_i)\Phi'(d_i) & (i = 2, ..., m-1) \\ \Phi'(u_m)\Phi'(d_{m-1}) & = & \Phi'(\gamma') & \end{array}$$

where from lemma 3

$$\tilde{\Phi}'(u_1), ..., \tilde{\Phi}'(u_m), \tilde{\Phi}'(v_1), ..., \tilde{\Phi}'(v_{m-1}) \in W(\tilde{\Phi}'(U')) \subseteq W(U)$$

This proves that $\tilde{\Phi}'(\gamma')$ is an element of $Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$ which connects $\Phi'\Phi(F)$ with $\Phi'\Phi(G)$.

Consider the commutative diagram



From the definition of U above and from lemma 2 there is a morphism from $\gamma : H_y \circ F \to \Phi'\Phi(F) \circ H_x$ that belongs to $Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$. From lemma 1

$$H_y^{-1} \circ \gamma : H_y^{-1} \circ H_y \circ F \to H_y^{-1} \circ \Phi'\Phi(F) \circ H_x \quad (1)$$

belongs to $Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$. Further, from lemma 1 we get that

$$H_y^{-1} \circ \tilde{\Phi}'(\gamma') \circ H_x : H_y^{-1} \circ \Phi'\Phi(F) \circ H_x \to H_y^{-1} \circ \Phi'\Phi(G) \circ H_x \quad (2)$$

is a morphism in $Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$. The commutativity of the above diagram and lemmas 1 and 2 imply the existence of a morphism in $Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$

$$H_y^{-1} \circ \Phi'\Phi(G) \circ H_x \to H_y^{-1} \circ H_y \circ G. \quad (3)$$

Composing now morphisms in (1), (2) and (3) we obtain a morphism

$$H_y^{-1} \circ H_y \circ F \to H_y^{-1} \circ H_y \circ G \quad (4)$$

in $Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$. Finally, from the definition of $W(U)$, $(id_y, H_y^{-1} \circ H_y) \in W(U)$, hence $(F, H_y^{-1} \circ H_y \circ F) \in W(U)$. In the same way we see that there is a morphism $(H_y^{-1} \circ H_y \circ G, G) \in W(U)$. After composing the morphism in (4) with these two last morphisms we obtain a morphism in $Dom_{W(U)}\mathcal{D}(\mathbf{x}, \mathbf{r})$ which connects F with G as desired. □

Proposition 1. *If a monoid presentation* $(\mathbf{x}, \mathbf{r})$ *is FDT, then it is* $FDOT$.

Proof. Assume that $(\mathbf{x}, \mathbf{r})$ is of type FDT, then there is a finite homotopy base $\mathcal{B}$ with the property that for every pair of parallel paths (ρ_1, ρ_2) from $M(\mathbf{x}, \mathbf{r})$, ρ_1 can be transformed to ρ_2 by using homotopy moves of the forms $p \circ p^{-1} \to id_{\iota(p)}$, $id_{\iota(p)} \to p \circ p^{-1}$ where p is a path in $M(\mathbf{x}, \mathbf{r})$, or homotopy moves $u.\beta_1.v \to u.\beta_2.v$ or $u.\beta_2.v \to u.\beta_1.v$ where $(\beta_1, \beta_2) \in \mathcal{B}$ and $u, v \in \mathbf{x}^*$. If for every $(\beta_1, \beta_2) \in \mathcal{B}$ we denote by $u_{(\beta_1, \beta_2)} = (id_{\iota(\beta_1)}, \beta_1, \beta_2, id_{\tau(\beta_1)})$ the edge in $\mathcal{D}(\mathbf{x}, \mathbf{r})$ that connects β_1 with β_2, then the set $U = \{u_{(\beta_1, \beta_2)} : (\beta_1, \beta_2) \in \mathcal{B}\}$ has the property that any two objects of $\mathcal{D}(\mathbf{x}, \mathbf{r})$ can be connected by a morphism in $W(U) \subseteq Dom_{\mathcal{D}(\mathbf{x}, \mathbf{r})}(W(U))$ which in turn means that $(\mathbf{x}, \mathbf{r})$ is FDOT. □

Question 1. *Is there any example of a finite monoid presentation which is FDOT but not FDT?*

3 Conclusions and suggestions

In the hierarchy of the finiteness condition that appears so far in the literature, Squier's FDT and Kobayashi's finite domination type are distinct, but both have homological equivalents which are similar to each other. The first is equivalent with bi-FP_3 for groups, and the second is equivalent with bi-FP_1 for all monoids. There are no known invariants so far that stay in between the two. The purpose of this paper was to propose such an invariant. We were able to define a property of the monoid presentation called finite domination type for monoid presentations (FDOT) and proved that this is an intrinsic property of the monoid which generalizes FDT since it is proved that FDT implies FDOT. The Kobayashi's definition of finite domination type is similar to our FDOT but it is applicable for monoids rather than for monoid presentations as it is our FDOT. Comparing between the two, we see that the role of Kobayashi's M is taken in our definition by the category $\mathcal{D}(\mathbf{x}, \mathbf{r})$, and that of S is taken by the subcategory $W(U)$. We hope that considering FDOT might have an advantage since it involves data like generators and relations giving the monoid, and likewise FDT, the involvement of such data might enable one to relate FDOT with properties of the monoid of homological nature. The Kobayashi's finite domination type for the monoid M turns out to be equivalent with the homological property bi-FP_1 for M, but this condition is very restrictive, since one can find monoids of type bi-FP_1 which are not of type bi-FP_n for $n \geq 2$. Being a close relative to FDT, our finite domination type has a stronger homotopical flavor, and this gives points to the idea that it might be used to relate the monoid presentation with stronger finiteness conditions than the property bi-FP_1. Finally, we note that it would be interesting to explore two more things. Firstly, it is beneficial to look for any possible connection that FDOT might have with first order Dehn functions for monoids and groups. Many interesting results relating Dehn functions with FDT can be found in [7]. Secondly, it is well known that FDT and bi-FP_3 are equivalent for groups (see [4]) and it would be of interest to ask if this is the case with FDOT to. A positive answer of the above would imply that FDOT and FDT coincide in groups.

REFERENCES

[1] Isbell, J., R., ”Epimorphisms and dominions Ill”, Amer. J. Math. vol. 90, pp. 1025-1030, 1968. https://doi.org/10.2307/2373286

[2] Kobayashi, Y., ”The homological finiteness property FP_1 and finite generation of monoids”, Internat. J. Algebra Comput., vol. 17, no. 3, pp. 593-605, 2007. https://doi.org/10.1142/S0218196707003743

[3] Lafon, Y., ”A new finiteness condition for monoids presented by a complete rewriting system (after Craig C. Squier)”, JPAA, vol. 98, no. 3, pp. 229-244, 1995. https://doi.org/10.1016/0022-4049(94)00043-I

[4] Pride, S.J., *Low-dimensional homotopy theory for monoids II: Groups*, Glasgow Math. J., vol. 41, no. 1, pp. 1-11, 1999. https://doi.org/10.1017/S0017089599970179

[5] Squier, C.C., Otto, F., Kobayashi, Y., ”A finiteness condition for rewriting systems”, Theoret. Comput. Sci. vol. 131, no. 2, pp. 271-294, 1994. https://doi.org/10.1016/0304-3975(94)90175-9

[6] Street, R., ”Categorical Structures”, Handbook of Algebra, vol. 1, Elsevier Sciences, 1996, pp. 529-577, https://doi.org/10.1016/S1570-7954(96)80019-2

[7] Wang, X., and Pride, S.J., ”Second order Dehn functions of groups and monoids”, Int. Journal of Algebra and Computation, vol. 10, no. 4, pp. 425-456, 2000. https://doi.org/10.1142/S0218196700000200