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Numerical Solution of the Two-Dimensional Elasticity Problem in Strains

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Abstract Usually, the boundary value problems of the theory of elasticity are formulated with respect to displacements, and are reduced to the well-known Lame equations. Strains and stresses can be calculated from displacements as a solution to Lame's equation. Also known are the Beltrami Mitchell equations, which make it possible to formulate the boundary value problem of the theory of elasticity with respect to stresses. Currently, the boundary value problems of the theory of elasticity in stresses are studied in more detail in the two-dimensional case, and usually solved numerically with the introduction of the Airy stress function. But, the direct solution of boundary value problems of elasticity theory with respect to stresses requires further researches. This work, similarly to the boundary value problem in stresses, is devoted to the formulation and numerical solution of boundary value problems of the theory of elasticity with respect to deformations. The proposed boundary value problem consists of six Beltrami-Mitchell-type equations depending on strains and three equations of the equilibrium equation expressed with respect to deformations. As boundary conditions, in addition to the usual conditions for surface forces, three additional conditions are also introduced based on the equilibrium equations. The boundary value problem is considered in detail for a rectangular area. The discrete analogue of the boundary value problem is composed by the finite difference method. The convergence of difference schemes and an iterative method for their solution are studied. Software has been developed in the C++ environment for solving boundary value problems in the theory of elasticity and deformation. A number of boundary value problems on the deformation of a rectangular plate are solved numerically under various boundary conditions. The reliability of the obtained results is substantiated by comparing the numerical results, with the exact solution, as well as with the known solutions of the plate tension problems with parabolic and uniformly distributed edge loads.

Keywords Compatibility Equation, Strains, Stresses, Equilibrium Equations, Hooke's Law, Boundary Value Problem, Deformation, Difference Schemes, Iterative Method

1. Introduction

It is known that when modeling the process of deformation of solid bodies, the main parameters are the displacement vector, and strain and stress tensors. Usually, in solid mechanics, boundary value problems describing the process of deformation of solid bodies are formulated mainly with respect to displacements in the form of well-known Lame equations. Deformations and stresses can be calculated from known displacements. Determination the safety margins and reliability of structures is also associated with the calculation of strains and stresses. Therefore, the formulation of boundary value problems of the theory of elasticity with respect to stresses and strains is an actual problem in solid mechanics.

It is known that boundary value problems with respect to stresses in solid mechanics are reduced to the well-known Beltrami-Mitchell equations [14]. The boundary value problem in stresses consists of six Beltrami-Mitchell equations and three equilibrium equations, as well as ordinary three surfaces and three additional boundary conditions. Additional boundary conditions, following the work of Pobedrya [7], are obtained by considering the equilibrium equation on the boundary of a given domain.

Usually, the boundary value problem in stresses, in the two-dimensional case, is solved by introducing the Air's stress function that satisfies the equilibrium equation and is reduced to solving the biharmonic equation [17]. The boundary value problem in stresses, in the three-dimensional case, was solved in the work of Filonenko-Borodich [12] by a variation method based on the expansion of the stress tensor in a series with respect to trigonometric functions, etc. [11,13]. In [1], the Beltrami Mitchell equations were solved using the integral Fourier transform. In [3,10], for solving three-dimensional problems in stresses, Maxwell-type stress functions were used. In the works of Pobedrya [7], a new formulation of the boundary value problem in stresses is proposed, from which, in a particular case follows the classical Beltrami-Mitchell equations. The works [2,13] are also devoted to the study of the Pobedrya's new boundary value problem in stresses. The issues of existence and uniqueness of the solution in stresses, and the equivalence of new and classical formulations are considered in the following works [9,21]. Dynamic boundary value problems with respect to stresses are considered in the works of Konovalov [18]. Despite the progress made, the solution of boundary value problems in stresses is far from completion.

Boundary value problems of solid mechanics as well can be formulated with respect to strains. For that, the Saint-Venan compatibility conditions using the equilibrium equation and Hooke's law can be written as a six Beltrami-Michel-type equations with respect to strains. The received strain equations, in combination with the equilibrium equations and with a usual surface and additional boundary conditions may be considered as a boundary value problem in strains. In this case, as additional boundary conditions, one can consider the equilibrium equation expressed with respect to deformations at the boundary of a given region. Formulation of the boundary value problem in strains is also considered in the work of Pobedrya [15]. Research on the formulation and solution of boundary value problems of solid mechanics with respect to deformations is still at the beginning of its development.

This work is devoted to the formulation of boundary value problems of the theory of elasticity with respect to strains, and their numerical solution. The work consists of six sections.

In the second section, boundary value problems of the theory of elasticity in strains are formulated. The boundary value problem in strains consists of six differential equations with respect to strains, and three equilibrium equations expressed in terms of strains, with usual surface and additional boundary conditions. To ensure the correctness of boundary value problems, the equilibrium equations are considered as additional boundary conditions.

In the third section, a two dimensional boundary value problem regarding the strains in a rectangular domain is considered. It is shown that the 2D boundary problem consists of one strain compatibility equation and two equilibrium equations expressed with respect to strains with appropriate surface and additional boundary conditions.

In the fourth section, finite-difference equations for two-dimensional boundary value problem in strains are constructed. Grid equations are written for internal and boundary nodal points in a convenient form for applying the iterative method.

The fifth section is devoted to the numerical solution of the plane boundary value problems of the theory of elasticity with respect to strains. A number of elasticity problems on the equilibrium of a rectangular plate under the action of various loads are numerically solved. The reliability of the results is ensured by comparing the numerical results of the boundary value problems in strains with the exact solution, as well as with the known solutions of the plate tension problem under parabolic and uniformly distributed loads.

2. Boundary Value Problem of the Theory of Elasticity in Strains

Similar to the Beltrami-Mitchell equations, the compatibility conditions can be written in the form of differential equations with respect to strains. That is why we consider the compatibility condition [14]

$$\nabla^2\varepsilon_{ij} + \theta,_{ij} - \varepsilon_{ik,kj} - \varepsilon_{jk,ki} = 0 \tag{2.1}$$

which, taking into account the following relation

$$\varepsilon_{ij,j} = -\frac{\lambda}{2\mu}\theta_{,i} - \frac{1}{2\mu}X_{i,j} \tag{2.2}$$

obtained from the equilibrium equation

$$\sigma_{ij,j} + X_i = 0 \tag{2.3}$$

and Hooke's law

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} \;, \tag{2.4}$$

can be written as a system of six differential equations with respect to the strain tensor [14]

$$\mu\nabla^2\varepsilon_{ij}+(\lambda+\mu)\theta,_{ij}+\frac{1}{2}(X_{i,j}+X_{j,i})=0 \quad (2.5)$$

Adding to (2.5) three equilibrium equations expressed in terms of deformations

$$\lambda\theta_{,i}+2\mu\varepsilon_{ij,j}+X_i=0 \quad (2.6)$$

and boundary conditions

$$(\lambda\theta\delta_{ij}+2\mu\varepsilon_{ij})n_j\,|_{\Sigma_2}=S_i \quad (2.7)$$

we obtain the boundary value problem of the theory of elasticity in strains. For to complete the system of differential equations, following [7] we add to them as the missing boundary conditions, three equilibrium equations expressed with respect to deformations, i.e.

$$(\lambda\theta_{,i}+2\mu\varepsilon_{ij,j}+X_i)\big|_{\Sigma}=0 \quad (2.8)$$

Thus, equations (2.5-2.8) represent the boundary value problem of the theory of elasticity in strains, where ∇^2- Laplace operator, $\sigma_{ij}-$ stress tensor, $\varepsilon_{ij}-$ strain tensor, λ,μ elastic Lame constants,eij,θ- the spherical part of the strain tensor, S_i- surface load, Xi body forces, δij Kronecker symbol.

Thus, the boundary value problem (2.5-2.8) consists of nine partial differential equations with six boundary conditions. Note that, it is sufficient to consider three of the six differential equations (2.5).

3. Plane Strain Case

Equation (2.5), in the case of plane deformation i.e. at $\varepsilon_{33}=\varepsilon_{31}=\varepsilon_{32}=0,$ and in the absence of body forces has the form

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{11}}{\partial x^2}+\mu\frac{\partial^2\varepsilon_{11}}{\partial y^2}+(\lambda+\mu)\frac{\partial^2\varepsilon_{22}}{\partial x^2}=0 \quad (3.1)$$

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{22}}{\partial y^2}+\mu\frac{\partial^2\varepsilon_{22}}{\partial x^2}+(\lambda+\mu)\frac{\partial^2\varepsilon_{11}}{\partial y^2}=0 \quad (3.2)$$

$$\mu(\frac{\partial^2\varepsilon_{12}}{\partial x^2}+\frac{\partial^2\varepsilon_{12}}{\partial y^2})+(\lambda+\mu)(\frac{\partial^2\varepsilon_{11}}{\partial x\partial y}+\frac{\partial^2\varepsilon_{22}}{\partial x\partial y})=0 \quad (3.3)$$

Using 2D equilibrium equations expressed relative to deformations

$$(\lambda+2\mu)\frac{\partial\varepsilon_{11}}{\partial x}+\lambda\frac{\partial\varepsilon_{22}}{\partial x}+2\mu\frac{\partial\varepsilon_{12}}{\partial y}=0 \quad (3.4)$$

$$(\lambda+2\mu)\frac{\partial\varepsilon_{22}}{\partial y}+\lambda\frac{\partial\varepsilon_{11}}{\partial y}+2\mu\frac{\partial^2\varepsilon_{12}}{\partial x}=0 \quad (3.5)$$

can be shown that equations (3.1) and (3.2) are equivalent to the well-known strain compatibility equation [14]

$$\frac{\partial^2\varepsilon_{11}}{\partial y^2}+\frac{\partial^2\varepsilon_{22}}{\partial x^2}=2\frac{\partial^2\varepsilon_{12}}{\partial x\partial y} \quad (3.6)$$

Differentiating the equilibrium equations (3.4) and (3.5) with respect to x and y, we find, respectively, i.e.

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{11}}{\partial x^2}+\lambda\frac{\partial^2\varepsilon_{22}}{\partial x^2}=-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y} \quad (3.7)$$

$$(\lambda+2\mu)\frac{\partial^2\varepsilon_{22}}{\partial y^2}+\lambda\frac{\partial^2\varepsilon_{11}}{\partial y^2}=-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y} \quad (3.8)$$

Taking into account relations (3.7-3.8), equations (3.1) and (3.2) can be reduced to the form

$$-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}+\mu\frac{\partial^2\varepsilon_{11}}{\partial y^2}+\mu\frac{\partial^2\varepsilon_{22}}{\partial x^2}=0 \quad (3.9)$$

$$-2\mu\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}+\mu\frac{\partial^2\varepsilon_{22}}{\partial x^2}+\mu\frac{\partial^2\varepsilon_{11}}{\partial y^2}=0 \quad (3.10)$$

Adding these two equations, we can find the well-known compatibility condition (3.6). This ensures the equivalence of the equations (3.1-3.2) and (3.6).

On the other hand, adding equations (3.7) and (3.8) i.e.

$$2\frac{\partial^2\varepsilon_{12}}{\partial x\partial y}=-(1+\frac{\lambda}{2\mu})(\frac{\partial^2\varepsilon_{11}}{\partial x^2}+\frac{\partial^2\varepsilon_{22}}{\partial y^2})-\frac{\lambda}{2\mu}(\frac{\partial^2\varepsilon_{11}}{\partial y^2}+\frac{\partial^2\varepsilon_{22}}{\partial x^2}) \quad (3.11)$$

and, substituting the resulting expression into (3.6), the compatibility condition (3.6) can be written as the following harmonic equation

$$\nabla^2(\varepsilon_{11}+\varepsilon_{22})=0 \quad (3.12)$$

The harmonic equation (3.12) in combination with the equilibrium equations (3.4-3.5) with appropriate boundary conditions can be considered as a boundary value problem of the theory of elasticity with respect to deformations. Note that a similar boundary value problem for stresses is well known in the theory of elasticity and is usually solved by introducing the Airy stress function [17].

We will consider equation (3.3) together with two equilibrium equations as a system of three differential equations with respect to deformations, i.e.

$$\begin{aligned}&(\lambda+2\mu)\frac{\partial\varepsilon_{11}}{\partial x}+\lambda\frac{\partial\varepsilon_{22}}{\partial x}+2\mu\frac{\partial\varepsilon_{12}}{\partial y}=0\\&(\lambda+2\mu)\frac{\partial\varepsilon_{22}}{\partial y}+\lambda\frac{\partial\varepsilon_{11}}{\partial y}+2\mu\frac{\partial\varepsilon_{21}}{\partial x}=0\\&\mu(\frac{\partial^2\varepsilon_{12}}{\partial x^2}+\frac{\partial^2\varepsilon_{12}}{\partial y^2})+(\lambda+\mu)(\frac{\partial^2\varepsilon_{11}}{\partial x\partial y}+\frac{\partial^2\varepsilon_{22}}{\partial x\partial y})=0\end{aligned} \quad (3.13)$$

with surface boundary conditions (2.7)

$$\begin{aligned}&\left[(\lambda(\varepsilon_{11}+\varepsilon_{22})+2\mu\varepsilon_{11})n_1+2\mu\varepsilon_{12}n_2\right]|_{\Gamma}=S_1\\&\left[(2\mu\varepsilon_{12}n_1+(\lambda(\varepsilon_{11}+\varepsilon_{22})+2\mu\varepsilon_{22}n_2)\right]|_{\Gamma}=S_2\end{aligned} \quad (3.14)$$

In the boundary value problem (3.13-3.14), the number of boundary conditions is less than the number of unknowns. Following [7], as the missing boundary condition, we consider the equilibrium equations (3.4) and (3.5) on the boundary Γ of the given domain, i.e.

$$\left[(\lambda+2\mu)\frac{\partial\varepsilon_{11}}{\partial x}+\lambda\frac{\partial\varepsilon_{22}}{\partial x}+2\mu\frac{\partial\varepsilon_{12}}{\partial y}\right]\Big|_{\Gamma}=0$$
$$\left[(\lambda+2\mu)\frac{\partial\varepsilon_{22}}{\partial y}+\lambda\frac{\partial\varepsilon_{11}}{\partial y}+2\mu\frac{\partial\varepsilon_{21}}{\partial x}\right]\Big|_{\Gamma}=0 \quad (3.15)$$

and, we will call them additional boundary conditions. Thus, equations (3.13-3.15) represent a plane boundary value problem of the theory of elasticity in strains. Similar equations were written for the boundary value problem in stresses [6].

Now, we consider the boundary conditions in more detail with respect to a rectangular area. Let the rectangle be under the action of tensile forces from both sides along the *OX* axis, the other sides are free from loads (Fig. 1) i.e.

$$x=\pm a:\quad \sigma_{11}\big|_{x=\pm a}=S(y),\ \ \sigma_{12}\big|_{x=\pm a_1}=0$$
$$y=\pm b:\quad \sigma_{22}\big|_{y=\pm b}=0,\ \ \sigma_{21}\big|_{y=\pm b}=0 \quad (3.16)$$

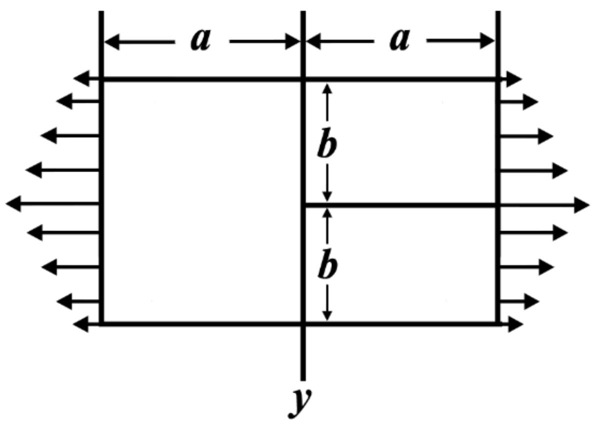

Figure 1. Stretching of a rectangular plate under the action of a parabolic force

Boundary conditions (3.16), using Hooke's law, can be written with respect to deformations:

$$\varepsilon_{11}\big|_{x=\pm a}=\frac{1}{E}\sigma_{11},\ \ \varepsilon_{12}\big|_{x=\pm a}=0,$$
$$\varepsilon_{22}\big|_{y=\pm b}=0,\qquad \varepsilon_{12}\big|_{y=\pm b}=0 \quad (3.17)$$

Then the additional boundary conditions (3.15) with respect to the rectangular domain have the form

$$\left[\frac{\partial\varepsilon_{22}}{\partial x}\right]\Big|_{x=\pm a}=-\left[\frac{2\mu}{\lambda}\frac{\partial\varepsilon_{12}}{\partial y}\right]\Big|_{x=\pm a}$$
$$\left[\frac{\partial\varepsilon_{11}}{\partial y}\right]\Big|_{y=\pm b}=-\left[\frac{2\mu}{\lambda}\frac{\partial\varepsilon_{21}}{\partial x}\right]\Big|_{y=\pm b} \quad (3.18)$$

Thus, according to the boundary conditions (3.17) and (3.18), each side of the rectangle has two boundary conditions and one additional boundary condition. For example, when $x=a$:

$$\varepsilon_{11}\big|_{x=a}=\frac{1}{E}\sigma_{11},\ \ \varepsilon_{12}\big|_{x=a}=0,$$

and

$$\left[\frac{\partial\varepsilon_{22}}{\partial x}\right]\Big|_{x=a}=-\left[\frac{2\mu}{\lambda}\frac{\partial\varepsilon_{12}}{\partial y}\right]\Big|_{x=a} \quad (3.19)$$

4. Finite Difference Equations

We consider the boundary value problem (3.13-3.15) in a rectangle $0\le x\le l_1,\ \ 0\le y\le l_2$ and draw two families of parallel lines $x_i=ih_1\ (i=\overline{0,n})$, $y_j=jh_2\ (j=\overline{0,n})$, where $h_k=l_k/N_k, k=1,2.$ In equation (3.13) replacing the derivatives with the corresponding difference ratios, we obtain the following discrete equations

$$2\varepsilon^{11}_{i,j}=\varepsilon^{11}_{i+1,j}+\varepsilon^{11}_{i-1,j} \quad (4.1)$$

$$2\varepsilon^{22}_{i,j}=\varepsilon^{22}_{i,j+1}+\varepsilon^{22}_{i-1,j} \quad (4.2)$$

$$\frac{\varepsilon^{12}_{i+1,j}-2\varepsilon^{12}_{i,j}+\varepsilon^{12}_{i-1,j}}{h_1^2}+K\frac{\varepsilon^{11}_{i+1,j+1}-\varepsilon^{11}_{i+1,j-1}-\varepsilon^{11}_{i-1,j+1}+\varepsilon^{11}_{i+1,j+1}}{4h_1h_2}+$$
$$+\frac{\varepsilon^{12}_{i,j+1}-2\varepsilon^{12}_{i,j}+\varepsilon^{12}_{i,j-1}}{h_2^2}+K\frac{\varepsilon^{22}_{i+1,j+1}-\varepsilon^{22}_{i+1,j-1}-\varepsilon^{22}_{i-1,j+1}+\varepsilon^{22}_{i+1,j+1}}{4h_1h_2}=0 \quad (4.3)$$

Note that the difference equations (4.1-4.2) are found by adding two difference schemes obtained by replacing the derivatives in the first term of each equilibrium equation by the right and left difference ratios, respectively. Note that the finite-difference schemes of boundary value problems are symmetric. It is easy to see that the order of approximation of difference equations (4.1-4.3) $O(h_1^2,h_2^2)$. Boundary conditions (3.17) with respect to the nodal points are satisfied exactly, i.e.

$$\varepsilon^{11}_{0j}=\frac{1}{E}S_j,\ \varepsilon^{11}_{n_1j}=\frac{1}{E}S_j,\ \varepsilon^{12}_{0j}=0,\ \varepsilon^{12}_{n_1j}=0,$$
$$\varepsilon^{22}_{i0}=0,\quad \varepsilon^{22}_{in_2}=0,\quad \varepsilon^{12}_{i0}=0,\quad \varepsilon^{12}_{in_2}=0. \quad (4.4)$$

Additional boundary conditions (3.18) depend on the first derivatives, therefore it has the first order of approximation. Then the general order of approximation of difference equations and boundary conditions will be $O(h_1,h_2)$. By the convergence theorem for difference schemes, the stability of a difference scheme is proved [22]. We confine ourselves to ensuring the convergence of the iterative method for solving difference equations.

By solving the finite difference equations (4.1), (4.2) and (4.3) with respect to the sought values ε^{11}_{ij}, ε^{22}_{ij} and ε^{12}_{ij}, the following recurrent type relations can be found, solved by the iterative method [15,16]:

$$\varepsilon^{(k+1)}_{11\,i,j} = \frac{\varepsilon^{(k)}_{11\,i+1,j} + \varepsilon^{(k)}_{11\,i-1,j}}{2}$$
$$\varepsilon^{(k+1)}_{22\,i,j} = \frac{\varepsilon^{(k)}_{22\,i,j+1} + \varepsilon^{(k)}_{22\,i-1,j}}{2} \quad (4.5)$$

$$\varepsilon^{(k+1)}_{12\,i,j} = (K\frac{\varepsilon^{(k)}_{11\,i+1,j+1} - \varepsilon^{(k)}_{11\,i+1,j-1} - \varepsilon^{(k)}_{11\,i-1,j+1} + \varepsilon^{(k)}_{11\,i+1,j+1}}{4h_1h_2} +$$
$$+\frac{\varepsilon^{(k)}_{12\,i+1,j} + \varepsilon^{(k)}_{12\,i-1,j}}{h_1^2} + +\frac{\varepsilon^{(k)}_{12\,i,j+1} + \varepsilon^{(k)}_{12\,i,j-1}}{h_2^2} + \quad (4.6)$$
$$+K\frac{\varepsilon^{(k)}_{22\,i+1,j+1} - \varepsilon^{(k)}_{22\,i+1,j-1} - \varepsilon^{(k)}_{22\,i-1,j+1} + \varepsilon^{(k)}_{22\,i+1,j+1}}{4h_1h_2}) / (\frac{2}{h_1^2} + \frac{2}{h_2^2})$$

where $K = 1 + \frac{\lambda}{\mu}$, $k-$ number of iterations. Following (3.17), the boundary conditions have the form:
for *x=0* and $x=l_1$

$$\varepsilon^{(0)}_{11\ 0j} = \frac{1}{E}\sigma_{11},\ \varepsilon^{(0)}_{12\ 0j} = 0,$$
$$\varepsilon^{(0)}_{11\ N_1j} = \frac{1}{E}\sigma_{11},\ \varepsilon^{(0)}_{12\ N_1j} = 0 \quad (4.7)$$

for *y=0* and $y=l_2$

$$\varepsilon^{(0)}_{22\ i0} = 0, \qquad \varepsilon^{(0)}_{12\ i0} = 0,$$
$$\varepsilon^{(0)}_{22\ iN_2} = 0, \qquad \varepsilon^{(0)}_{12\ iN_2} = 0 \quad (4.8)$$

According to relations (3.18), additional boundary conditions have the form:
for *y=0* and $y=l_2$

$$\varepsilon^{(0)}_{11\ i0} = \varepsilon^{(0)}_{11\ i1} + \frac{\mu h_2}{\lambda}\frac{\varepsilon^{(0)}_{12\ i+1,0} - \varepsilon^{(0)}_{12\ i-1,0}}{h_1}$$
$$\varepsilon^{(0)}_{11\ iN_2} = \varepsilon^{(0)}_{11\ i,N_2-1} - \frac{\mu h_2}{\lambda}\frac{\varepsilon^{(0)}_{12\ i+1,N_2} - \varepsilon^{(0)}_{12\ i-1,N_2}}{h_1} \quad (4.9)$$

for *x=0* and $x=l_1$

$$\varepsilon^{(0)}_{22\ 0,j} = \varepsilon^{(0)}_{22\ 1,j} + \frac{\mu h_1}{\lambda}\frac{\varepsilon^{12}_{0,j+1} - \varepsilon^{12}_{0,j-1}}{h_2}$$
$$\varepsilon^{(0)}_{22\ N_1,j} = \varepsilon^{(0)}_{22\ N_1-1,j} - \frac{\mu h_1}{\lambda}\frac{\varepsilon^{(0)}_{12\ N_1,j+1} - \varepsilon^{(0)}_{12\ N_1,j-1}}{h_2} \quad (4.10)$$

It is easy to check that the following inequalities hold for the coefficients of the difference equations (4.5–4.6) [16]

$$\sum_{\substack{i=1,\\ i\neq j}}^{n} \frac{|a_{ij}|}{|a_{ii}|} \le 1 \quad (4.11)$$

The last inequality is the diagonal dominance condition for the coefficients for each equation. The fulfillment of inequality (4.11) ensures the convergence of the iterative process (4.5–4.10). Note that inequality (4.11) is a consequence of the general convergence theorem for iterative methods [15,16].

5. Numerical examples

This section is devoted to justifying the validity of the formulated boundary value problem (BVP) in strains (3.13-3.15) and the proposed numerical method for solving finite-difference equations (4.4-4.9). Note that the so-called additional boundary conditions (4.8-4.9) are, in fact, equations for the required quantities at the boundary nodal points. Thus, the equations defined at internal points (4.4-4.5) and on the boundary (4.8-4.9) of a given area, as a system of equations, are solved by the iterative method. In the initial approximation, the values of the unknown quantities are considered to be zero. On the basis of the above technique using the recurrent relations (4.5-4.10) software developed in C++ environment, and a number of two-dimensional strain problems for a rectangular region with different boundary conditions are solved.

5.1. Rectangular Plate under Uniaxial Parabolically Distributed Loads

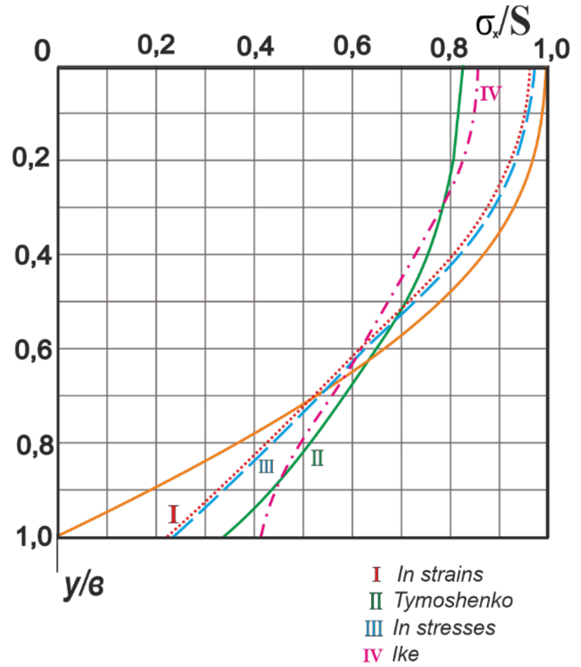

Figure 2. Distribution of σ_{11} over the cross-section *x=0* of a rectangular plate according to BVP in stresses (III) and strains (I), as well as according to [18] (II)

As an example, consider the problem of stretching a rectangular plate under the action of a parabolic force applied on opposite sides, considered in the work of Timoshenko and Goodier [17]. In this case, the boundary conditions have the form

for $x = \pm a$: $\sigma_{11} = S_0(1 - \frac{y^2}{a^2}), \quad \sigma_{12} = 0$ (5.1) for $y = \pm b: \quad \sigma_{22} = 0, \qquad \sigma_{12} = 0 \qquad (5.2)$

Taking into account relations (5.1) and (5.2), the boundary conditions with respect to deformations (4.6) can be found. Stress values σ_{11} calculated from the results of finite-difference equations (4.4-4.9) according to Hooke's law, as well as results obtained by other researchers are shown in Table 1, where *k* - is the number of iterations. At the same time, the initial data for all problems solved in this work had the following values: $\lambda = 0.78,\ \mu = 0.5,\ a = 1,\ b = 1,\ h_1 = h_2 = 0.2.$

Fig.2 shows the distribution of σ_{11} over the cross section *x=0* according to the results of the boundary value problem formulated with respect to strains(I) and stresses (III) as well as according to Timoshenko-Goodier [18] (II). The yellow solid curve represents part of the parabolic load (5.1) applied on the side $x = \pm a$.

5.2. Rectangular Plate under Uniformly Distributed Loads

Similarly, to the previous problem, the problem of stretching a rectangular plate with a uniformly distributed load applied on opposite sides of the rectangular, i.e.

$$\begin{aligned} x = \pm a: &\quad \sigma_{11} = S_0 = 1, \quad \sigma_{12} = 0 \\ y = \pm b: &\quad \sigma_{22} = 0, \qquad \sigma_{12} = 0 \end{aligned} \qquad (5.2)$$

Table 2 shows the numerical results obtained by solving the equations (3.13-3.15) under boundary conditions (5.2), as well as the results of the boundary value problem in stresses, based on the Beltrami-Mitchell equations. In this case, the initial data had the following values: $\lambda = 0.78,\ \mu = 0.5,\ a = 0.5,\ b = 0.5,\ h_1 = h_2 = 0.1$.

5.3. Comparison of Exact and Numerical Solutions

Let a plane boundary value problem in deformations (3.13-3.15) be considered in a rectangle $l_1 \times l_2$. Consider the functions

$$\begin{aligned} \varepsilon_{11} &= y(y - l_2) \\ \varepsilon_{22} &= x(x - l_1) \\ \varepsilon_{12} &= xy(x - l_1)(y - l_2) \end{aligned} \qquad (5.3)$$

satisfying equations (3.13), with the following right-hand sides (body forces):

$$\begin{aligned} X_1 &= -(x^2 - xl_1)(2y - l_2) \\ X_2 &= -(2x - l_1)(y^2 - yl_2) \\ X_{12} &= \frac{1}{2\mu}\left(\frac{\partial X_1}{\partial y} + \frac{\partial X_2}{\partial x}\right) \end{aligned} \qquad (5.4)$$

According to (5.3), the boundary and additional boundary conditions have the form

$$\begin{aligned} x = 0,\ l_1: &\quad \varepsilon_{11} = y(y - l_2), \quad \varepsilon_{12} = 0, \qquad \varepsilon_{22} = 0, \\ y = 0,\ l_2: &\quad \varepsilon_{22} = x(x - l_1), \quad \varepsilon_{12} = 0, \qquad \varepsilon_{11} = 0. \end{aligned} \qquad (5.5)$$

The first row of Table 3 shows the values of the strain tensor component ε_{11} found by equations (4.4-4.9) taking into account the boundary conditions (5.5). The second line is calculated according to the exact solution (5.3). The strain values ε_{11} given in the third line are calculated from the results of the boundary value problem in stresses, according to Hooke's law. Initial data as in the first task.

Table 1. Comparison of results with other results

BVP results	y=0	y=0.2	y=0.4	y=0.6	y=0.8	y=1
In strains (k=70)	0.2161	0.4351	0.6153	0.8076	0.9229	0.9614
Timoshenko [21]	0.3404	0.5166	0.6536	0.7515	0.8102	0.8298
In stresses [10] (k=76)	0.2297	0.4493	0.6217	0.8160	0.9326	0.9714
Ike [6]	0.4172	0.4961	0.6206	0.7334	0.8306	0.8619

Table 2. Comparison of results obtained in strains and stresses

BVP	y=0	y=0.1	y=0.2	y=0.3	y=0.4	y=0.5
In strains (k=85)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
In stresses [10] (k=73)	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 3. Comparison of results with the exact solution

BVP	x=0	x=0.1	x=0.2	x=0.3	x=0.4	x=0.5
In strains (k=74)	-0.2491	-0.2493	-0.2493	-0.2495	-0.2496	-0.2500
Exact	-0.2500	-0.2500	-0.2500	-0.2500	-0.2500	-0.2500
In stresses [10] (k=68)	-0.2490	-0.2490	-0.2491	-0.2493	-0.2494	-0.2496

6. Conclusions

The following conclusion can be made from this study:

I. The compatibility conditions, using the equilibrium equation and Hooke's law, are written in the form of differential equations with respect to strains.

II. Formulated elasticity boundary value problem in strains which consist of strain equations and equilibrium equations with surface and "additional" boundary conditions.

III. As "additional" boundary conditions, considered the equilibrium equations on the boundary of a given domain.

IV. In the two-dimensional case, the strain equations consist of three equations and allow us to formulate two variants of boundary value problems. It is shown that the first two strain equations are equivalent to the harmonic equation, which together with the two equilibrium equations, constitute the first boundary value problem. The second boundary value problem consists of two equilibrium equations and third strain equation.

V. Grid equations for second boundary value problem are constructed by the finite-difference method separately for internal and boundary nodal points. The convergence of difference schemes and the iterative method is studied. Software has been developed in the C++ environment for solving boundary value problems in the theory of elasticity under deformations.

VI. The problem of stretching a rectangular plate under the action of a parabolic load applied on opposite sides is solved and a good agreement with the well-known Timoshenko and Goodier results is obtained.

VII. The problem of compression of a rectangular plate under the action of a uniformly distributed load on opposite sides has been solved.

VIII. The coincidence of the numerical results with the exact solution of the problem obtained by the semi-inverse method for a rectangular plate is shown.

IX. The coincidence of the numerical results with the exact solution of the problem obtained by the semi-inverse method for a rectangular plate is shown.

X. Methods for setting boundary value problems in the theory of elasticity under deformations can be generalized for plastic, thermoplastic, and viscoplastic problems.

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