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Analysis of Hetrogeneous Feedback Queue Model in Stochastic and in Fuzzy Environment Using L-R Method

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Abstract In this paper, we analyse a feedback queue network in stochastic and in fuzzy environment. We consider a model with three heterogeneous servers which are commonly attached to a server in starting. At the initial stage, all queue performance measures are obtained in steady-state that is in stochastic environment. After that, work is extended to fuzzy environment because practically all characteristics of the system are not exact, they are uncertain in nature. In the present work we use probability generating function technique, triangular fuzzy numbers, classical formulae for the calculation of all queue characteristics and L-R method to calculate queue characteristics in fuzzy environment.

Keywords Feedback, Heterogeneous, Stochastic Environment, Fuzzy Environment, L-R Method

1. Introduction

Nowadays, a lot of research is done on queuing theory, on priority queues, feedback queues, single arrival, bulk arrival, queues in stochastic environment, queues in fuzzy environment and many more. In the present paper, we analyse feedback queues with heterogeneous servers instead of homogeneous servers because when we take identical servers then this situation does not look like realistic situation, it seems to be mechanized. In real life, servers are not always homogeneous in nature and as we all know feedback if possible, is necessary everywhere at this time. Kryshnamootrhy [1] analyzed Poisson queues with heterogeneous servers. Kusum [9] analyzed a heterogeneous feedback queue system. Recently V. Saini, Deepak Gupta [10] analyze heterogeneous queue system with at most one revisit and with the concept that whenever we revisit the system probability of visiting is changed. Queues in fuzzy environment are studied by R.J.Li, E.S.Lee [2], T.P.Singh, M.Mittal, D.Gupta [5][6] etc. J.P.Mukeba, R. Mabela and Ulengue [3] used L-R method to compute fuzzy queuing performance measures.W. Ritha and S. Josephine Vinnarsi [7] used L-R method for priority queuing model. J.P. Mukeba [4] proposed L-R method to solve triangular fuzzy numbers.

In this present paper, firstly the proposed model is analysed in steady state and queue performance measures are calculated. After that queue performance measures are obtained in fuzzy environment, in this paper L-R method is used to obtain performance measures in fuzzy environment. A numerical example is taken first for validation check in stochastic atmosphere after that same numerical is used for fuzzy environment by taking such triangular fuzzy numbers whose defuzzified values will approximately equal to same crisp values as we have taken in the example for stochastic environment. Such type of model can be applicable to super markets, hospital management, computer networks and business management.

2. Definitions

2.1. Fuzzy Set

Let U be a classical set. A fuzzy set $\tilde{A}$ which is a subset of U, is defined by the function $\eta_{\tilde{A}}$, which is called membership function of $\tilde{A}$ from U→[0,1]. Also $\eta_{\tilde{A}}(a)$ is called the membership of a for all a ∈ $\tilde{A}$ and for each x ∈ U such as $\eta_{\tilde{A}}(x) = 1$, x is called mean or modal value of $\tilde{A}$.

2.2. Triangular Fuzzy Number

A number of the form $\tilde{A} = (a_1, m, a_2)$, $a_1 < m < a_2$ defined on R, is said to be a triangular fuzzy number if its membership function can be defined as,

$$n_{\tilde{A}}(x) = \begin{cases} \dfrac{x-a}{b-a} \; if\; a \le x \le b \\ \dfrac{c-x}{c-b} \; if\; b < x \le c \\ 0 \qquad otherwise \end{cases}$$

2.3. L-R Fuzzy Number

A fuzzy number $\tilde{A} = (a_1, m, a_2)$ is said to be L-R fuzzy if and only if there exists three real numbers m, a>0, b>0 and two positive, continuous and decreasing functions L and R, from **R** to [0,1],such as L(0) = R(0) =1

$$L(1)=0,\ L(x)>0,\ \lim_{x\to\infty} L(x)=0$$

$$R(1)=0,\ R(x)>0,\ \lim_{x\to\infty} R(x)=0$$

$$n_{\tilde{A}}(x) = \begin{cases} L\left(\dfrac{m-x}{a}\right) if x \in [m-a, m] \\ R\left(\dfrac{x-m}{b}\right) if x \in [m, m+b] \\ 0, \qquad otherwise \end{cases}$$

We can denote L-R fuzzy number $\tilde{A}$ as, $\tilde{A}$ = $(m, a, b)_{LR}$ called L-R representation of $\tilde{A}$ where modal value is m, a is left spread and b is right spread of $\tilde{A}$.

Support of $\tilde{A}$ = (m-a, m-b)

2.4. L-R Fuzzy Arithmetic

Let $\tilde{B} = (m, a, b)_{LR}$, $\tilde{C} = (n, c, d)_{LR}$ be two L-R fuzzy numbers then we can define addition, subtraction, multiplication and division as,

$$\tilde{B} + \tilde{C} = (m + n, a + c, b + d)_{LR}$$

$$\tilde{B} - \tilde{C} = (m - n, a + d, b + c)_{LR}$$

$$\tilde{B}.\tilde{C} = (mn, mc + na - ac, md + nb + bd)_{LR}$$

$$\frac{\tilde{B}}{\tilde{C}} = \left(\frac{m}{n}, \frac{md}{n(n+d)} + \frac{a}{n} - \frac{ad}{n(n+d)}, \frac{mc}{n(n-c)} + \frac{b}{n} - \frac{bc}{n(n-c)}\right)$$

3. Notations

The used notations are given below in the form of a table.

Table 1. Notations used in whole paper

m = number of arriving customers	
λ = arrival rate	$\tilde{\lambda}$ = fuzzy arrival rate
$(m,a,b)_{LR}$ = L-R representation of fuzzy number	
μ = service rate	$\tilde{\mu}$ = fuzzy service rate
a,b,c,d=probabilities of leaving the servers at first time	a_1, b_1, c_1, d_1= probabilities of leaving the servers at second time
q_{ij} = probability of first time visit from one server to another in the state (i.j)	$\tilde{q_{ij}}$ = fuzzy probability of first time visit from one server to another in the state (i.j)
q'_{ij}= probability of second time visit from one server to another in the state (i.j)	$\tilde{q'_{ij}}$ = fuzzy probability of second time visit from one server to another in the state (i.j)
L = mean queue length of the system	$\tilde{L}$=fuzzy mean queue length of the system
L_i =partial queue length of the server, where i=1,2,3,4	$\tilde{L_i}$= fuzzy partial queue length of the server, where i=1,2,3,4

4. Assumptions

Some basic assumptions of queuing model are:

- The arrival process and service process are according to Poisson distribution.
- There is no unusual customer behavior.
- If any customer is unsatisfied or needs feedback then he may revisit the system at most once.
- Since the servers C_2, C_3, C_4 are heterogeneous in nature so the customer cannot move from one server to another.
- The service discipline is FIFO.
- There is no unusual server behavior.

5. Description of Model

The proposed model is comprised of four servers as shown in figure 1 given below. The first server C_1 is a common server. Other three servers C_2, C_3, C_4 are three parallel heterogeneous servers commonly connected to first server. m_1, m_2, m_3, m_4 be the number of units in front of the servers C_1,C_2,C_3, C_4 respectively which come with arrival rate $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Firstly a customer will come in front of the server C_1. After getting service from this server he can move any one of the servers C_2,C_3, C_4. After getting service from the servers either the customer will leave the system or he may revisit the system for feedback one time only. In each condition the following conditions will always be satisfied:

$$aq_{12}+aq_{13}+aq_{14}+a_1q_{12}^{'}+a_1q_{13}^{'}+a_1q_{14}^{'}=1,$$
$$bq_2+bq_{21}+b_1q_2^{'}=1,$$
$$cq_3+cq_{31}+c_1q_3^{'}=1, dq_4+dq_{41}+d_1q_4^{'}=1$$

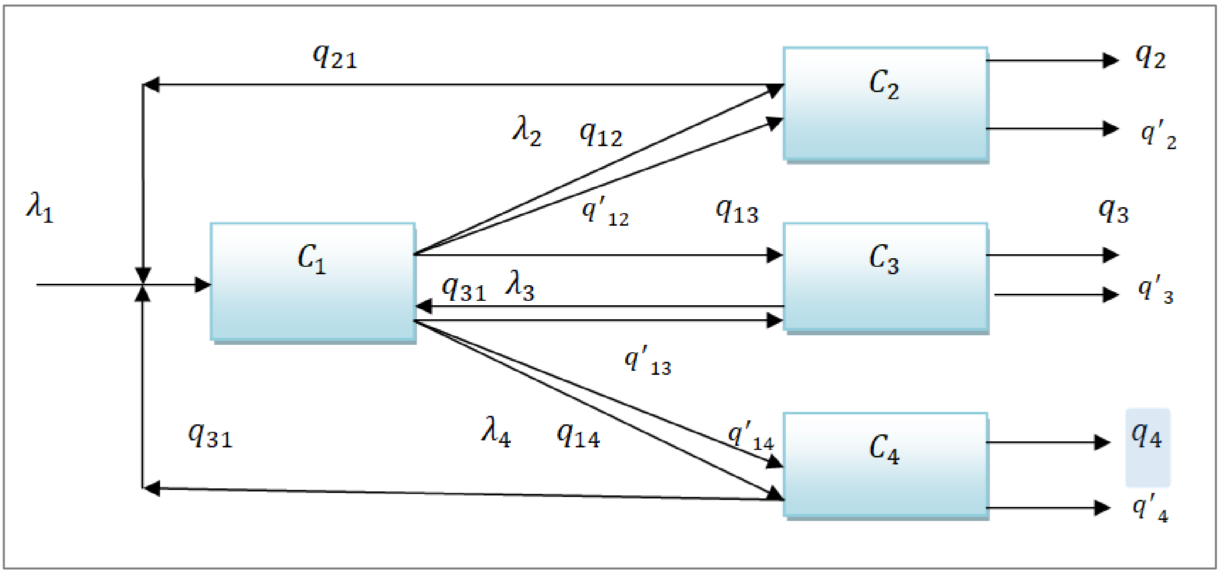

Figure 1. Flow of customersitems from one server to another

6. Mathematical Modelling

Suppose P_{m_1,m_2,m_3m_4} (t) denotes the probability of m_1, m_2, m_3, m_4 units/customers in front of the servers C_1, C_2, C_3, C_4 respectively, where $m_1, m_2, m_3, m_4 \geq 0$ .The differential difference equation for the model in steady state is as follows:

$$(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\mu_1+\mu_2+\mu_3+\mu_4)P_{m_1,m_2,m_3,m_4}$$
$$=\lambda_1 P_{m_1-1,m_2,m_3,m_4}+\lambda_2 P_{m_1,m_2-1,m_3,m_4}$$
$$+\lambda_3 P_{m_1,m_2,m_3-1,m_4}+\lambda_4 P_{m_1,m_2,m_3,m_4-1}$$
$$+\mu_1(aq_{12}+a_1q_{12}^{'})P_{m_1+1,m_2-1,m_3,m_4}$$
$$+\mu_1(aq_{13}+a_1q_{13}^{'})P_{m_1+1,m_2,m_3-1,m_4}$$
$$+\mu_1(aq_{14}+a_1q_{14}^{'})P_{m_1+1,m_2,m_3,m_4-1}$$
$$+\mu_2(bq_{21})P_{m_1-1,m_2+1,m_3,m_4}$$
$$+\mu_2(bq_2+b_1q_2^{'})P_{m_1,m_2+1,m_3,m_4}$$
$$+\mu_2(cq_{31})P_{m_1-1,m_2,m_3+1,m_4}$$
$$+\mu_3(cq_3+c_1q_3^{'})P_{m_1,m_2,m_3+1,m_4}$$
$$+\mu_4(dq_{41})P_{m_1-1,m_2,m_3,m_4+1}$$
$$+\mu_4(dq_4+d_1q_4^{'})P_{m_1,m_2,m_3,m_4+1}. \tag{1}$$

15 more equations will be formed after we considering all possible values of m_1, m_2, m_3, m_4

For the purpose of solving all these equations the generating function and partial generating functions can be defined as,

$$G(X,Y,Z,R)=$$
$$\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\sum_{m_4=0}^{\infty}P_{m1,m2,m3,m4}(X)^{m_1}(Y)^{m_2}(Z)^{m_3}(R)^{m_4},$$
$$|X|=1,|Y|=1,|Z|=1,|R|=1$$

$$G_{m_2,m_3,m_4}(X)=\sum_{m_1=0}^{\infty}P_{m_1,m_2,m_3,m_4}(X)^{m_1}$$

$$G_{m_3,m_4}(X,Y)=\sum_{m_1=0}^{\infty}G_{m_2,m_3,m_4}(X)(Y)^{m_2}$$

$$G_{m_4}(X,Y,Z)=\sum_{m_1=0}^{\infty}G_{m_3,m_4}(X,Y)(Z)^{m_3}$$

$$G(X,Y,Z,R)=\sum_{m_1=0}^{\infty}G_{m_4}(X,Y,Z)(R)^{m_4}$$

When we solve the equations with the help of above defined generating functions then we find the value as,

$$G = \frac{\begin{array}{l}\mu_1\left[\begin{array}{l}1-\frac{(aq_{12}+a_1q'_{12})Y}{X}-\frac{(aq_{13}+a_1q'_{13})Z}{X}\\-\frac{(aq_{14}+a_1q'_{14})R}{X}\end{array}\right]G_1\\+\mu_2\left[1-\frac{(bq_{21})X}{Y}-\frac{(bq_2+b_1q'_2)}{Y}-\right]G_2\\+\mu_3\left[1-\frac{(cq_{31})Y}{Z}-\frac{(cq_3+c_1q'_3)}{Z}\right]G_3\\+\mu_4\left[1-\frac{(dq_{41})X}{R}-\frac{(dq_4+d_1q'_4)}{R}\right]G_4\end{array}}{(1-X)\lambda_1+(1-Y)\lambda_2+(1-Z)\lambda_3+(1-R)\lambda_4}$$

$$+\mu_1\left[\begin{array}{l}1-\frac{(aq_{12}+a_1q'_{12})Y}{X}-\frac{(aq_{13}+a_1q'_{13})Z}{X}\\-\frac{(aq_{14}+a_1q'_{14})R}{X}\end{array}\right]$$

$$+\mu_2\left[1-\frac{(bq_{21})X}{Y}-\frac{(bq_2+b_1q'_2)}{Y}\right]$$

$$+\mu_3\left[1-\frac{(cq_{31})Y}{Z}-\frac{(cq_3+c_1q'_3)}{Z}\right]$$

$$+\mu_4\left[1-\frac{(dq_{41})X}{R}-\frac{(dq_4+d_1q'_4)}{R}\right]$$

$$G_1=G_0(Y,Z,R), G_2=G_0(X,Z,R),$$
$$G_3=G_0(X,Y,R), G_4=G_0(X,Y,Z),$$
$$G=G(X,Y,Z,R)$$

Taking X=Y=Z=R=1, G(X,Y,Z,R)=1and above equation reduces to indeterminate form then using L'Hospital rule and taking limit as X,Y,Z,R tends to 1 one by one then we get below equations,

$$-\lambda_1+\mu_1-(bq_{21})\mu_2-(cq_{31})\mu_3-(dq_{41})\mu_4$$
$$=\mu_1G_1-(bq_{21})\mu_2G_2-(cq_{31})\mu_3G_3$$
$$-(dq_{41})\mu_4G_4 \ldots\ldots\ldots\ldots(I)$$

$$\mu_2G_2=(aq_{12}+a_1q'_{12})\mu_1G_1-\lambda_2$$
$$-(aq_{12}+a_1q'_{12})\mu_1+\mu_2 \ldots\ldots\ldots\ldots(II)$$

$$\mu_3G_3=(aq_{13}+a_1q'_{13})\mu_1G_1-\lambda_3$$
$$-(aq_{13}+a_1q'_{13})\mu_1+\mu_3 \ldots\ldots\ldots\ldots(III)$$

$$\mu_4G_4=(aq_{14}+a_1q'_{14})\mu_1G_1-\lambda_4$$
$$-(aq_{14}+a_1q'_{14})\mu_1+\mu_4 \ldots\ldots\ldots\ldots(IV)$$

On solving above four equations we get,

$$G_1=1-\frac{1}{\mu_1}\left[\frac{\begin{array}{l}\lambda_1+(bq_{21})\lambda_2+(cq_{31})\lambda_3\\+(dq_{41})\lambda_4\end{array}}{\begin{array}{l}1-(bq_{21})(aq_{12}+a_1q'_{12})\\-(cq_{31})(aq_{13}+a_1q'_{13})\\-(dq_{41})(aq_{14}+a_1q'_{14})\end{array}}\right]$$

$$G_2=1-\frac{1}{\mu_2}\left[\frac{\begin{array}{l}(aq_{12}+a_1q'_{12})\lambda_1\\+\left(\begin{array}{l}1-(cq_{31})(aq_{13}+a_1q'_{13})\\-(dq_{41})(aq_{14}+a_1q'_{14})\end{array}\right)\lambda_2\\+(aq_{12}+a_1q'_{12})(cq_{31})\lambda_3\\+(aq_{12}+a_1q'_{12})(dq_{41})\lambda_4\end{array}}{\begin{array}{l}1-(bq_{21})(aq_{12}+a_1q'_{12})\\-(cq_{31})(aq_{13}+a_1q'_{13})\\-(dq_{41})(aq_{14}+a_1q'_{14})\end{array}}\right]$$

$$G_3=1-\frac{1}{\mu_3}\left[\frac{\begin{array}{l}(aq_{13}+a_1q'_{13})\lambda_1\\+(aq_{13}+a_1q'_{13})(bq_{21})\lambda_2\\+\begin{array}{l}(1-(bq_{21})(aq_{12}+a_1q'_{12})\\-(dq_{41})(aq_{14}+a_1q'_{14}))\lambda_3\end{array}\\+(aq_{13}+a_1q'_{13})(dq_{41})\lambda_4\end{array}}{\begin{array}{l}1-(bq_{21})(aq_{12}+a_1q'_{12})\\-(cq_{31})(aq_{13}+a_1q'_{13})\\-(dq_{41})(aq_{14}+a_1q'_{14})\end{array}}\right]$$

$$G_4=1-\frac{1}{\mu_4}\left[\frac{\begin{array}{l}(aq_{14}+a_1q'_{14})\lambda_1\\+(aq_{14}+a_1q'_{14})(bq_{21})\lambda_2\\+(aq_{14}+a_1q'_{14})(cq_{31})\lambda_3\\+\begin{array}{l}(1-(bq_{21})(aq_{12}+a_1q'_{12})\\-(cq_{31})(aq_{13}+a_1q'_{13}))\lambda_4\end{array}\end{array}}{\begin{array}{l}1-(bq_{21})(aq_{12}+a_1q'_{12})\\-(cq_{31})(aq_{13}+a_1q'_{13})\\-(dq_{41})(aq_{14}+a_1q'_{14})\end{array}}\right]$$

Solution of the model will be, $P_{m_1m_2m_3m_4}(t) =$ Where $(1-\rho_1)^{m_1}(1-\rho_2)^{m_2}(1-\rho_3)^{m_3}(1-\rho_4)^{m_4}\rho_1\rho_2\rho_3\rho_4$

$$\rho_1 = 1-G_1, \rho_2 = 1-G_2, \rho_3 = 1-G_3,$$
$$\rho_4 = 1-G_4 \text{.................}(*)$$

And solution for the model will exist if $\rho_1, \rho_2, \rho_3, \rho_4 < 0$

6.1. Numerical Illustration

Assuming particular values (crisp values) as,

Table 2. Paticular values (crisp values)

λ_1=5	λ_2=2	λ_3=2	λ_4=1
μ_1=8	μ_2=7	μ_3=6	μ_4=5
m_1=12	m_2=8	m_3=7	m_4=7
q_{12}=0.3 q_{13}=0.3 q_{14}=0.4 q'_{12}=0.2 q'_{13}=0.6 q'_{14}=0.2, a= 0.7 a_1=0.3	q_{21}=0.6 q_2=0.4 q'_2=1 b= 0.6 b_1=0.4	q_{31}=0.6 q_3=0.4 q'_3=1 c= 0.6 c_1=0.4	q_{41}=0.6 q_4=0.4 q'_4=1 d= 0.6 d_1=0.4

we obtain ρ_1 = 0.6597, ρ_2 = 0.4504, ρ_3=0.50875, ρ_4=0.4398

L_1= 1.9385, L_2= 0.8195, L_3= 1.0356, L_4= 0.7850

7. Fuzzified Model

Let us take $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \rho_1, \rho_2, \rho_3, \rho_4, \mu_1, \mu_2, \mu_3, \mu_4$ are approximately known crisp parameters and let us represent them by fuzzy numbers $\widetilde{\lambda_1}$, $\widetilde{\lambda_2}, \widetilde{\lambda_3}$, $\widetilde{\lambda_4}, \widetilde{\rho_1}$, $\widetilde{\rho_2}$, $\widetilde{\rho_3}, \widetilde{\rho_4}, \widetilde{\mu_1}, \widetilde{\mu_2}$, $\widetilde{\mu_3}, \widetilde{\mu_4}$ then from G_1, G_2, G_3, G_4 derived in stochastic environment and from equation (*), fuzzy utilization factor for all the four servers can be written as,

$$\widetilde{\rho_1} = \frac{1}{\widetilde{\mu_1}}\left[\frac{\widetilde{\lambda_1} + \widetilde{\lambda_2}(\tilde{b}\widetilde{q_{21}}) + \widetilde{\lambda_3}(\tilde{c}\widetilde{q_{31}}) + \widetilde{\lambda_4}(\tilde{d}\widetilde{q_{41}})}{1-(\tilde{b}\widetilde{q_{21}})(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}}) + (\tilde{c}\widetilde{q_{31}})(\tilde{b}\widetilde{q_{13}} + \widetilde{b_1}\widetilde{q'_{13}}) + (\tilde{d}\widetilde{q_{41}})(\tilde{c}\widetilde{q_{14}} + \widetilde{c_1}\widetilde{q'_{14}})}\right]$$

$$\widetilde{\rho_2} = \frac{1}{\widetilde{\mu_2}}\left[\frac{\widetilde{\lambda_1}(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}}) + \widetilde{\lambda_2}\begin{pmatrix}1-(\tilde{c}\widetilde{q_{31}})(\tilde{a}\widetilde{q_{13}} + \widetilde{a_1}\widetilde{q'_{13}}) \\ -(\tilde{d}\widetilde{q_{41}})(\tilde{c}\widetilde{q_{14}} + \widetilde{c_1}\widetilde{q'_{14}})\end{pmatrix} + \widetilde{\lambda_3}(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}})(\tilde{c}\widetilde{q_{31}}) + \widetilde{\lambda_4}(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}})(\tilde{d}\widetilde{q_{41}})}{1-(\tilde{b}\widetilde{q_{21}})(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}}) + (\tilde{c}\widetilde{q_{31}})(\tilde{b}\widetilde{q_{13}} + \widetilde{b_1}\widetilde{q'_{13}}) + (\tilde{d}\widetilde{q_{41}})(\tilde{c}\widetilde{q_{14}} + \widetilde{c_1}\widetilde{q'_{14}})}\right]$$

$$\widetilde{\rho_3} = \frac{1}{\widetilde{\mu_3}}\left[\frac{\widetilde{\lambda_1}(\tilde{a}\widetilde{q_{13}} + \widetilde{a_1}\widetilde{q'_{13}}) + \widetilde{\lambda_2}(\tilde{a}\widetilde{q_{13}} + \widetilde{a_1}\widetilde{q'_{13}})(\tilde{b}\widetilde{q_{21}}) + \widetilde{\lambda_3}\begin{pmatrix}1-(\tilde{b}\widetilde{q_{21}})(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}}) \\ -(\tilde{d}\widetilde{q_{41}})(\tilde{c}\widetilde{q_{14}} + \widetilde{c_1}\widetilde{q'_{14}})\end{pmatrix} + \widetilde{\lambda_4}(\tilde{a}\widetilde{q_{13}} + \widetilde{a_1}\widetilde{q'_{13}})(\tilde{d}\widetilde{q_{41}})}{1-(\tilde{b}\widetilde{q_{21}})(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}}) + (\tilde{c}\widetilde{q_{31}})(\tilde{b}\widetilde{q_{13}} + \widetilde{b_1}\widetilde{q'_{13}}) + (\tilde{d}\widetilde{q_{41}})(\tilde{c}\widetilde{q_{14}} + \widetilde{c_1}\widetilde{q'_{14}})}\right]$$

$$\widetilde{\rho_4} = \frac{1}{\widetilde{\mu_3}}\left[\frac{\widetilde{\lambda_1}(\tilde{a}\widetilde{q_{14}} + \widetilde{a_1}\widetilde{q'_{14}}) + \widetilde{\lambda_2}(\tilde{a}\widetilde{q_{14}} + \widetilde{a_1}\widetilde{q'_{14}})(\tilde{b}\widetilde{q_{21}}) + \widetilde{\lambda_3}(\tilde{a}\widetilde{q_{14}} + \widetilde{a_1}\widetilde{q'_{14}})(\tilde{c}\widetilde{q_{31}}) + \widetilde{\lambda_4}\begin{pmatrix}1-(\tilde{b}\widetilde{q_{21}})(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}}) \\ -(\tilde{c}\widetilde{q_{31}})(\tilde{b}\widetilde{q_{13}} + \widetilde{b_1}\widetilde{q'_{13}})\end{pmatrix}}{1-(\tilde{b}\widetilde{q_{21}})(\tilde{a}\widetilde{q_{12}} + \widetilde{a_1}\widetilde{q'_{12}}) + (\tilde{c}\widetilde{q_{31}})(\tilde{b}\widetilde{q_{13}} + \widetilde{b_1}\widetilde{q'_{13}}) + (\tilde{d}\widetilde{q_{41}})(\tilde{c}\widetilde{q_{14}} + \widetilde{c_1}\widetilde{q'_{14}})}\right]$$

Now partial fuzzy queue lengths are,

$$\widetilde{L_1} = \frac{\widetilde{\rho_1}}{1-\widetilde{\rho_1}}, \widetilde{L_2} = \frac{\widetilde{\rho_2}}{1-\widetilde{\rho_2}}, \widetilde{L_3} = \frac{\widetilde{\rho_3}}{1-\widetilde{\rho_3}}, \widetilde{L_4} = \frac{\widetilde{\rho_4}}{1-\widetilde{\rho_4}}$$

Also mean fuzzy queue length = $\tilde{L} = \widetilde{L_1} + \widetilde{L_2} + \widetilde{L_3} + \widetilde{L_4}$

7.1. Numerical Illustration

By taking the value of each fuzzy parameter given in table below satisfying the conditions,

$$\tilde{a}\widetilde{q_{12}} + \tilde{a}\widetilde{q_{13}} + \tilde{a}\widetilde{q_{14}} + \widetilde{a_1}\widetilde{q'_{12}} + \widetilde{a_1}\widetilde{q'_{13}} + \widetilde{a_1}\widetilde{q'_{14}} = 1$$

$$\tilde{b}\widetilde{q_2} + \tilde{b}\widetilde{p_{21}} + \widetilde{b_1}\widetilde{q'_2} = 1$$

$$\tilde{c}\widetilde{q_3} + \tilde{c}\widetilde{p_{31}} + \widetilde{c_1}\widetilde{q'_3} = 1$$

$$\tilde{d}\widetilde{q_4} + \tilde{d}\widetilde{d_{41}} + \widetilde{d_1}\widetilde{q'_4} = 1$$

Table 3. Paticular values of fuzzy parameters

No. of customers	Arrival Rate	Service Rate	Probabilities		
m_1=12	$\widetilde{\lambda_1}$=(5,4,3)	$\widetilde{\mu_1}$=(16,18,20)	$\widetilde{q_{12}}$ =(.4,.3,.2)	$\widetilde{q'_{12}}$ =(.3,.2,.1)	$\tilde{a}$= (.6,.7,.8)
m_2=8	$\widetilde{\lambda_2}$=(3,4,5)	$\widetilde{\mu_2}$=(15,16,17)	$\widetilde{q_{13}}$= (.4,.3,.2)	$\widetilde{q'_{13}}$= (.4,.6,.8)	$\widetilde{a_1}$= (.4,.3,.2)
m_3=7	$\widetilde{\lambda_3}$=(2,3,4)	$\widetilde{\mu_3}$=(14,15,16)	$\widetilde{q_{14}}$ =(.2,.4,.6)	$\widetilde{q'_{14}}$ =(.3,.2,.1)	$\tilde{b}$= (.7,.6,.5)
m_4=7	$\widetilde{\lambda_4}$=(2,3,4)	$\widetilde{\mu_4}$=(14,16,18)	$\widetilde{q_{21}}$ =(.8,.6,.4)	$\widetilde{q'_2}$ =(1,1,1)	$\widetilde{b_1}$= (.3,.4,.5)
			$\widetilde{q_2}$ =(.2,.4,.6)	$\widetilde{q'_3}$ =(1,1,1)	$\tilde{c}$= (.4,.6,.8)
			$\widetilde{q_{31}}$ =(.5,.6,.7)	$\widetilde{q'_4}$=(1,1,1)	$\widetilde{c_1}$= (.6,.4,.2)
			$\widetilde{q_3}$ =(.5,.4,.3)		$\tilde{d}$=(.5,.6,.7)
			$\widetilde{q_{41}}$ =(.4,.6,.8)		$\widetilde{d_1}$=(.5,.4,.3)
			$\widetilde{q_4}$ =(.6,.4,.2)		

Table 4. L-R representation of fuzzy parameters

Arrival Rate	Service Rate	Probabilities		
$\widetilde{\lambda_1}$=(4,1,1)$_{LR}$	$\widetilde{\mu_1}$=(18,2,2)$_{LR}$	$\widetilde{q_{12}}$ =(.3,.1,.1)$_{LR}$	$\widetilde{q'_{12}}$ =(.2,.1,.1)$_{LR}$	$\tilde{a}$= (.7,.1,.1)$_{LR}$
$\widetilde{\lambda_2}$=(4,1,1)$_{LR}$	$\widetilde{\mu_2}$=(16,1,1)$_{LR}$	$\widetilde{q_{13}}$= (.3,.1,.1)$_{LR}$	$\widetilde{q'_{13}}$= (.6,.2,.2)$_{LR}$	$\widetilde{a_1}$= (.3,.1,.1)$_{LR}$
$\widetilde{\lambda_3}$=(3,1,1)$_{LR}$	$\widetilde{\mu_3}$=(15,1,1)$_{LR}$	$\widetilde{q_{14}}$ =(.4,.2,.2)$_{LR}$	$\widetilde{q'_{14}}$ =(.2,.1,.1)$_{LR}$	$\tilde{b}$= (.6,.1,.1)$_{LR}$
$\widetilde{\lambda_4}$=(3,1,1)$_{LR}$	$\widetilde{\mu_4}$=(16,2,2)$_{LR}$	$\widetilde{q_{21}}$ =(.6,.2,.2)$_{LR}$	$\widetilde{q'_2}$ =(1,0.0)$_{LR}$	$\widetilde{b_1}$= (.4,.1,.1)$_{LR}$
		$\widetilde{q_2}$ =(.4,.2,.2)$_{LR}$	q'_3 =(1,0,0)$_{LR}$	$\tilde{c}$= (.6,.2,.2)$_{LR}$
		$\widetilde{q_{31}}$ =(.6,.1,.1)$_{LR}$	q'_4=(1,0,0)$_{LR}$	$\widetilde{c_1}$= (.4,.2,.2)$_{LR}$
		$\widetilde{q_3}$ =(.4,.1,.1)$_{LR}$	$\widetilde{q_{41}}$ =(.6,.2,.2)$_{LR}$	$\tilde{d}$=(.6,.1,.1)$_{LR}$
		$\widetilde{q_4}$ =(.4,.2,.2)$_{LR}$		$\widetilde{d_1}$=(.4,.1,.1)$_{LR}$

Using these numerical values we get,

$$\widetilde{\rho_1}= (0.6597,0.4652,1.2485)_{LR}$$
$$\widetilde{\rho_2} = (0.4211,0.31244,1.15899)_{LR}$$
$$\widetilde{\rho_3} = (0.303125, 0.25581,1.15282)_{LR}$$
$$\widetilde{\rho_4}=(0.4486,0.359848,1.44889)_{LR}$$

Modal values of $\widetilde{\rho_1}$, $\widetilde{\rho_2}$, $\widetilde{\rho_3}$, $\widetilde{\rho_4}$ are 0.6597, 0.4211, 0.303125, 0.4486 and for $\widetilde{L_1}$, $\widetilde{L_2}$, $\widetilde{L_3}$, $\widetilde{L_4}$ are 1.93858, 0.7274, 0.4349, 0.81356 respectively.

Supp($\widetilde{\rho_1}$)=(0.6597-0.4652, 0.6597+1.2485) = (0.1945, 1.9082)

Supp($\widetilde{\rho_2}$)=(0.4211-0.31244, 0.4211+1.15899) = (0.10866, 1.58009)

Supp($\widetilde{\rho_3}$)=(0.303125-0.25581, 0.303125+1.15282) = (0.047315, 1.455945)

Supp($\widetilde{\rho_4}$)=(0.4486-0.359848, 0.4486+1.44889) = (0.088752, 1.89749)

8. Results

- Utilization factor of first server lies between 0.1945 and 1.9082. Most possible value of utilization factor and partial queue length will be 0.6597, 1.93858 respectively.
- Utilization factor of second server lies between 0.10866 and 1.58009. Most possible value of utilization factor and partial queue length will be 0.4211, 0.7274 respectively.
- For third server utilization factor will lie between 0.047315 and 1.455945. Also most possible value of utilization factor and partial queue length for this server will be 0.303125 and 0.4349 respectively.
- Utilization factor for fourth server will be between 0.088752 and 1.89749, where most possible value of utilization factor and partial queue length for this server will be 0.4486 and 0.81356 respectively.

9. Conclusion

In the present work we analyse the behaviour of feedback queues in case of heterogeneous servers and in stochastic environment and fuzzy environment. We calculate utilization factor and partial queue length using probability generating function and classical formulae. In case of fuzzy environment we use L-R method to calculate numerical values of utilization factors and partial queue lengths. Numerical illustrations are given to check the validation of the study. Also since we try to use approximately same crisp and fuzzy data we see that in case of crisp data the percentage usage of four servers are 65.97%, 45%, 50.87%, 43.98% respectively. But in case of fuzzy data the percentage usage of four servers C_1 , C_2 , C_3, C_4 are 65.97%, 42.11%, 30.31%, 44.86% respectively. Form this we can see that there is little bit change in utilization of 1st, 2nd and 4th server but in crisp data 3rd server is utilized 50.87% but in case of fuzzy data utilization is 30.31% only.

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