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Fuzzy Norm on Fuzzy *n*-Normed Space

Mashadi[1], Abdul Hadi[1], Sukono[2,*]

[1]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Riau University, Indonesia
[2]Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjajaran, Indonesia

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Abstract In various articles, fuzzy *n*-normed space concept for $n \geq 2$ is constructed from fuzzy normed space which uses intuitionistic approach or *t*-norm approach concept. However, fuzzy normed space can be approached using fuzzy point too. This paper shows that fuzzy *n*-normed space for $n \geq 2$ can be constructed from fuzzy normed space using fuzzy point approach of fuzzy set. Furthermore, for $m = 1, 2, \ldots, n-1$, it is also discussed how to construct fuzzy (*n*-*m*)-normed space from fuzzy *n*-normed space using fuzzy point approach. The method that can be used is as follows. From fuzzy normed space, we construct a norm function that satisfies properties of fuzzy *n*-normed, so that fuzzy *n*-normed space is derived. Conversely, from fuzzy *n*-normed space, we construct a normed function that satisfies properties of fuzzy (*n*-*m*)-normed, so that fuzzy (*n*-*m*)-normed space is obtained. Finally, we get two new theorems that state that a fuzzy *n*-normed space from any fuzzy normed space and fuzzy (*n*-*m*)-normed space for *m*=1, 2,…, *n*-1 from fuzzy *n*-normed space using fuzzy point of fuzzy set always can be constructed.

Keywords Norm Space, *N*-Normed Space, Fuzzy Normed Space, Fuzzy *N*-Normed Space

1. Introduction

Many researchers have development concept of fuzzy normed space (FNS) using various approaches. Likewise with fuzzy 2-normed space (F2-NS) to fuzzy *n*-normed space (Fn-NS). One of them is intuitionistic approach which uses the concept of the continuous *t*-norm and the continuous *t*-conorm [2,4,5,11,12,18,20], [25-33]. However, the concept of FNS which uses the fuzzy point approach of the fuzzy set is also used by many authors [1], [14-17], [22-24]. Intuitionistic approach using fuzzy point is also used to construct F2-NS and generally, construct Fn-NS, for $n \geq 2$ as discussed in [5,12,25,28,31,32].

In [22], the concept of FNS has been developed to F2-NS and in [21], it is generalized to Fn-NS, $n \geq 2$ using fuzzy point approach of the fuzzy set. The study about the relationship among FNS, F2-NS, Fn-NS, norm space (NS), 2-normed space (2-NS), *n*-normed space (n-NS), fuzzy inner product space (FIPS), fuzzy 2-inner product space (F2-IPS), fuzzy *n*- inner product space (F*n*-IPS), inner product space (IPS), 2-inner product space (2-IPS) and *n*-inner product space (*n*-IPS) has been done by researchers and many results are gotten as explained on figure 1. Blue arrows in figure 1 show the new theorem in this study. Arrow symbol can be interpreted as an implication relation.

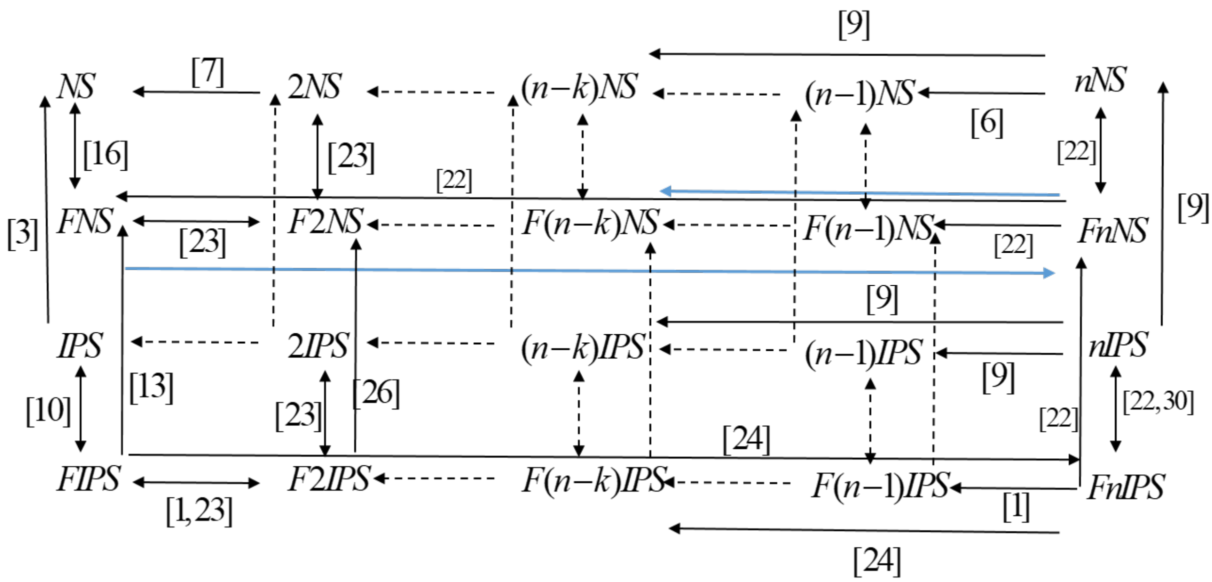

Figure 1. The relevancy among FNS, F2-NS to Fn-NS.

Based on the condition above, we need to analyze how to construct directly Fn-NS from any FNS and vice versa, reduce any Fn-NS to derive fuzzy (*n*-*m*)-normed space (F(*n*-*m*)-NS), for *m*=1, 2,…, *n*-1. If we succeed to construct it, then all the properties on FNS and Fn-NS will be equivalent.

Hereafter, in [22], it shows that norm spaces and FNS are equivalent. Then, based on this result study, it means norm space equivalent to FNS and Fn-NS. Consequently, all the properties of norm space can be applied to FNS and Fn-NS.

2. Preliminaries

The fuzzy set concept is first introduced by Zadeh in [20], that as follows.

Definition 2.1. For any nonempty set *Y*, and $I=[0,1]$, it is defined function $\mu_{\tilde{B}}: Y \to I$, or it can be stated as

$$\tilde{B}=\{(y,\mu_{\tilde{B}}(y)) \mid \mu_{\tilde{B}}(y)\in I, y\in Y\}.$$

Then $\tilde{B}$ is called fuzzy set on *Y*.

Various concepts related to fuzzy set are developed based on this definition 2.1. Among other are the development of fuzzy point that is discussed in [1], [13-18], [22-24] and [32]. The concept of fuzzy point is as follows.

Definition 2.2. Membership functions for a fuzzy point P_y in *Y* are defined by

$$\mu_{P_y}(x)=\begin{cases}0 & if \;\; x\neq y\\ b & if \;\; x=y\end{cases}$$

for every $x\in Y$ and $b\in(0,1]$. Fuzzy points are denoted as (y,b) or y_b.

Based on definition 2.2 above, fuzzy points x_a equal to y_b if $a=b$ where $a,b\in(0,1]$ and *x*=*y*.

Definition 2.3. Let $\tilde{B}$ be a fuzzy set on *Y* and y_b be fuzzy point. If $b\le\mu_{\tilde{B}}(y)$, then y_b is said to be in $\tilde{B}$ or belong to $\tilde{B}$. It is denoted by $y_b\in\tilde{B}$.

Definition 2.4. Let $\tilde{B}$ be a fuzzy set in a vector space *Y* over a field *L*. If for all $y,x\in Y$ and $\lambda\in L$ satisfy $\mu_{\tilde{B}}(y+x)\ge\min\{\mu_{\tilde{B}}(y),\mu_{\tilde{B}}(x)\}$ and $\mu_{\tilde{A}}(\lambda y)\ge\mu_{\tilde{A}}(y)$ then $\tilde{B}$ is said to be a fuzzy subspace in *Y*.

There are many ways to define the FNS, including definition that is approached based on fuzzy point and intuitionistic. The next, definition of FNS and F2-NS that is approached based on the fuzzy point is given [15]. This definition is disparate from definition in [21], [29], [33] and [34-36].

Definition 2.5. For any vector space *Y* over a field *L* and $\tilde{B}$ be a fuzzy set in *Y*, it is defined function $\|\cdot\|_f:\tilde{B}\to R^+\cup\{0\}$ which point $y_\beta\in\tilde{B}, b\in(0,1]$, is mapped to real number $\|y_b\|_f\ge 0$ so that for every $x_a,y_b\in\tilde{B}$, $\lambda\in L$ satisfy

(FN1) $y_b=0$ if only if $\|y_b\|_f=0$.

(FN2) $\|\lambda y_b\|_f=|\lambda|\,\|y_b\|_f$

(FN3) $\|y_b+x_a\|_f\le\|y_b\|_f+\|x_a\|_f$

(FN4) If $0<\rho\le b<1$, then $\|y_b\|_f\le\|y_\rho\|_f$ and there is a sequence (b_n) with condition $0<b_n\le b<1$ and $\lim_{n\to\infty}\|y_b\|_f=\|y_b\|_f$, then function $\|\cdot\|_f$ is called a fuzzy norm on Y and a pair $(Y, \|\cdot\|_f)$ is called a fuzzy norm space (FNS).

Definition 2.6. For any vector space Y over a field L and $\tilde{B}$ be a fuzzy set in Y, it is defined function $\|\cdot,\cdot\|_f : \tilde{B}\times\tilde{B} \to R^+ \cup \{0\}$ in which each point (y_b, x_a) in $\tilde{B}\times\tilde{B}$ is mapped to real number $\|y_b, x_a\|_f \ge 0$ so that for every $x_a, y_b, z_c \in \tilde{B}, \lambda \in L$ satisfy

(F2N1) y_b, x_a is linearly independent if only if $\|y_b, x_a\|_f \neq 0$

(F2N2) $\|y_b, x_a\|_f = \|x_a, y_b\|_f$

(F2N3) $\|\lambda y_b, x_a\|_f = |\lambda| \|y_b, x_a\|_f$

(F2N4) $\|y_b + z_c, x_a\|_f \le \|y_b, x_a\|_f + \|z_c, x_a\|_f$

(F2N5) If $0 < \sigma \le a < 1$ and $0 < \rho \le b < 1$ then $\|y_b, x_a\|_f \le \|y_\rho, x_\sigma\|_f$ and there are two sequences $(a_n), (b_n)$ where $0 < a_n \le a < 1$ and $0 < b_n \le b < 1$ so that $\lim_{n\to\infty} \|y_{b_n}, x_{a_n}\|_f = \|y_b, x_a\|_f$,

then function $\|\cdot,\cdot\|_f$ is called fuzzy 2-norm on Y and a pair $(Y, \|\cdot,\cdot\|_f)$ is called a fuzzy 2-norm space (F2-NS).

The next, the definition of Fn-NS that is approached using fuzzy point is given as discussed on [22]. Clearly, it is not the same from definition in [4], [18] and [37-38].

Definition 2.7. For fuzzy set $\tilde{B}$, it is defined function

$$\|\cdot,\dots,\cdot\|_f : \tilde{B}^n \to R^+ \cup \{0\}$$

where $(y^{(1)}_{b_1},\dots,y^{(n)}_{b_n}) \in \tilde{B}^n$ with $b_j \in (0,1]$, $j \in N_n = \{1,2\dots,n\}$ is mapped to $\|y^{(1)}_{b_1},\dots,y^{(n)}_{b_n}\|_f \in R^+ \cup \{0\}$. The function $\|\cdot,\dots,\cdot\|_f$ is called fuzzy n-normed if it satisfies the conditions:

For all $y^{(j)}_{b_j}, x_a \in \tilde{B}$ and $r \in R$, with $a, b_j \in (0,1]$, $j \in N_n$,

(FnN1) $y^{(1)}_{b_1},\dots,y^{(n)}_{b_n}$ is linearly independent if only if $\|y^{(1)}_{b_1},\dots,y^{(n)}_{b_n}\|_f \neq 0$

(FnN2) $\|y^{(1)}_{b_1},\dots,y^{(n)}_{b_n}\|_f$ is invariant under permutation.

(FnN3) $\|ry^{(1)}_{b_1},\dots,y^{(n)}_{b_n}\|_f = |r| \|y^{(1)}_{b_1},\dots,y^{(n)}_{b_n}\|_f$

(FnN4) $\|y^{(1)}_{b_1} + x_a, y^{(2)}_{b_2},\dots,y^{(n)}_{b_n}\|_f \le \|y^{(1)}_{b_1}, y^{(2)}_{b_2},\dots,y^{(n)}_{b_n}\|_f + \|x_a, y^{(2)}_{b_2},\dots,y^{(n)}_{b_n}\|_f$

(FnN5) If $0 < \rho_j \le b^{(j)} < 1$, for $j \in N_n$ then

$$\|y^{(1)}_{b_1},\dots,y^{(n)}_{b_n}\|_f \le \|y^{(1)}_{\rho_1},\dots,y^{(n)}_{\rho_n}\|_f$$

and there exists $0 < b^{(j)}_k \le b^{(j)} < 1$ for $k \in N$, $j \in N_n$, such that

$$\lim_{k\to\infty} \|y^{(1)}_{b^{(1)}_k},\dots,y^{(n)}_{b^{(n)}_k}\|_f = \|y^{(1)}_{b^{(1)}},\dots,y^{(n)}_{b^{(n)}}\|_f .$$

In this case, $(Y, \|\cdot,\dots,\cdot\|_f)$ is called as fuzzy n-normed space (Fn-NS).

3. Results and Discussion

In [6], Hendra states that in any n-NS with $n \ge 2$, an (n-1)-NS always can be explicitly derived from n-NS or, more generally, an (n–k)-NS from the n-NS, for each $k \in N_{n-1}$. While in fuzzy concept in [20], it is showed that in Fn-NS, F(n-1)-NS can be constructed. The following, a new theorem is introduced which states that in any Fn-NS with $n \ge 2$, can be explicitly derived an fuzzy norm or, more generally, an F(n-m)-N from the Fn-NS, for each $m \in N_{n-1}$.

Theorem 3.1.

Let $(Y, \|\cdot,\dots,\cdot\|_f)$ be a Fn-NS with $n \ge 2$ and fix $\{u^{(1)}_1, u^{(2)}_1,\dots,u^{(n)}_1\}$ is linearly independent in $\tilde{B}$. For $m \in N_{n-1}$, it is defined function $\|\cdot,\dots,\cdot\|^*_f : \tilde{B}^{n-m} \to R^+ \cup \{0\}$ by

$$\|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}},\dots,y^{(n-m)}_{b^{(n-m)}}\|^*_f := \left(\sum_{\{j_1,j_2,\dots,j_m\}\subseteq N_n} \|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}},\dots,y^{(n-m)}_{b^{(n-m)}}, u^{(j_1)}_1,\dots,u^{(j_m)}_1\|^2_f \right)^{\frac{1}{2}}$$

for all $y^{(j)}_{b^{(j)}}$ in $\tilde{B}$ with $b^{(j)} \in (0,1]$, $j \in N_{n-m}$. Then $(Y, \|\cdot,\dots,\cdot\|^*_f)$ is F(n-m)-NS. Especially, for $m = n-1$,

$$\|y_b\|^*_f := \left(\sum_{\{i_1,i_2,\dots,i_m\}\subseteq\{1,2,\dots,n\}} \|y_b, u^{(j_1)}_1,\dots,u^{(j_m)}_1\|^2_f \right)^{\frac{1}{2}}.$$

Then $(Y, \|\cdot\|^*_f)$ is a FNS.

Proof. (F(n-m)N1). If $\{y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}},\dots,y^{(n-m)}_{b^{(n-m)}}\}$ is linearly dependent, then $\{y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}},\dots,y^{(n-m)}_{b^{(n-m)}}, u^{(j_1)}_1,\dots,u^{(j_m)}_1\}$ is linearly dependent too. Hence, it is obtained $\|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}},\dots,y^{(n-m)}_{b^{(n-m)}}\|^*_f = \|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}},\dots,y^{(n-m)}_{b^{(n-m)}}, u^{(j_1)}_1,\dots,u^{(j_m)}_1\| = 0.$ Conversely, if $\|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}},\dots,y^{(n-m)}_{b^{(n-m)}}\|^*_f = 0$ then

$\left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2$ =0 for any permutation of $\{u_1^{(i_1)}, ..., u_1^{(i_m)}\}$. Consequently, $\{y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(i_1)}, ..., u_1^{(i_m)}\}$ is linearly dependent. Clearly, this mean that $\{y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}\}$ is linearly dependent.

(F(*n-m*)N2). Since $\left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f$ is invariant under any permutations, and then $\left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)} \right\|_f^*$ is invariant under any permutations too.

(F(n-m)N3). Furthermore, let $C = \{j_1, ..., j_m\}$ and $C \subseteq N_n$.

For all $y_{b_i}^{(j)}, x_a \in \tilde{B}$ with $a, b_j \in (0,1],\ j \in N_{n-m},\ r \in R,$

$$\left\| r y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)} \right\|_f^* = \left(\sum_{C \subseteq N_n} \left\| r y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{C \subseteq N_n} r^2 \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left(r^2 \sum_{C \subseteq N_n} \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= | r | \left(\sum_{C \subseteq N_n} \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= | r | \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)} \right\|_f^*$$

(F(n-m)N4).

$$\left\| x_a + y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)} \right\|_f^* = \left(\sum_{C \subseteq N_n} \left\| x_a + y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$\le \left(\sum_{C \subseteq N_n} \left(\left\| x_a, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f + \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f \right)^2 \right)^{\frac{1}{2}}$$

$$\le \left(\sum_{C \subseteq N} \left\| x_a, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}} + \left(\sum_{C \subseteq D} \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left\| x_a, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)} \right\|_f^* + \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)} \right\|_f^*$$

(F(n-m)N5). If $0 < \sigma_j \le b^{(j)} < 1$ for $j \in N_{n-m}$, then

$$\left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)} \right\|_f^* = \left(\sum_{C \subseteq N_n} \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$\le \left(\sum_{C \subseteq N_n} \left\| y_{\sigma_1}^{(1)}, y_{\sigma_2}^{(2)}, ..., y_{\sigma_{n-m}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left\| y_{\sigma_1}^{(1)}, y_{\sigma_2}^{(2)}, ..., y_{\sigma_{n-m}}^{(n-m)} \right\|_f^*$$

and there exist $0 < b_k^{(j)} \le b^{(j)}$ for $j \in N_{n-m}$, $k \in N$ and $(b_k^{(j)})$ converge to $b^{(j)}$ such that

$$\lim_{k\to\infty} \left\| y_{b_k^{(1)}}^{(1)}, y_{b_k^{(2)}}^{(2)}, ..., y_{b_k^{(n-m)}}^{(n-m)} \right\|_f^* = \lim_{k\to\infty} \left(\sum_{C \subseteq N_n} \left\| y_{b_k^{(1)}}^{(1)}, y_{b_k^{(2)}}^{(2)}, ..., y_{b_k^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left(\lim_{k\to\infty} \sum_{C \subseteq N_n} \left\| y_{b_k^{(1)}}^{(1)}, y_{b_k^{(2)}}^{(2)}, ..., y_{b_k^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{C \subseteq N_n} \lim_{k\to\infty} \left\| y_{b_k^{(1)}}^{(1)}, y_{b_k^{(2)}}^{(2)}, ..., y_{b_k^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{C \subseteq N_n} \left(\lim_{k\to\infty} \left\| y_{b_k^{(1)}}^{(1)}, y_{b_k^{(2)}}^{(2)}, ..., y_{b_k^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f \right)^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{C \subseteq N_n} \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^2 \right)^{\frac{1}{2}}$$

$$= \left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n-m)}}^{(n-m)}, u_1^{(j_1)}, ..., u_1^{(j_m)} \right\|_f^* . \qquad \square$$

In [23], it is showed that in any FNS, we can derive F2-NS from FNS. Hereafter, we give a new theorem which states that generally we can derive Fn-NS from FNS.

Theorem 3.2.

Let $(Y, \|\cdot\|_f)$ be a FNS. It is defined

$$\left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n)}}^{(n)} \right\|_f := \begin{cases} 0 & \text{if } \ y^{(j)} \in \operatorname{span}_{i \ne j}\{y^{(i)}\} \\ \left\| y_{b^{(1)}}^{(1)} \right\|_f \left\| y_{b^{(2)}}^{(2)} \right\|_f \cdots \left\| y_{b^{(n)}}^{(n)} \right\|_f & \text{otherwise} \end{cases}$$

for all $y_{b^{(j)}}^{(j)} \in \tilde{B},\ j \in N_n$. Then $(Y, \|\cdot, ..., \cdot\|_f)$ is a Fn-NS.

Proof. Let $y_{b^{(j)}}^{(j)}, x_a \in \tilde{B}$ with $b_j, a \in (0,1],\ j \in N_n$ and $r \in R$.

(FnN1). it is clear from definition that $\left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n)}}^{(n)} \right\|_f = 0$ if only if $\left\{ y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n)}}^{(n)} \right\}$ is linearly dependent.

(FnN2). $\left\| y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n)}}^{(n)} \right\|_f = \left\| y_{b^{(1)}}^{(1)} \right\|_f \left\| y_{b^{(2)}}^{(2)} \right\|_f \cdots \left\| y_{b^{(n)}}^{(n)} \right\|_f$ is invariant inder permutation of $\left\{ y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n)}}^{(n)} \right\}$.

(FnN3).

$$\left\| r y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n)}}^{(n)} \right\|_f = \left\| r y_{b^{(1)}}^{(1)} \right\|_f \left\| y_{b^{(2)}}^{(2)} \right\|_f \cdots \left\| y_{b^{(n)}}^{(n)} \right\|_f$$

$$= | r | \left\| y_{b^{(1)}}^{(1)} \right\|_f \left\| y_{b^{(2)}}^{(2)} \right\|_f \cdots \left\| y_{b^{(n)}}^{(n)} \right\|_f$$

(FnN4).

$$\left\| x_a + y_{b^{(1)}}^{(1)}, y_{b^{(2)}}^{(2)}, ..., y_{b^{(n)}}^{(n)} \right\|_f = \left\| x_a + y_{b^{(1)}}^{(1)} \right\|_f \left\| y_{b^{(2)}}^{(2)} \right\|_f \cdots \left\| y_{b^{(n)}}^{(n)} \right\|_f$$

$$\leq \left(\|x_a\|_f + \|y^{(1)}_{b^{(1)}}\|_f \right) \|y^{(2)}_{b^{(2)}}\|_f \cdots \|y^{(n)}_{b^{(n)}}\|_f$$
$$= \|x_a\|_f \|y^{(2)}_{b^{(2)}}\|_f \cdots \|y^{(n)}_{b^{(n)}}\|_f + \|y^{(1)}_{b^{(1)}}\|_f \|y^{(2)}_{b^{(2)}}\|_f \cdots \|y^{(n)}_{b^{(n)}}\|_f$$
$$= \|x_a, y^{(2)}_{b^{(2)}}, ..., y^{(n)}_{b^{(n)}}\|_f + \|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}}, ..., y^{(n)}_{b^{(n)}}\|_f$$

(FnN5) If $0 < \sigma_j \leq b^{(j)} < 1$ for $j \in N_n$ then $\frac{1}{b^{(j)}} \leq \frac{1}{\sigma_j}$, so $\|y^{(j)}_{b^{(j)}}\|_f = \frac{1}{b^{(j)}}\|y^{(j)}\|_f \leq \frac{1}{\sigma_j}\|y^{(j)}\|_f = \|y^{(j)}_{\sigma_j}\|_f$.

and

$$\|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}}, ..., y^{(n)}_{b^{(n)}}\|_f = \|y^{(1)}_{b^{(1)}}\|_f \|y^{(2)}_{b^{(2)}}\|_f \cdots \|y^{(n)}_{b^{(n)}}\|_f$$
$$\leq \|y^{(1)}_{\sigma_1}\|_f \|y^{(2)}_{\sigma_2}\|_f \cdots \|y^{(n)}_{\sigma_n}\|_f$$
$$= \|y^{(1)}_{\sigma_1}, y^{(2)}_{\sigma_2}, ..., y^{(n)}_{\sigma_n}\|_f .$$

The next, let $k \in N$ and $b^{(j)}_k = (1 - \frac{1}{k})b^{(j)}$ where $j \in N_n$. Then the sequence $(b^{(j)}_k)$ converge to $b^{(j)}$ when $k \to \infty$ so that

$$\lim_{k\to\infty} \|y^{(1)}_{b^{(1)}_k}, y^{(2)}_{b^{(2)}_k}, ..., y^{(n)}_{b^{(n)}_k}\|_f = \lim_{k\to\infty} \|y^{(1)}_{b^{(1)}_k}\|_f \|y^{(2)}_{b^{(2)}_k}\|_f \cdots \|y^{(n)}_{b^{(n)}_k}\|_f$$
$$= \lim_{k\to\infty} \frac{1}{b^{(1)}_k}\|y^{(1)}\| \frac{1}{b^{(2)}_k}\|y^{(2)}\| \cdots \frac{1}{b^{(n)}_k}\|y^{(n)}\|$$
$$= \frac{1}{b^{(1)}}\|y^{(1)}\| \frac{1}{b^{(2)}}\|y^{(2)}\| \cdots \frac{1}{b^{(n)}}\|y^{(n)}\|$$
$$= \|y^{(1)}_{b^{(1)}}\|_f \|y^{(2)}_{b^{(2)}}\|_f \cdots \|y^{(n)}_{b^{(n)}}\|_f$$
$$= \|y^{(1)}_{b^{(1)}}, y^{(2)}_{b^{(2)}}, ..., y^{(n)}_{b^{(n)}}\|_f . \quad \square$$

4. Conclusion

On fuzzy normed space, we always can construct a Fn-N from FNS and conversely, construct a F(n-m)-N from Fn-NS for each $m \in N_{n-1}$. Therefore, FNS can be considered as Fn-NS and vice versa so that all the properties of FNS can be applied to Fn-NS and vice versa.

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