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Henstock - Kurzweil Integral for Banach Valued Function

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Abstract In this paper, we have studied the Henstock - Kurzweil integral which is a generalized Riemann integral means. Henstock - Kurzweil integral is the natural extension of Riemann integral. We defined Henstock - Kurzweil integral of Banach space valued function with respect to a function of bounded variation which is an extension of real valued Henstock - Kurzweil integral with respect to an increasing function. We investigated elementary properties of the Henstock - Kurzweil integral of Banach space valued function with respect to a function of bounded variation. We proved the convergence theorems and Saks - Henstock lemma of the Henstock - Kurzweil integral of Banach valued functions with respect to a function of bounded vari-ation. Equi-integrability with respect to Banach space valued function is defined and equi-integrable theorem of Henstock - Kurzweil integral of Banach space valued function with respect to a function of bounded variation is proved. Finally Bochner Henstock - Kurzweil integral of Banach valued function with respect to a function of bounded variation is defined and the relation between Bochner Henstock - Kurzweil integral and Henstock - Kurzweil integral is exhibited.

Keywords λ Henstock - Kurzweil Integral, Banach Space, Bounded Variation Function, Bochner Integral

Mathematics Subject Classification 2020: 26A42, 46B07, 26A45.

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1 Introduction

The $\mathcal{HK}$ integral is the generalized Riemann integral, which has similar definition as Riemann integral. $\mathcal{HK}$ integral is just extension of the Riemann integral, $\mathcal{HK}$ integral first introduced by or developed by R. Henstock in 1960. Apostal T.M. in [1] developed integration theory Also firstly the theory of Banach space valued functions for $\mathcal{HK}$ integral is developed by S. S. Cao [2]. In the last two decades many authors studied $\mathcal{HK}$ integral for real valued functions on Banach spaces Kurzweil J [3], Lim J. S., Yoon J. H. and Eun G. S. [4] studied Henstock - Stieltjes integral for Banach space, Alternative definition of $\mathcal{HK}$ with primitives gives in [5], Schwabik S. [6] studied Banach space integration, that all integration theories motivate us to studying $\mathcal{HK}$ integral with respect to Banach space and bounded variation functions. $\mathcal{HK}$ integral or Gauge integration widely studied in Swartz C. [7]. In [8],[9] here integral transform generalized with $\mathcal{HK}$ integral and boundary value problem solved by using $\mathcal{DHK}$, integral in this way$\mathcal{HK}$ integral developed in many areas, In this paper we studied the $\mathcal{HK}$ integral for Banach space valued functions, that function must be bounded variation. Also that function is the more convenient and real valued $\mathcal{HK}$ integrable increasing function. Also some properties of Banach space valued function for $\mathcal{HK}$ integral is given.

2 Basic Terminology and Definitions

In this paper we consider $(X, ||\cdot||_X)$ be a Banach space and X^* its dual.

Asume $\tau = (\delta_i, [x_i, x_{i+1}])_{i=0}^t$ be a finite collection of non overlapping fine tagged intervals in $I_{[p,q]}$. Let $g : I_{[p,q]} \to X$ and $\psi : I_{[p,q]} \to R$ functions are of bounded variation.

We set,

$$\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) = \sum_{i=0}^{t} g(\delta_i)(\psi(x_{i+1}) - \psi(x_i))$$

This is the $\mathcal{HK}$ sum with respect to ψ where ψ is the function of bounded variation.

2.1 Definition of Bounded Variation Function

Let $g : [p, q] \to R$,If $\pi = \{\tau = x_0 < x_1 < x_2 - - - - - < x_n = q\}$ is a partition of [p,q], the variation of g over π is

$$Var(g : \pi) = \sum_{i=0}^{n-1} |g(x_{i+1}) - g(x_i)|$$

and the variation of g over $[p, q]$ is

$$Var(g : [p, q]) = supVar(g : \pi)$$

where, the Supremum is taken over all possible partitions, π of $[p, q]$.
If

$$Var(g : [p, q]) < \infty,$$

then g is said to be a function of bounded variation. The class of all functions of bounded variatons is denoted by $BV[p, q]$

Now we define the $\mathcal{HK}$ integral for Banach valued functions and equiintegrability for the same.

2.2 Definition of Henstock - Kurzweil Integral for Banach valued functions

Let, $\psi : I_{[p,q]} \to R$ be a function of bounded variation, a function $g : [p, q] \to X$ is $\mathcal{HK}$ integrable on $I_{[p,q]}$ w.r.t.ψ if there exists a vector $A \in X$ with for every $\epsilon > 0$ there exist a gauge ξ on $I_{[p,q]}$ s.t.

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - A\|_X < \epsilon$$

when τ is a ξ fine tagged partition of $I_{[p,q]}$. A number I is called the gauge integral of g over $I = [p, q]$ and is denoted by $\int_p^q g$. The function g is $\mathcal{HK}$ integrable on a measurable set $E \subset I$ with respect to ψ if $g\chi_E$ is $\mathcal{HK}$ integrable w.r.t. ψ on I.

2.3 Definition of Equiintegrability

A collection $\Im$ of functions $g : I_{[p,q]} \to X$ is called $\mathcal{HK}$ euiintegrable on $I_{[p,q]}$ w.r.t. ψ if there exist a gauge ξ on $I_{[p,q]}$ s.t. for every $\epsilon > 0$ $\exists$ ξ fine partition τ of $I_{[p,q]}$ s.t.

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int_{I[p,q]} gd\psi\|_X < \epsilon$$

for each $g \in \Im$.

3 Properties of $\mathcal{HK}$ Integral for Banach Space valued functions

Theorem 3.1 *Consider $g : I_{[p,q]} \to X$ is $\mathcal{HK}$ integrable w.r.t. to ψ on I, ψ is defined as $\psi : I_{[p,q]} \to R$ which is of bounded variation, then*

$$\| \int_{I[p,q]} gd\psi\|_X \leq sup_{x\in[p,q]}\|g(x)\|_X BV(\psi, I)$$

Proof. Consider ξ on I such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int_{I_{[p,q]}} gd\psi\|_X < \epsilon$$

where $\epsilon > 0$

$$\begin{aligned}\| \int_{I_{[p,q]}} gd\psi\|_X &= \| \int_{I_{[p,q]}} gd\psi - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) + \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau)\|_X \\ &\leq \|gd\psi - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau)\|_X + \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau)\|_X \\ &< \epsilon + \sum_{i=0}^{p} \|g(\delta_i)\|_X \cdot |\psi(x_{i+1}) - \psi x_i| \\ &\leq \epsilon + sup_{x\in I_{[p,q]}}\|g(x)\|_X \cdot BV(\psi, I)\end{aligned}$$

hence

$$\| \int\limits_{I_{[p,q]}} gd\psi\|_X \leq sup_{x\in[p,q]}\|g(x)\|_X BV(\psi, I)$$

Theorem 3.2

If $g, f : I \to X$ be Henstock-Kurzweil integrable function w.r.t. to ψ on I and $\psi : I \to R$ be a bounded variation, then

a) *$m \in R$, mg is $\mathcal{HK}$ integrable with function of bounded variation ψ on I and*

$$\int_{I_{[p,q]}} mgd\psi = m \int_{I_{[p,q]}} gd\psi$$

b) *$g + f$ is $\mathcal{HK}$ integrable with bounded variation function ψ on I and*

$$\int\limits_{I}(g + f)d\psi = \int\limits_{I_{[p,q]}} gd\psi + \int\limits_{I_{[p,q]}} fd\psi$$

Proof (a)

We have

$g : I_{[p,q]} \to X$ be $\mathcal{HK}$ integrable with respect to function of bounded variation ψ on $I_{[p,q]}$

Case I If $m = 0$ then result is obvious.

Case II If $m \neq 0$, there exist ξ on I for $\epsilon > 0$ such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int\limits_{I_{[p,q]}} gd\psi\|_X < \frac{\epsilon}{|m|},$$

where τ is the ξ tagged partitions of $[p, q]$.

$$\text{Thus, } |m \cdot |\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int\limits_{I_{[p,q]}} gd\psi\|_X < \epsilon$$

Here we write

$$\|\mathcal{S}_{\mathcal{HK}}(mg, d\psi, \tau) - k \int\limits_{I_{[p,q]}} gd\psi\|_X < \epsilon$$

by this equation we say that mg is $\mathcal{HK}$ integrable w.r.t. bounded variation function on $I_{[p,q]}$ and

$$\int\limits_{I_{[p,q]}} mgd\psi = m \int\limits_{I_{[p,q]}} gd\psi$$

Hence proved.

Proof (b)

Let $g, f : I_{[p,q]} \to X$ be $\mathcal{HK}$ integrable w.r.t. function of bounded variation ψ on $I_{[p,q]}$ so there exist ξ on $I_{[p,q]}$, such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \int\limits_{I_{[p,q]}} g d\psi\|_X < \frac{\epsilon}{2}$$

where τ_1 is the ξ_1 fine tagged fine partition of $I_{[p,q]}$ and a ξ_2 on $I_{[p,q]}$. Also

$$\|\mathcal{S}_{\mathcal{HK}}(f, d\psi, \tau_2) - \int\limits_{I_{[p,q]}} f d\psi\|_X < \frac{\epsilon}{2}$$

where τ_2 is the ξ_2 fine tagged fine partition of $I_{[p,q]}$ and a ξ_2 on $I_{[p,q]}$.

Consider ξ, such that

$$\xi(x) = min\{\xi_1(x), \xi_2(x)\}$$

Let τ be ξ fine tagged partition of $I_{[p,q]}$, so that

$$\begin{aligned}
\|\mathcal{S}_{\mathcal{HK}}(g+f, d\psi, \tau) - \left(\int\limits_{I_{[p,q]}} g d\psi + \int\limits_{I_{[p,q]}} f d\psi\right)\|_X &= \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \int\limits_{I_{p,q}} g d\psi \\
&+ \mathcal{HK}(f, d\psi, \tau_2) - \int\limits_{I[p,q]} f d\psi\|_X \\
&\leq \|\mathcal{HK}(g, d\psi, \tau_1) - \int\limits_{I[p,q]} g d\psi\|_X \\
&+ \|\mathcal{HK}(f, d\psi, \tau_2\|_X \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

Hence we write

$$\|\mathcal{S}_{\mathcal{HK}}(g+f, d\psi, \tau) + \left(\int\limits_{I_{[p,q]}} g d\psi + \int\limits_{I_{[p,q]}} f d\psi\right)\|_X < \epsilon$$

therefore $g + f$ is $\mathcal{HK}$ integrable w.r.t. bounded variation function ψ on $I_{[p,q]}$.

Theorem 3.3

Let $g : I_{p,q} \to X, s \in I, \psi$ is the function of bounded variation. If g is $\mathcal{HK}$ integrable on ψ, so each of $[p, s]$ and $[s, q]$, then g is $\mathcal{HK}$ integrable on $I_{[p,q]}$ with respect to ψ and

$$\int\limits_{I_{[p,q]}} g d\psi = \int\limits_p^s g d\psi + \int\limits_s^q g d\psi$$

Proof

Lets $g : I_{[p,q]} \to X$ and $s \in [p, q]$.

Assume that for each of the subintervals $[p, s]$ and $[s, q]$, g is $\mathcal{HK}$ integrable on ψ.

For $\epsilon > 0$ and the subinterval $[p, s]$ there exist ξ_1 such that ,

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \int\limits_p^s g d\psi\|_X < \tfrac{\epsilon}{2} \ ,$$

where τ_1 is ξ_1 fine tagged partition on $[p, s]$ Similarly for $\epsilon > 0$ and the subinterval $[s, q]$ there exist ξ_2 such that;

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2) - \int_s^q g d\psi\|_X < \tfrac{\epsilon}{2},$$

where, τ_2 is ξ_2 fine tagged partition on $[s, q]$.
Consider a gauge ξ in the interval I such that,

$$\begin{aligned}
&\xi(x) = min\{\xi_1(x), s - x\} \qquad if \qquad p \leq x \leq s\\
&\xi(x) = min\{\xi_1(x), \xi_2(x)\} \qquad if \qquad x = s\\
&\xi(x) = min\{x - s, \xi_2(x)\} \qquad when \qquad s < x \leq q
\end{aligned}$$

Here τ be the fine tagged ξ partition on $[p, q]$ and we consider each partition occurs once. So we write

$$\tau = \tau_p \cup (s, [m, n]) \cup \tau_q$$

where $\tau_p < s$ and $\tau_q > s$.
Assume that,

$$\begin{aligned}
\tau_1 &= \tau_p \cup (s, [m, n])\\
\tau_2 &= (s, [m, n]) \cup \tau_q
\end{aligned}$$

so that τ_1 is ξ_1 tagged fine partition on $[p, s]$ and τ_2 is ξ_2 tagged fine partition on $[s, q]$.
Therefore

$$\begin{aligned}
\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) &= \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1 + \tau_2)\\
&= \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) + \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)
\end{aligned}$$

Now, we have

$$\begin{aligned}
\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \left(\int_p^s g d\psi + \int_s^q g d\psi\right)\|_X &= \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) + \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)\\
&\qquad - \int_p^s g d\psi - \int_s^q g d\psi\|_X\\
&\leq \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \int_p^s g d\psi\|_X\\
&\qquad + \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)\\
&\qquad - \int_s^q g d\psi\|_X\\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2}\\
&= \epsilon
\end{aligned}$$

So for $\epsilon > o$, we have gauge ξ fine partition on $[p, q]$ such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \left(\int_p^s g d\psi + \int_s^q g d\psi\right)\|_X < \epsilon$$

Hence

$$\int_{I_{[p,q]}} g d\psi = \int_p^s g d\psi + \int_s^q g d\psi$$

Converse can be proved easily.

Theorem 3.4 *Assume $\psi : I_{[p,q]} \to R$ be a function of bounded variation, $g : I_{[p,q]} \to X$ is $\mathcal{HK}$ integrable with respect to ψ on $I_{[p,q]}$ for every positive ϵ, there exist ξ on the interval $I_{[p,q]}$ such that*

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)\|_X < \epsilon$$

where τ_1 and τ_2 are the fine tagged partitions on $I_{[p,q]}$.

Proof

Let $\epsilon > 0$ and g is the $\mathcal{HK}$ integral with respect to ψ on $I_{[p,q]}$ such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \int\limits_{I_{[p,q]}} g d\psi\|_X < \frac{\epsilon}{2}$$
$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2) - \int\limits_{I_{[p,q]}} g d\psi\|_X < \frac{\epsilon}{2}$$

where τ_1 and τ_2 are the ξ fine tagged partitions on intervals $I_{[p,q]}$

Thus, we obtain

$$\begin{aligned}\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)\|_X &= \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \int\limits_{I[p,q]} g d\psi \\ &+ \int\limits_{I_{[p,q]}} g d\psi - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)\|_X \\ &\leq \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) \\ &\quad - \int\limits_{I_{[p,q]}} g d\psi\|_X + \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2) \\ &\quad - \int\limits_{I_{[p,q]}} g d\psi\|_X \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

Hence

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)\|_X < \epsilon$$

Cauchy criteria is satisfied.

Conversely,

If we consider that the Cauchy criterion is satisfied, we choose gauge ξ_k fine partition of $I_{[p,q]}$ for each positive integer k. such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_1) - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_2)\|_X < \tfrac{1}{k}$$

where τ_1 and τ_2 are ξ_k partitions of $I_{[p,q]}$.

Let us consider non increasing sequence $\{\xi_k\}$, For k, τ_k be ξ_k fine partition of $I_{[p,q]}$.

$l, k \geq K, \quad K \in N$,

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_k) - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_l)\|_X < \tfrac{1}{K}$$

So $\{\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_k)\}$ is Cauchy sequence in X. Hence $\{\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_k)\}$ is convergent.

Assume given sequence have limit L and for a given $\epsilon > 0$ we take positive integer K such that $\frac{1}{K} < \frac{\epsilon}{2}$ and

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_k) - L\|_X < \tfrac{\epsilon}{2} \quad \forall k \geq K$$

where τ be ξ_K fine partition of $I_{[p,q]}$
Thus

$$\begin{aligned}\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - L\|_x &= \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_K) + \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_K) - L\|_X \\ &\leq \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_K)\|_X + \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau_K) - L\|_X \\ &\leq \frac{1}{K} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

Hence

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - L\|_X < \epsilon$$

Hence the function g is $\mathcal{HK}$ integrable with respect to ψ on interval $I_{[p,q]}$.

Theorem 3.5 *Assume $g : I_{[p,q]} \to X$ is a regulated function and $\psi : I_{[p,q]} \to R$ be a function of bounded variation, then $\mathcal{HK}$ integral*

$$\int\limits_{I_{[p,q]}} g d\psi$$

exist.

Proof

Given, $g : I_{[p,q]} \to X$ be regulated function,
For each $\epsilon > 0$ there exist $\tau : p = x_0 < x_1 < - - - - < x_{(t+1)} = q$ of interval $I = [p, q]$, such that

$$\|g(\gamma_j) - g(\eta_j)\|_X < \epsilon.$$

For $\gamma_j, \eta_j \in (x_j, x_{j+1}), j = 0, 1, - - -, t$ define ξ Gauge on $I_{[p,q]}$ and

$$\tau = \{(\gamma_i, [x_j, x_{j+1}])\}_{i=0}^{t}$$

is the ξ fine partition of $I_{[p,q]}$, then

$$\begin{aligned}\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int\limits_{I_{[p,q]}} g d\psi\|_X &= \|\sum_{j=0}^{t} g(\gamma_i)(\psi(x_{j+1}) - \psi(x_j)) - \sum_{j=0}^{t} \int\limits_{x_j}^{x_{j+1}} g d\psi\|_X \\ &\leq \|g(\gamma_i) - g(\eta_i)\|_X \cdot \sup_{X_j, x_{j+1} \in I_{[p,q]}} \sum_{j=0}^{t} |\psi(x_{j+1}) - \psi(x_j)| \\ &< \epsilon BV(\psi, I_{[p,q]}) \\ &= \epsilon_1\end{aligned}$$

there exist ξ gauge on I for each $\epsilon_1 > 0$, such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int\limits_{I_{[p,q]}} g d\psi\|_X < \epsilon_1$$

where

$$\tau = \{(\gamma_j, [x_j, x_{j+1}])\}_{i=0}^{t}$$

which is ξ gauge fine partition on I, thus $\mathcal{HK}$ integral

$$\int\limits_{I_{[p,q]}} g d\psi$$

exists.

4 Convergence Theorems

Theorem 4.1 *Let $\{g_k\}$ be a sequence of $\mathcal{HK}$ integrable functions w.r.t. ψ on $I_{[p,q]}$ and $\psi : I_{[p,q]} \to R$ be a function of bounded variation. Let $\{g_k\}$ converges uniformly to g on $I_{[p,q]}$ ($\lim\limits_{k\to\infty} \|g_k(x)-g(x)\|_X \to 0$) then g is $\mathcal{HK}$ integrable w.r.t. ψ on $I_{[p,q]}$ and*

$$\int_{I_{[p,q]}} g(x)d\psi(x) = \lim_{k\to\infty} \int_{I_{[p,q]}} g_k(x)d\psi(x)$$

Proof
Let $\epsilon > 0$ be given. since sequence $\{g_k\}$ converges to g uniformly on $I_{[p,q]}$,
there exist the positive integer K; for $k > K$ and $x \in I$.

$$\|g_k(x) - g(x)\|_X < \frac{\epsilon}{6[BV(\psi,I)+1]}.$$

For $k, l > K$ and $x \in I$ we have

$$\begin{aligned}\|g_k(x) - g_l(x)\|_x &= \|g_k(x) - g(x) + g(x) - g_l(x)\|_X \\ &\le \|g_k(x) - g(x)\|_X + \|g(x) - g_l(x)\|_X \\ &< \frac{\epsilon}{6[BV(\psi,I)+1]} + \frac{\epsilon}{6[BV(\psi,I)+1]} \\ &< \frac{\epsilon}{3[BV(\psi,I)+1]}\end{aligned}$$

Then

$$\begin{aligned}\|\int_{I_{[p,q]}} g_k(x)d\psi(x) - \int_{I_{[p,q]}} g_l(x)d\psi(x)\|_X &= \|\int_{I_{[p,q]}} [g_k(x) - g_l(x)]d\psi(x)\|_X \\ &\le \sup_{x\to I[p,q]} \|g_k(x) - g_l(x)\|_X \cdot BV(\psi,I) \\ &< \frac{\epsilon}{3[BV(\psi,I)+1]} \cdot BV(\psi,I) \\ &< \frac{\epsilon}{3}\end{aligned}$$

That is

$$\|\int_{I_{[p,q]}} g_k(x)d\psi(x) - \int_{I_{[p,q]}} g_l(x)d\psi(x)\|_X < \frac{\epsilon}{3}$$

Thus $\{g_k\}$ is Cauchy sequence in X, hence convergent.
Consider, $A \in X$,

$$\lim_{k\to\infty} \int_{I_{[p,q]}} g_n(x)d\psi(x) = A$$

and $K_1 \in N$ for $k > K_1$, then

$$\|\int_{I_{[p,q]}} g_k(x)d\psi(x) - A\|_X < \frac{\epsilon}{3}$$

Consider $l > \max\{K, K_1\}$, therefore $\int_{I_{[p,q]}} g_l x d\psi(x)$ integral exist, and
So ξ gauge on $I_{[p,q]}$ exist such that,

$$\|\mathcal{S}_{\mathcal{HK}}(g_l, d\psi, \tau) - \int_{I_{[p,q]}} g_l(x)d\psi(x)\|_X < \frac{\epsilon}{3}$$

where τ is the ξ fine partitions on $I_{[p,q]}$.
Then,

$$\begin{aligned}
\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - A\|_X &= \|\sum_{i=0}^{t} g(\gamma_i)(\psi(x_{i+1}) - \psi(x_i)) - A\|_X \\
&= \|\sum_{i=0}^{t} g(\gamma_i)(\psi(x_{(i+1)} - (\psi(x_i) - \sum_{i=0}^{t} g_l(\gamma_i)(\psi(x_{i+1}) - \psi(x_i)) \\
&+ \sum_{i=0}^{t} g_l(\gamma_i)(\psi(x_{i+1}) - \psi(x_i)) - \int_{I_{[p,q]}} g_l(x) d\psi(x) \\
&+ \int_{I_{[p,q]}} g_l(x), d\psi(x) - A\|_X \\
&\leq \|\sum_{i=0}^{t} g(\gamma_i)(\psi(x_{i+1} - \psi(x_i)) - \sum_{i=0}^{t} g_l(\gamma i)(\psi(x_{i+1}) - \psi(x_i)\|_X \\
&+ \|\sum_{i=0}^{t} g_l(\gamma_i)(\psi(x_{i+1}) - \psi(x_i)) - \int_{I_{[p,q]}} g_l(x) d\psi(x)\|_X \\
&+ \|\int_{I_{[p,q]}} g_l(x) d\psi(x) - A\|_X \\
&= \|\sum_{i=0}^{p} [g(\gamma_i) - g_l(\gamma_i)] 9\psi(x_{i+1)} - \psi(x_i))\|_X + \frac{\epsilon}{3} + \frac{\epsilon}{3}
\end{aligned}$$

Now

$$\begin{aligned}
\|\sum_{i=0}^{t} [g(\gamma_i)](\psi(x_{i+1}) - \psi(x_i))]\|_X &\leq \sum_{i=0}^{t} \|g(\gamma_i) - g_l(\gamma_i)\|_X \cdot |\psi(x_{i+1}) - \psi(x_i)| \\
&\leq max\|g(\gamma_i) - g_l(\gamma_i)\|_X \cdot \sum_{i=0}^{t} |\psi(x_{i+1}) - \psi(x_i)| \\
&= max\|g(\gamma_i) - g_l(\gamma_i)\|_X \cdot Bv(\psi, I) \\
&< \frac{\epsilon}{3}
\end{aligned}$$

Hence

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - A\|_X < \epsilon$$

Thus Henstock -Kurzweil integral exist and

$$\text{Let} \int_{I_{[p,q]}} g(x) d\psi(x) = \lim_{k \to \infty} \int_{I_{[p,q]}} g_k(x) d\psi(x)$$

Theorem 4.2 *Let $\{g_k\}$ be a sequence of X -valued $\mathcal{HK}$ integrable function w.r.t.ψ on $I_{[p,q]}$, ψ is function of Bounded variation*

$$\psi : I_{[p,q]} \to R$$

If $\{g_k\}$ converges point wise to g on $I_{[p,q]}$, then g is $\mathcal{HK}$ integrable w.r.t. ψ on I

$$\lim_{k \to \infty} \int_{I_{[p,q]}} g_k d\psi = \int_{I_{[p,q]}} g d\psi$$

Proof
By using the definition of the HK integrability of $\{g_k\}$ w.r.t. the bounded variation function ψ,
we get

$$\|\mathcal{S}_{\mathcal{HK}}(g_k, d\psi, \tau) - \int\limits_{I_{[p,q]}} g_k d\psi\|_X < \epsilon \text{ for all } k \in N$$

where; τ is the ξ fine partition on $I_{[p,q]}$
Let g which is the limit of $\{g_k\}$, The pointwise convergence $g_k \to g$ yields, the sequence $\mathcal{S}_{\mathcal{HK}}(g_k, d\psi, \tau)$ if the partition τ is fixed and that sequence converges to $\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau)$.
Then we got,

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int\limits_{I[p,q]} g d\psi\|_X < \epsilon$$

Also

$$\begin{aligned}\|\int\limits_{I_{[p,q]}} g_k d\psi - \int\limits_{I_{[p,q]}} g d\psi\|_X &\leq \|\int\limits_{I_{[p,q]}} g_k d\psi - \mathcal{S}_{\mathcal{HK}}(g_k, d\psi, \tau)\|_X + \|\mathcal{S}_{\mathcal{HK}}(g_{k,d\psi,\tau}) \\ &- \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau)\|_X + \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int\limits_{I_{[p,q]}} g d\psi\|_X \\ &< \epsilon + \epsilon + \epsilon \\ &< 3\epsilon\end{aligned}$$

for k
Since

$$\|\int\limits_{I_{[p,q]}} g_k d\psi - \int\limits_{I_{[p,q]}} g d\psi\|_X < 3\epsilon$$

for k
Hence

$$\lim_{k\to\infty} \int\limits_{I_{[p,q]}} g_k d\psi = \int\limits_{I_{[p,q]}} g d\psi.$$

Theorem 4.3 *Let $g : I_{[p,q]} \to X$ be $\mathcal{HK}$ integrable w.r.t.ψ on $I_{[p,q]}$ and $\psi : I_{[p,q]} \to R$ is of bounded variation then for any $x^* \in X^*$, the $x^* f$ is the $\mathcal{HK}$ integrable w.r.t. ψ on $I_{[p,q]}$ and*

$$\int\limits_{I_{[p,q]}} x^* g d\psi = x^* \int\limits_{I_{[p,q]}} g d\psi$$

also if $B(X^*)$ is the unit ball in X^*
then $\{x^* g : x^* \in B(X^*)\}$ is $\mathcal{HK}$ equi - integrable with respect to ψ on I

Proof
Let $g : I_{[p,q]} \to X$ be $\mathcal{HK}$ integrable on I with respect to ψ then there is ξ gague on I, such that

$$\|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - (\int\limits_{I_{[p,q]}} g d\psi)\|_X < \epsilon$$

where τ is the ξ fine partition on $I_{[p,q]}$, we have $x^* \in X^{*'}$ then

$$\begin{aligned}\|\mathcal{S}_{\mathcal{HK}}(x^* g, d\psi, \tau) - x^* \int\limits_{I_{[p,q]}} g d\psi\|_X &= \|x^*\|_{X^*} \|\mathcal{S}_{\mathcal{HK}}(g, d\psi, \tau) - \int\limits_{I_{[p,q]}} g d\psi\|_X \\ &< \|x^*\|_{X^*\epsilon}\end{aligned}$$

Then there exist ξ Gauge on $I_{[p,q]}$ such that

$$\|\mathcal{S}_{\mathcal{HK}}(x^*g, d\psi, \tau) - x^* \int\limits_{I_{[p,q]}} g d\psi\|_X < \epsilon^{'}$$

where τ is the ξ fine partition of $I[p,q]$ so the x^*g is the $\mathcal{HK}$ integrable on $I_{[p,q]}$ with respect to ψ,Also if $x^* \in B(X^*)$ then

$$\|\mathcal{S}_{\mathcal{HK}}(x^*g, d\psi, \tau) - x^* \int\limits_{I_{[p,q]}} g d\psi\|_X < \epsilon$$

for any $x^* \in B(X^*)$
Hence $\{x^*f : x^* \in B(X^*)\}$ is $\mathcal{HK}$ equi - integrable on $I_{[p,q]}$ with respect to ψ.

5 Relation Between $\mathcal{HK}$ Integral and Bochner Integral

5.1 Definition of Bochner $\mathcal{HK}$ Integral

Let The measurable function $g : I_{[p,q]} \to X$ is called Bochner $\mathcal{HK}$ integrable for Banach valued function w.r.t. ψ on $I_{[p,q]}$ if there exist a sequence $\{g_k\}$ of simple functions such that $\lim\limits_{k\to\infty} \int_{I_{[p,q]}} \|g_k - g\|_X d\psi = 0$ then $\int\limits_{I_{[p,q]}} f d\psi = \lim\limits_{k\to\infty} \int\limits_{I_{[p,q]}} g_k d\psi$.
Assume that the integral of g w.r.t. ψ is independent on the sequence $\{g_k\}$.

Theorem 5.1 *Let $\psi : I_{[p,q]} \to R$ be a function of bounded variation. A function $g : I_{[p,q]} \to X$ is $\mathcal{HK}$ integrable with respect to ψ if and only if it is Bochner - $\mathcal{HK}$ integrable with respect to ψ and the integral coincides.*

Proof
Let a function $g : I_{[p,q]} \to X$ be integrable if and only if it is a.e. and uniform limit of a sequence of countably valued measurable functions with respect to ψ.
By this we consider a sequence of simple functions $\{f_k\}$ by assuming

$$\int\limits_{I[p,q]} \|g - f_k\|_X d\psi < \frac{1}{k}$$

and

$$\|f_k\|_X \leq \|g\|_X + \frac{1}{k} \qquad \text{for } k \geq 1$$

Since

$$\lim_{k\to\infty} \int\limits_{I_{[p,q]}} \|g - f_k\|_X d\psi = 0$$

We consider a subsequence $\{f_{k_z}\}$ such that $\{f_{k_z}\} \to g$
Now $\{f_{k_z}\}$ are $\mathcal{HK}$ integrable and

$$\|\{f_{k_z}\}\|_X \leq \|g\|_X + \frac{1}{k_z}$$

By using dominated convergence theorem g is integrable with respect to ψ thus g is measurable and $\|g\|_X$ is Lebesgue - integrable with respect to ψ.

6 Conclusion

Defined Henstock - Kurzweil integral of Banach space valued function with respect to a function of bounded variation, investigated elementary properties of Henstock - Kurzweil integral, proved convergence theorems, equi-integrability, Saks - Henstock lemma and finally exhibit relation between Henstock - Kurzweil integral and Bochner Henstock - Kurzweil integral.

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