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Mathematical Analysis of Dynamic Models of Suspension Bridges with Delayed Damping

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Abstract Suspension bridges are a type of construction in which the deck is suspended under a series of suspension cables that are on vertical hangers. The first modern example of this project began to appear in the early 1800s. Modern suspension bridges are lightweight, aesthetically pleasing and can span longer distances than any other bridge form. Many papers have been devoted to the modelling of suspension bridges, for instance, Lazer and McKenna studied the problem of nonlinear oscillation in a suspension bridge. They introduced a (one-dimensional) mathematical model for the bridge that takes into account of the fact that the coupling provided by the stays connecting the main cable to the deck of the road bed is fundamentally nonlinear, that is, they gave rise to the system of semi linear hyperbolic equation, where the first equation describes the vibration of the road bed in the vertical plain and the second equation describes that of the main cable from which the road bed is suspended by the tie cables. Recently, interest in this field has been increasing at a high rate. In this paper, we investigate some mathematical models of suspension bridges with a strong delay in linear aerodynamic resistance force. We establish the exponential decay of the solution for the corresponding homogeneous system and prove the existence of an absorbing set as well as a bounded attractor.

Keywords Suspension Bridges, Dynamic Models, Stability, Strong Delay, Absorbing Set

AMS Subject Classifications 93D05, 58D25, 35Q72

1 Introduction

The mathematical model of the suspension bridge proposed by Lazer and McKenna [1, 2] is brought to the system of hyperbolic equations with a specific form, one of which is of the fourth-order and the other of the second-order. There are several research studies on various issues for the dynamical system generated by the corresponding system describing the motion of a suspension bridge, including the asymptotics of the energy function and the existence of a bounded absorbing set and attractor (see [3, 4, 6, 10, 11]).

In recent years, increasing attention has been focused on the study of differential equations with a time-varying delay (see [6, 7, 8]). Time delays occur in many applications, since in most cases the state of the system naturally depends not only on the current physical, chemical and thermal parameters, but also on some past values of these parameters. In fact, it has been shown in many cases that delay can be a source of instability, and even an arbitrarily small delay can destabilise a system that is uniformly asymptotically stable in the absence of delay, unless additional conditions are met [9]. Therefore, the study of differential equations with delay effects is both of theoretical and practical importance.

This article is a continuation of the work [17], where the existence and uniqueness of the solution of the corresponding mixed problem for systems of suspension bridge equations with a strong delay in the linear aerodynamic resistance forces are investigated. The purpose of this article is to obtain a result on the absorbing set and global attractor for system of suspension bridge equations with time delay. For more information on global attractors for systems of bridge equations without delay, see [3, 4, 5, 6, 11].

The scope of this paper is as follows: In Section 2, some notation and definitions are given, and a theorem on the existence of global solutions is formulated; in Section 3, the problem under consideration is reduced to an operator-differential equation and a scheme for proving existence theorems is given; in Section 4, relations between certain energy functionals are obtained; in Section 5, exponential decay solutions of the corresponding homogeneous linear system are shown, in Section 6, the existence of an absorbing set for the corresponding non-linear semigroup is established, in Section 7, the existence of a minimal global attractor is proved.

2 Statement of the problem and the main results

We consider the following mathematical model for the oscillations of the bridge with strong delay

$$u_{tt}(x,t)+u_{xxxx}(x,t)+[u-v]_+ +\lambda_1 u_t(x,t)+\lambda_2 u_t(x,t-\tau_1)+$$
$$+g_1(x,u(x,t),v(x,t))=h_1(x,t), \tag{2.1}$$
$$v_{tt}(x,t)-v_{xx}(x,t)-[u-v]_+ +\mu_1 v_t(x,t)+ +\mu_2 v_t(x,t-\tau_2)+$$
$$+g_2(x,u(x,t),v(x,t))=h_2(x,t), \tag{2.2}$$

where $0<x<l,\ t>0$, $u(x,t)$ is state function of the road bed and $v(x,t)$ is that of the main cable; $\tau_1,\ \tau_2>0$ represents the time delay, λ_1, λ_2, μ_1, μ_2 are real numbers, and $g_1(u,v)=\lambda(x)|u|^{p-1}|v|^{p+1}u$, $g_2(u,v)=\lambda(x)|u|^{p+1}|v|^{p-1}v$, $p\geq 1$; $\lambda(\cdot)\in C[0,l],\ \lambda(x)\geq 0,\ 0\leq x\leq l$, $h_1(\cdot), h_2(\cdot)$ are real-valued functions. Here $[a]_+=\max\{a,0\}$.
Let's define the following initial and boundary conditions for the system (2.1),(2.2).

$$u(0,t)=u_{xx}(0,t)=u(l,t)=u_{xx}(l,t)=v(0,t)=v(l,t)=0,\ t>0, \tag{2.3}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_1(x),\quad x\in(0,l), \tag{2.4}$$

$$u_t(x,t-\tau_1)=f_1(x,t-\tau_1),\ \ x\in(0,l),\ \ t\in(0,\tau_1), \tag{2.5}$$

$$v(x,0)=v_0(x), v'(x,0)=v_1(x),\ \ x\in(0,l), \tag{2.6}$$

$$v_t(x,t-\tau_2)=f_2(x,t-\tau_2),\ \ x\in(0,l),\ \ t\in(0,\tau_2). \tag{2.7}$$

For investigating the problem (2.1)-(2.7), we introduce the following notations:

$$\mathrm{H}^{\mathrm{k}}(0,l)=\left\{y:\ y,y',...,y^{(k)}\in L_2(0,l)\right\},$$

$$\widehat{\mathrm{H}}^{\mathrm{k}}=\widehat{\mathrm{H}}^{\mathrm{k}}(0,l)=\left\{y:\ \ y\in \mathrm{H}^{\mathrm{k}}(0,l),\ \ \ y^{(2s)}(0)=y^{(2s)}(l)=0,\ \ \ s=0,1,...,\left[\frac{k-1}{2}\right]\right\},$$

where k=1,2, ... $[r]$ is the integer part of the number r .

The following theorem about the existence and uniqueness of the global solution to the problem (2.1)-(2.7) is true. We will give the proof scheme of this theorem in Section 3.

Theorem 2.1. *Assume that*

$$\lambda_i,\ \mu_i\in\mathbb{R},\ i=1,2 \tag{2.8}$$
$$h_i(\cdot)\in W_2^1([0,+\infty);L_2(0,l)),\ i=1,2. \tag{2.9}$$

Then for any $u_0\in\widehat{\mathrm{H}}^2$, $u_1\in L_2(0,1)$, $v_0\in\widehat{\mathrm{H}}^1$, $v_1\in L_2(0,1)$,
$f_{0i}(\cdot,-\cdot\tau_1)\in L_2((0,l)\times(0,1)),\ i=1,2$ *, the problem* (2.1)-(2.7) *has a unique solution* $(\mathrm{u(x,t)},v(x,t))$ *where*

$$\mathrm{u}(\cdot)\in \mathrm{C}([0,+\infty),\widehat{\mathrm{H}}^2)\cap \mathrm{C}^1([0,+\infty),L_2(0,1)),$$

$$v(\cdot)\in \mathrm{C}([0,+\infty),\widehat{\mathrm{H}}^1)\cap \mathrm{C}^1([0,+\infty),L_2(0,1)),$$

Moreover, if $u_0 \in \widehat{\mathrm{H}}^4$, $u_1 \in \widehat{\mathrm{H}}^2$, $v_0 \in \widehat{\mathrm{H}}^2$, $v_1 \in \widehat{\mathrm{H}}^1$, $f_{0i}(\cdot, -\cdot\tau_1) \in L_2((0,l) \times (0,1))$, $f_{0i\rho}(\cdot, -\cdot\tau_1) \in L_2((0,l) \times (0,1))$, $i = 1, 2$, *then the solution of* (2.1),(2.2) *satisfies*

$$\mathrm{u}(\cdot)\in\mathrm{C}([0,+\infty), \widehat{\mathrm{H}}^4)\cap\mathrm{C}^1([0,+\infty), \widehat{\mathrm{H}}^2)\cap\mathrm{C}^2([0,+\infty), L_2(0,1)),$$

$$v(\cdot)\in\mathrm{C}([0,+\infty), \widehat{\mathrm{H}}^2)\cap\mathrm{C}^1([0,+\infty), L_2(0,1))\cap\mathrm{C}^2([0,+\infty), L_2(0,1)).$$

The main goal of this paper is to investigate the behavior of energy function

$$E(t) = \frac{1}{2}\int_0^l {u_t}^2(x,t)\,dx + \frac{1}{2}\int_0^l u_{xx}^2(x,t)\,dx + \frac{1}{2}\int_0^l v_t^2(x,t)\,dx +$$

$$+\frac{1}{2}\int_0^l v_x^2(x,t)\,dx + \frac{\tau_1|\lambda_2|}{2}\int_{t-\tau_1}^t\int_0^l {u_s}^2(x,s)\,dxds +$$

$$+\frac{\tau_2|\mu_2|}{2}\int_{t-\tau_2}^t\int_0^l v_s^2(x,s)\,dx, \tag{2.10}$$

to the problem (2.1)-(2.7) as $t \to +\infty$; where $u_s(x,s) = f_1(x,s)$, $v_s(x,s) = f_2(x,s)\ \ if\ \ 0 \le x \le l,\ -\tau_1 < s < 0$.
Let us first recall some well-known concepts in the literature [3, 6, 10, 11, 14, 16]. Assume that the X is some Banach space, $\{S_t,\ t \ge 0\}$ is a strongly continuous semigroup, that is, semigroup of type C_0 in a Banach space X.

Definition 1. [19] A bounded closed subset $A \subset X$ is called a global attractor for (X, S_t), if it is fully invariant and uniformly attracting, that is, $S_t(\mathrm{A}) = \mathrm{A}$ for all $\mathrm{t} > 0$ and for every bounded subset $\mathrm{B}\subset\mathrm{X}$, $\lim_{t\to+\infty}\mathrm{dist}_X(S_t(B), \mathrm{A}) = 0$ where $\mathrm{dist}_X(S_t(B), \mathrm{A}) = \sup_{y\in S_t(B)}\inf_{z\in\mathrm{A}}\|y-z\|_X$ is the Hausdorff semidistance between $S_t(B)$ and A in X.

Definition 2. [19] A bounded set $\mathrm{B}_0 \subset \mathrm{A}$ is a bounded absorbing set for $\{S_t,\ t \ge 0\}$ if every bounded set $\mathrm{B}\subset\mathrm{X}$ there exists t_0 such that from $\mathrm{t} \ge \mathrm{t}_0$ implies $\mathrm{S}_t(\mathrm{B})\subset\mathrm{B}_0$.

Definition 3. [19] A semigroup S_t is said to be exponentially stabilizable if there exist positive numbers $M > 0$ and $\omega > 0$, such that

$$\|S_t(w)\|_X \le Me^{-\omega t}\|w\|_X,\ \ t > 0$$

3 Reduction of the system to the operator equation. Proof scheme of Theorem (2.1).

As in Dafermos [7, 8] , we introduce the new variables:

$$z_1(x,\rho,t) = u_t(x, t-\tau_1\rho), \rho \in (0,1), x \in (0,l), t > 0$$

$$z_2(x,\rho,t) = v_t(x, t-\tau_2\rho), \rho \in (0,1), x \in (0,l), t > 0$$

Obviously, z_1 and z_2 are solutions to the following problems:

$$\tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0 \quad z_1(x,\rho,0) = f_1(x, -\rho\tau_1), \tag{3.1}$$

$$\tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0, \quad z_2(x,\rho,0) = f_2(x, -\rho\tau_2) \tag{3.2}$$

where $\rho \in (0,1),\ x \in (0,l),\ t > 0$. Thus, problem (2.1)-(2.7) is reduced to a mixed problem for systems

$$u_{tt}(x,t) + u_{xxxx}(x,t) + [u-v]_+ + \lambda_1 u_t(x,t) + \lambda_2 z_1(x,1,t) +$$

$$+g_1(u(x,t), v(x,t)) = h_1(x,t), \qquad x \in (0,l),\ \ t > 0 \tag{3.3}$$

$$v_{tt}(x,t) - v_{xx}(x,t) - [u-v]_+ + \mu_1 v_t(x,t) + \mu_2 z_2(x,1,t) +$$

$$+g_2(u(x,t), v(x,t)) = h_2(x,t), \qquad x \in (0,l),\ \ t > 0 \tag{3.4}$$

$$\tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0, \qquad \rho \in (0,1),\ x \in (0,l),\ t > 0 \tag{3.5}$$

$$\tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0, \qquad \rho \in (0,1),\ x \in (0,l),\ t > 0 \tag{3.6}$$

with boundary conditions

$$u(0,t) = u_{xx}(0,t) = u(l,t) = u_{xx}(l,t) = 0, \quad t > 0 \tag{3.7}$$

$$v(0,t) = v(l,t) = 0, \quad t > 0 \tag{3.8}$$

$$z_i(0,\rho,t) = z_i(l,\rho,t) = 0, \quad t > 0 \tag{3.9}$$

$$z_2(0,\rho,t) = z_2(l,\rho,t) = 0, \quad t > 0 \tag{3.10}$$

and initial conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \;\; x \in (0,l), \tag{3.11}$$

$$z_1(x,\rho,0) = f_1(x, -\rho\tau_1), \quad x \in (0,l), \;\rho \in (0,1), \tag{3.12}$$

$$v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \;\; x \in (0,l), \tag{3.13}$$

$$z_2(x,\rho,0) = f_2(x, -\rho\tau_2), \;\; x \in (0,l), \;\rho \in (0,1)\,. \tag{3.14}$$

The problem (3.1)-(3.14) will be studied in the space

$$\mathcal{H} = \widehat{\mathrm{H}}^2 \times L_2(0,l) \times \widehat{\mathrm{H}}^1 \times L_2(0,l) \times L_2((0,1)\times(0,l)) \;\; \times L_2((0,1)\times(0,l)).$$

$\mathcal{H}$ is a Hilbert space equipped with the inner product

$$\langle \omega, \widetilde{\omega} \rangle = \int_0^l u_{1xx}\tilde{u}_{1xx}dx + \int_0^l u_2\tilde{u}_2dx + \int_0^l u_3\tilde{u}_3dx + \int_0^l u_4\tilde{u}_4dx +$$

$$+\tau\,|\lambda_2| \int_0^l \int_0^1 z_1\tilde{z}_1 d\rho dx + \tau\,|\mu_2| \int_0^l \int_0^1 z_2\tilde{z}_2 d\rho dx,$$

where $\omega = (u_1,\; u_2, u_3, u_4,\; z_1, z_2)^T, \widetilde{\omega} = (\;\tilde{u}_1,\; \tilde{u}_2,\;\; \tilde{u}_3,\;\; \tilde{u}_4,\; \tilde{z}_1, \widetilde{z_2})^T \in \mathcal{H}$.

In the space $\mathcal{H}$, we define a linear operator A_0 :

$$D(A_0) = \Big\{\omega : \omega = (u_1, u_2, u_3, u_4, z_1, z_2)^T \in \mathcal{H}, u_1 \in \widehat{H}^4, u_2 \in \widehat{H}^2, u_3 \in \widehat{H}^2,$$

$$u_4 \in \widehat{H}^1, z_i, z_{i\rho} \in L_2((0,1) \times L_2(0,l)),\; z_i(\cdot,1) \in L_2((0,1)\times(0,l)),\; i = 1,2.\Big\},$$

$$A_0\omega = (-u_2, u_{1xxxx} + \lambda_1 u_2 + \lambda_2 z_1(\cdot,1)\,,$$

$$-u_4, -\,u_{3xx} + \mu_1 u_4 + \mu_2 z_2(\cdot,1)\,,\; \frac{1}{\tau_1}z_{1\rho},\; \frac{1}{\tau_2}z_{2\rho}),$$

$$\omega = (u_1, u_2, u_3, u_4, z_1, z_2)^T \in D(A_0).$$

We also define the nonlinear operators $A_1(\cdot)$ and $F(\cdot)$, acting from $\mathcal{H}$ to $\mathcal{H}$, respectively:

$$A_1(\omega) = (0, [u_1 - u_3]_+,\; 0,\; -\,[u_1 - u_3]_+, 0\;, 0),$$

$$F(t,\omega) = (0,\; g_1(\cdot,u,v) + h_1(\cdot,t),\; 0,\; g_2(\cdot,u,v) + h_2(\cdot,t),\; 0,\; 0).$$

The problem (3.3)-(3.14) can be written as the Cauchy problem in the Hilbert space $\mathcal{H}$:

$$\begin{cases} \omega^{'} + A_0\omega + A_1(\omega) + F(\omega) = 0, \\ \omega(0) = \omega_0, \end{cases} \tag{3.15}$$

where

$$\omega = \omega(t) = (u_1(t),\; u_2(t), u_3(t), u_4(t),\; z_1(t), z_2(t))^T,$$

$$\omega(0) = \omega_0 = (u_{10},\; u_{20},\; u_{30}, u_{40},\; z_1(\cdot - \rho\tau), z_2(\cdot - \rho\tau)),$$

$$u_1(t) = u(\cdot t)\,, u_2(t) = u_t(\cdot t)\,, u_3(t) = v(\cdot t)\,, u_4(t) = v_t(\cdot t)\,,$$

$$z_1(t) = z_1(\cdot,t), z_2(t) = z_2(\cdot,t).$$

Using the methods of operator-differential equations (see [12, 13, 15, 16]), we obtain the following statement about the existence and uniqueness of the solution to problem (3.15):

Theorem 3.1. . *Assume that conditions* (2.8) *and* (2.9) *are satisfied.*
Then for any $\omega_0 \in \mathcal{H}$ *, there exists* $T' > 0$ *such that the problem* (3.15) *has a unique solution*

$$\omega(\cdot)\in \mathrm{C}(\,[0,T']\,,\ \mathcal{H}\,).$$

Moreover, if $\omega_0 \in D\,(A_0)$*, then*

$$\omega(\cdot) \in C^1(\,[0,\mathrm{T}']\,,\ \mathcal{H}\,)\cap \mathrm{C}(\,[0,T']\,,D\,(A_0)\,).$$

If $T_{max}= \sup \mathrm{T}'$*, i.e.,* T_{max} *is the length of the maximal existence interval of the solution* $\omega(\cdot)\in \mathrm{C}(\,[0,T_{max})]\,,\ \mathcal{H}\,)$*, then either (i)* $T_{max} = +\infty$*, or (ii)* $T_{max}\limsup_{t\to+\infty} \|\omega(\mathrm{t})\|_{\mathcal{H}} = +\infty$

We don't give a proof of this theorem, because it is carried out in the same way as in the proof of Theorem 2.1 in [17].
According to Theorem (3.1), if we obtain a priori estimate

$$\|\omega(\mathrm{t})\|_{\mathcal{H}} \le c,\ \ t \in [0, T_{max}), \tag{3.16}$$

then the local solution can be extended to the global one.
It follows from the definition of the functions $z_1\,(x,\rho,t)$ and $z_2\,(x,\rho,t)$ that

$$E\,(t) = \frac{1}{2}\int_0^l {u_t}^2\,(x,t)\,dx + \frac{1}{2}\int_0^l u_{xx}^2\,(x,t)\,dx\ + \frac{1}{2}\int_0^l v_t^2\,(x,t)\,dx +$$
$$+\frac{1}{2}\int_0^l v_x^2\,(x,t)\,dx + \frac{\tau_1\,|\lambda_2|}{2}\int_0^l\int_0^1 z_1^2 d\rho dx + \frac{\tau_2\,|\mu_2|}{2}\int_0^l\int_0^1 z_2^2 d\rho dx$$

Using integration by parts and recalling boundary conditions from (3.3)-(3.6) we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_0^l {u_t}^2\,(x,t)\,dx + \frac{1}{2}\frac{d}{dt}\int_0^l u_{xx}^2\,(x,t)\,dx + \lambda_1\int_0^l u_t^2\,(x,t)\,dx +$$
$$+\int_0^l (u\,(x,t) - \upsilon\,(x,t))_+\,u_t\,(x,t) + \lambda_2\int_0^l z_1\,(x,1,t)\,u_t\,(x,t)\,dx =$$
$$= \int_0^l \tilde{g}_1\,(x,t)\,u_t\,(x,t)\,dx \tag{3.17}$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^l v_t^2\,(x,t)\,dx + \frac{1}{2}\frac{d}{dt}\int_0^l v_x^2\,(x,t)\,dx\ + \mu_1\int_0^l v_t^2\,(x,t)\,dx -$$
$$-\int_0^l (u\,(x,t) - \upsilon\,(x,t))_+\,\upsilon_t\,(x,t) + \mu_2\int_0^l z_2\,(x,1,t)\,\upsilon_t\,(x,t)\,dx =$$
$$= \int_0^l \tilde{g}_2\,(x,t)\,\upsilon_t\,(x,t)\,dx \tag{3.18}$$

where $\tilde{g}_i\,(x,t) = -\ g_i(x,u\,(x,t)\,,v\,(x,t)) + h_i(x,t),\ x \in\ (0,l)\,,\ t \in (0,\infty)\,,$
$i = 1, 2.$
Analogously, multiplying (3.5) by $|\lambda_2|\,z_1$, (3.6) by $|\mu_2|\,z_2$ and integrating both over $(0,1)\times(0,l)$, using integration by parts, we obtain respectively

$$\frac{\tau_1\,|\lambda_2|}{2}\frac{d}{dt}\int_0^l\int_0^1 z_1^2\,(x,\rho,t)\,d\rho dx + \frac{|\lambda_2|}{2}\int_0^l z_1^2\,(x,1,t)\,dx - \frac{|\lambda_2|}{2}\int_0^l u_t^2\,(x,t)\,dx = 0, \tag{3.19}$$

$$\frac{\tau_2\,|\mu_2|}{2}\frac{d}{dt}\int_0^l\int_0^1 z_2^2\,(x,\rho,t)\,d\rho dx + \frac{|\mu_2|}{2}\int_0^l z_2^2\,(x,1,t)\,dx - \frac{|\mu_2|}{2}\int_0^l v_t^2\,(x,t)\,dx = 0. \tag{3.20}$$

Summing up (3.17)-(3.20), we have

$$\frac{d}{dt}E_1(t) = -(\lambda_1 - \frac{|\lambda_2|}{2})\int_0^l u_t^2\,(x,t)\,dx - (\mu_1 - \frac{|\mu_2|}{2})\int_0^l v_t^2\,(x,t)\,dx +$$
$$-\lambda_2\int_0^l z_1\,(x,1,t)\,u_t\,(x,t)\,dx - \mu_2\int_0^l z_2\,(x,1,t)\,v_t\,(x,t)\,dx -$$
$$-\frac{|\lambda_2|}{2}\int_0^l z_1^2\,(x,1,t)\,dx - \frac{|\mu_2|}{2}\int_0^l z_2^2\,(x,1,t)\,dx + +$$

$$+\int_0^l \tilde{g}_1(x,t)\,u_t(x,t)\,dx+\int_0^l \tilde{g}_2(x,t)\,v_t(x,t)\,dx. \tag{3.21}$$

where $E_1(t)=E(t)+\frac{1}{2}\int_0^l \left|(u(x,t)-v(x,t))_+\right|^2 dx.$

It is clear from the expression of the functions $\tilde{g}_1(u,v)$ and $\tilde{g}_2(u,v)$ that

$$\int_0^l \tilde{g}_1(x,u(x,t),v(x,t))\,u_t(x,t)\,dx+\int_0^l \tilde{g}_2(x,u(x,t),v(x,t))\,v_t(x,t)\,dx=$$

$$=-\frac{1}{p+1}\frac{d}{dt}\int_0^l \lambda(x)|u(x,t)v(x,t)|^{p+1}dx+\int_0^l h_1(x,t)\,u_t(x,t)\,dx+$$

$$+\int_0^l h_2(x,t)\,v_t(x,t)\,dx. \tag{3.22}$$

By virtue of Hölder's and Young's inequality[], the following inequalities are true

$$\left|\int_0^l h_1(x,t)\,u_t(x,t)\,dx\right|\le\frac{1}{2}\int_0^l u_t^2(x,t)\,dx+\frac{1}{2}\int_0^l h_1^2(x,t)\,dx, \tag{3.23}$$

$$\left|\int_0^l h_2(x,t)\,v_t(x,t)\,dx\right|\le\frac{1}{2}\int_0^l v_t^2(x,t)\,dx+\frac{1}{2}\int_0^l h_2^2(x,t)\,dx. \tag{3.24}$$

$$\left\lceil\int_0^l z_1(x,1,t)\,u_t(x,t)\,dx\right\rceil\le\frac{1}{2}\int_0^l z_1^2(x,1,t)\,dx+\frac{1}{2}\int_0^l u_t^2(x,t)\,dx, \tag{3.25}$$

$$\left\lceil\mu_2\int_0^l z_2(x,1,t)\,v_t(x,t)\,dx\right\rceil\le\frac{1}{2}\int_0^l z_2^2(x,1,t)\,dx+\frac{1}{2}\int_0^l v_t^2(x,t)\,dx. \tag{3.26}$$

Denoting $E_2(t)=E_1(t)+\frac{1}{p+1}\frac{d}{dt}\int_0^l \lambda(x)|u(x,t)v(x,t)|^{p+1}dx$, from
(3.22)-(3.26) we have

$$\frac{d}{dt}E_2(t)=(\frac{1}{2}-\lambda_1+|\lambda_2|)\int_0^l u_t^2(x,t)\,dx+(\frac{1}{2}-\mu_1+|\mu_2|)\int_0^l v_t^2(x,t)\,dx+$$

$$+\frac{1}{2}\int_0^l h_1^2(x,t)\,dx+\frac{1}{2}\int_0^l h_2^2(x,t)\,dx. \tag{3.27}$$

Using Hölder's inequality and Gronwall's lemma, from (3.27) we obtain a priori estimate (3.16). It follows that $T_{max}=+\infty$, i.e. for any $\omega_0\in\mathcal{H}$, the problem (3.15) has a unique solution $\omega(\cdot)\in C([0,+\infty)\,,\,\mathcal{H})$.
So, problem (3.15) corresponds to some non-linear strongly continuous semigroup, which will be denoted by W_t.
Thus, our next aim is to investigate the asymptotic behavior of semigroup W_t at $t\to+\infty$.

4 Relations between some energy functionals

First, we introduce some parameters $\eta_1,\ \eta_2,\ \delta_1,\ \delta_2$ and ε as follows:

$$0<\eta_1<\min\{\frac{1}{2},\frac{1}{p_0^4(\lambda_1+|\lambda_2|)}\},\ 0<\eta_2<\min\{\frac{1}{2},\frac{1}{p_0^2(\mu_1+|\mu_2|)}\}, \tag{4.1}$$

$$\delta_1=e^{\tau_1}\varepsilon\frac{|\lambda_2|}{2},\ \delta_2=e^{\tau_2}\varepsilon\frac{|\mu_2|}{2}, \tag{4.2}$$

$$0<\varepsilon<min\left\{\frac{2\eta_1}{p_0^4},\frac{2\eta_2}{p_0^2},\ \frac{2\eta_1 p_0^4(\lambda_1^2-\lambda_2^2)}{2+\lambda_1+p_0^4(\lambda_1+|\lambda_2|)e^{\tau_1}},\ \frac{2\eta_2 p_0^2(\mu_1^2-\mu_2^2)}{2+\mu_1+p_0^2(\mu_1+|\mu_2|)e^{\tau_1}}\right\}, \tag{4.3}$$

where p_0 is the smallest number for which Poincaré's inequality

$$\int_0^l |u|^2dx\ \le p_0\int_0^l |u_x|^2dx \tag{4.4}$$

holds. Further, we define the functional

$$\mathcal{E}_\varepsilon(t) = E_1(t) + \varepsilon X(t) + Y(t), \tag{4.5}$$

where $\varepsilon,\ \delta_1\ and\ \delta_2$ fulfill the conditions (4.1)-(4.3). The functional $X(t)$ and $Y(t)$ are defined as follows:

$$X\left(t\right) = \int_0^l u_t\left(x,t\right) u\left(x,t\right) dx\ + \int_0^l v_t\left(x,t\right) v\left(x,t\right) dx\ , \tag{4.6}$$

and

$$Y(t) = Y_{\delta_1\delta_2}(t) = \sum_{i=1}^{2} \delta_i \tau_i \int_0^l \int_0^1 e^{-\rho\tau_i} |z_i(x,\rho,t)|^2 d\rho dx\ .$$

Differentiating $X(t)$ in (3.3)-(3.10), we get

$$\begin{aligned} \frac{dX(t)}{dt} &= \int_0^l |u_t|^2 dx\ + \int_0^l |v_t|^2 dx\ - \int_0^l |u_{xx}|^2 dx\ - \int_0^l |v_x|^2 dx - \\ &- \lambda_1 \int_0^l u_t\left(x,t\right) u\left(x,t\right) dx\ - \lambda_2 \int_0^l z_1\left(x,1,t\right) u\left(x,t\right) dx\ - \mu_1 \int_0^l v_t\left(x,t\right) v\left(x,t\right) dx - \\ &- \mu_2 \int_0^l z_2\left(x,1,t\right) v\left(x,t\right) dx - \int_0^l \left|\left(u\left(x,t\right) - \upsilon\left(x,t\right)\right)_+\right|^2 dx + \\ &+ \int_0^l \tilde{g}_1\left(x,t\right) u\left(x,t\right) dx + \int_0^l \tilde{g}_1\left(x,t\right) v\left(x,t\right) dx\ . \end{aligned} \tag{4.7}$$

Similarly, differentiating $Y(t)$ in (3.1), (3.2), (3.5) and (3.6), we obtain

$$\begin{aligned} \frac{dY(t)}{dt} &= -\sum_{i=1}^{2} 2\delta_i \int_0^l \int_0^1 e^{-\rho\tau_i} z_i(x,\rho,t) z_{i\rho}(x,\rho,t) d\rho dx\ = \\ &= -\sum_{i=1}^{2} \delta_i \int_0^l e^{-\tau_i} z_i^2(x,1,t) dx + \sum_{i=1}^{2} \delta_i \int_0^l z_i^2(x,0,t) dx\ - \\ &- \sum_{i=1}^{2} \delta_i \tau_i \int_0^l \int_0^1 e^{-\rho\tau_i} z_i^2(x,\rho,t) d\rho dx\ = \\ &= -\sum_{i=1}^{2} \delta_i \int_0^l e^{-\tau_i} z_i^2(x,1,t) dx\ - \sum_{i=1}^{2} \delta_i \tau_i \int_0^l \int_0^1 e^{-\rho\tau_i} z_i^2(x,\rho,t) d\rho dx\ + \\ &+ \delta_1 \int_0^l {u_t}^2\left(x,t\right) dx + \delta_2 \int_0^l v_t^2\left(x,t\right) dx\ . \end{aligned} \tag{4.8}$$

Lemma 4.1. *Assume that conditions* (4.1) - (4.3) *are satisfied. Then there exist* $0 < c_1 < c_2$ *such that*

$$c_1 E_1(t) \le \mathcal{E}_\varepsilon(t) \le c_2 E_1(t), \quad t \ge 0\ . \tag{4.9}$$

Proof. By virtue of the Poincaré inequality (4.4)

$$\int_0^l |v\left(x,t\right)|^2 dx \le p_0^2 \int_0^l |v_x\left(x,t\right)|^2 dx, \tag{4.10}$$

$$\int_0^l |u\left(x,t\right)|^2 dx \le p_0^4 \int_0^l |u_{xx}\left(x,t\right)|^2 dx\ . \tag{4.11}$$

In addition, using Holder's and Young's inequalities, and (4.10), (4.11), we obtain that

$$\left|\int_0^l u_t\left(x,t\right) u\left(x,t\right) dx\right| \le \frac{p_0^4}{4\eta_1} \int_0^l |u_t\left(x,t\right)|^2 dx + \eta_1 \int_0^l |u_{xx}\left(x,t\right)|^2 dx, \tag{4.12}$$

$$\left|\int_0^l v_t\left(x,t\right) v\left(x,t\right) dx\right| \le \frac{p_0^2}{4\eta_2} \int_0^l |v_t\left(x,t\right)|^2 dx + \eta_2 \int_0^l |v_x\left(x,t\right)|^2 dx, \tag{4.13}$$

Considering (4.12), (4.13) in (4.5), we get

$$E_1(t) - \frac{p_0^4}{4\eta_1}\int_0^l |u_t(x,t)|^2 dx - \eta_1 \int_0^l |u_{xx}(x,t)|^2 dx, -$$

$$-\frac{p_0^2}{4\eta_2}\int_0^l |v_t(x,t)|^2 dx - \eta_2 \int_0^l |v_x(x,t)|^2 dx +$$

$$+\sum_{i=1}^{2}\delta_i\tau_i e^{-\tau_i}\int_0^l\int_0^1 z_i^2(x,\rho,t)d\rho dx \le E_{\varepsilon}(t) \le$$

$$\le E_1(t) + \frac{p_0^4}{4\eta_1}\int_0^l |u_t(x,t)|^2 dx + \eta_1 \int_0^l |u_{xx}(x,t)|^2 dx +$$

$$+\frac{p_0^2}{4\eta_2}\int_0^l |v_t(x,t)|^2 dx + \eta_2 \int_0^l |v_x(x,t)|^2 dx + \sum_{i=1}^{2}\delta_i\tau_i\int_0^l\int_0^1 z_i^2(x,\rho,t)d\rho dx \ .$$

As can be seen from here, if we choose

$$c_1 = min\{\frac{1}{2} - \frac{\varepsilon p_0^4}{4\eta_1}, \frac{1}{2} - \frac{\varepsilon p_0^2}{4\eta_2},\ \frac{1}{2} - \varepsilon\eta_1,\ \frac{1}{2} - \varepsilon\eta_2,\ \frac{1}{2} - \varepsilon\eta_1\ , \tau_1\delta_1 e^{-\tau_1},\ \tau_2\delta_2 e^{-\tau_2}\}$$

$$c_2 = max\{\frac{1}{2} + \frac{\varepsilon p_0^4}{4\eta_1}, \frac{1}{2} + \frac{\varepsilon p_0^2}{4\eta_2},\ \frac{1}{2} + \varepsilon\eta_1,\ \frac{1}{2} + \varepsilon\eta_2,\ \frac{1}{2} + \varepsilon\eta_1\ \tau_1\delta_1, \tau_2\delta_2\}$$

then inequality (4.9) is satisfied. The proof is complete. □

Lemma 4.2. *Assume that conditions* (4.1) - (4.3) *are satisfied. Then the following inequality holds:*

$$\frac{d\mathcal{E}_{\varepsilon}(t)}{dt} \le -E_3(t) + \int_0^l \tilde{g}_1(x,t)\,u_t(x,t)\,dx + \int_0^l \tilde{g}_1(x,t)\,v_t(x,t)\,dx +$$

$$+\varepsilon\int_0^l \tilde{g}_1(x,t)\,u(x,t)\,dx + \varepsilon\int_0^l \tilde{g}_1(x,t)\,v(x,t)\,dx, \tag{4.14}$$

where

$$E_3(t) = \varepsilon(1 - \frac{p_0^4\eta_1}{2}(\lambda_1 + |\lambda_2|))\int_0^l |u_{xx}|^2 dx +$$

$$+\varepsilon\left(1 - \frac{p_0^2\eta_2}{2}(\mu_1 + |\mu_2|)\right)\int_0^l |v_x|^2 dx +$$

$$+[\lambda_1 - |\lambda_2| - \varepsilon(1 + \frac{\lambda_1}{2\eta_1}) - \delta_1]\int_0^l |u_t|^2 dx +$$

$$+\left[\mu_1 - |\mu_2| - \varepsilon\left(1 + \frac{\mu_1}{2\eta_2}\right) - \delta_2\right]\int_0^l |v_t|^2 dx +$$

$$+\int_0^l \left|(u(x,t) - \upsilon(x,t))_+\right|^2 dx +$$

$$+\sum_{i=1}^{2}\tau_i\delta_i\int_0^l\int_0^1 e^{-\rho\tau_i} z_i^2(x,\rho,t)d\rho dx$$

Proof. From (3.15)-(3.18), we have

$$\frac{d\mathcal{E}_{\varepsilon}(t)}{dt} = -[\lambda_1 - \frac{|\lambda_2|}{2} - \varepsilon]\int_0^l u_t^2(x,t)\,dx - \lambda_2\int_0^l z_1(x,1,t)[\,u_t(x,t) + \varepsilon u(x,t)]dx$$

$$-\varepsilon\lambda_1\int_0^l u_t(x,t)\,u(x,t)\,dx - \varepsilon\mu_1\int_0^l v_t(x,t)\,v(x,t)\,dx -$$

$$-[\mu_1 - \frac{|\mu_2|}{2} - \varepsilon]\int_0^l v_t^2(x,t)\,dx - \mu_2\int_0^l z_2(x,1,t)\,[v_t(x,t) + \varepsilon v(x,t)]dx -$$

$$-\frac{|\lambda_2|}{2}\int_0^l z_1^2\left(x,1,t\right)dx-\frac{|\mu_2|}{2}\int_0^l z_2^2\left(x,1,t\right)dx-$$

$$-\varepsilon\int_0^l |u_{xx}|^2dx \quad -\varepsilon\int_0^l |v_x|^2dx -\varepsilon\int_0^l \left|\left(u\left(x,t\right)-v\left(x,t\right)\right)_+\right|^2dx+$$

$$-\sum_{i=1}^{2}\delta_i\int_0^l e^{-\tau_i}z_i^2(x,1,t)dx-\sum_{i=1}^{2}\delta_i\tau_i\int_0^l\int_0^1 e^{-\rho\tau_i}z_i^2(x,\rho,t)d\rho dx \;+$$

$$+\delta_1\int_0^l {u_t}^2\left(x,t\right)dx+\delta_2\int_0^l v_t^2\left(x,t\right)dx \quad +$$

$$+\int_0^l \tilde{g}_1\left(x,t\right)\left[u_t\left(x,t\right)+\varepsilon u\left(x,t\right)\right]dx+\int_0^l \tilde{g}_1\left(x,t\right)\left[v_t\left(x,t\right)+\varepsilon v\left(x,t\right)\right]dx. \tag{4.15}$$

Using Holder's and Young's inequalities, and (4.10), (4.11), we obtain that

$$\left|\int_0^l z_1\left(x,1,t\right)u\left(x,t\right)dx\right|\le\frac{p_0^4}{4\eta_3}\int_0^l |z_1\left(x,1,t\right)|^2dx+\eta_3\int_0^l |u_{xx}\left(x,t\right)|^2dx, \tag{4.16}$$

$$\left|\int_0^l z_2\left(x,1,t\right)v\left(x,t\right)dx\right|\le\frac{p_0^2}{4\eta_3}\int_0^l |z_2\left(x,1,t\right)|^2dx+\eta_3\int_0^l |v_x\left(x,t\right)|^2dx. \tag{4.17}$$

Taking into account (3.25), (3.26), (4.16), (4.17) in (4.15), we get (4.14)

□

5 Exponential stability

In this section, we investigate the exponential stability for the solution of the problem (2.1)-(2.7) in the case of $h_i(x,t)=0$ and $g_i\left(x,u,v\right)=0$, $i=1,2$.

Theorem 5.1. *Suppose that conditions* (2.8), (2.9) *are satisfied and* $h_i(x,t)=0$, *and* $\lambda(x)=0,\ x\in(0,l)$, $t>0$, *then there exist positive constants* M *and* ω, *such that the solution of the problem* (2.1)-(2.7) *satisfies the estimate*

$$E\left(t\right)\le ME\left(0\right)e^{-\omega t},\ t\ge 0, \tag{5.1}$$

i.e.,the corresponding semigroup stabilized exponentially.

Proof. Let $h_1\left(x\right)=h_2\left(x\right)=\lambda(x)=0,\ \ 0\le x\le l$, then from (4.14) we obtain that

$$\frac{d\mathcal{E}_\varepsilon\left(t\right)}{dt}\le -E_3\left(t\right),\quad t\ge 0. \tag{5.2}$$

Lemma 5.2. *Assume that conditions* (2.8), (2.9) *are satisfied. Then there exist* $0<c_3<c_4$, *such that*

$$c_3E_3(t)\le\mathcal{E}_\varepsilon(t)\le c_4E_3(t),\quad t\ge 0\ . \tag{5.3}$$

The proof of this lemma is carried out in the same way as the proof of Lemma (4.1).

It follows from (5.2) and (5.3) that

$$E_\varepsilon\left(t\right)\le ME_\varepsilon\left(0\right)e^{-\omega t}\,,t\ge 0. \tag{5.4}$$

On the other hand, as in Lemma 1 and Lemma 2, we prove the following inequality

$$c_5E\left(t\right)\le E_\varepsilon\left(t\right)\le c_6E\left(t\right),\qquad t\ge 0, \tag{5.5}$$

where $0<c_5<c_6$.

The inequality (5.1) follows from the inequalities (5.4) and (5.5) □

Remark 5.3. Note that the proof of the exponential stabilization of the linear semigroup e^{-tA_0} is not difficult.

6 Existence of the bounded absorbing set.

According to Theorem (2.1), a dynamical system corresponding to the problem (2.1)-(2.7) generates a continuous semigroup $\mathrm{W_t}$ in $\mathcal{H}$.The main goal of the paper is to show the existence of an absorbing set and a minimal global attractor for $\mathrm{W_t}$. In this section, we show the existence of an absorbing set. In the next section, the existence of a minimal global attractor is investigated. Based on Definition 2, a bounded set $\mathrm{B_0} \subset \mathcal{H}$ is a bounded absorbing set for $\mathrm{W_t}$ if every bounded $\mathrm{B} \subset \mathcal{H}$ there exists t_0 such that from $t \geq t_0$ implies $\mathrm{W_t B \subset B_0}$.

Theorem 6.1. *Suppose that conditions* (2.8), (2.9) *are satisfied,* $\lambda(\cdot) \in C\,[0, l]$,
$\lambda(x) \geq 0$ *and*

$$h_i(x,t) = h_i(x),\ 0 \leq x \leq l,\ t > 0\ where\ h_i(\cdot) \in L_2(0,l), i = 1,2. \tag{6.1}$$

Then corresponding semigroup $\mathrm{W_t}$ *has a bounded absorbing set* $\mathrm{B_0} \subset \mathcal{H}$.

Proof. It is clear from the expression of the functions $\tilde{g}_1\,(u,v)$ and $\tilde{g}_2\,(u,v)$ that

$$\int_0^l \tilde{g}_1\,(x,u(x,t),v(x,t))\,u\,(x,t)\,dx + \int_0^l \tilde{g}_2\,(x,u(x,t),v(x,t))\,v\,(x,t)\,dx =$$

$$= -2\int_0^l \lambda(x)\,|u(x,t)v\,(x,t)|^{p+1}dx + \int_0^l h_1\,(x)\,u_t\,(x,t)\,dx + \int_0^l h_2\,(x)\,v_t\,(x,t)\,dx. \tag{6.2}$$

Using Holder's and Young's inequalities [14, 18] and (4.10),(4.11), we get

$$\int_0^l h_1\,(x)\,u\,(x,t)\,dx + \int_0^l h_2\,(x)\,v\,(x,t)\,dx \leq \frac{p_0^4}{4\eta_3}\int_0^l |h_1\,(x)|^2dx + \frac{p_0^2}{4\eta_3}\int_0^l |h_2\,(x)|^2dx$$

$$+\eta_3\int_0^l |u_{xx}\,(x,t)|^2dx + \eta_3\int_0^l |v_x\,(x,t)|^2dx. \tag{6.3}$$

and

$$\int_0^l h_1\,(x)\,u_t\,(x,t)\,dx + \int_0^l h_2\,(x)\,v_t\,(x,t)\,dx \leq \frac{1}{4\eta_3}\int_0^l |h_1\,(x)|^2dx + \frac{1}{4\eta_3}\int_0^l |h_2\,(x)|^2dx$$

$$+\eta_3\int_0^l |u_t\,(x,t)|^2dx + \eta_3\int_0^l |v_t\,(x,t)|^2dx, \tag{6.4}$$

Considering (6.2), (6.3) and (6.4) in (4.14), we obtain the following inequality

$$\frac{dE_4\,(t)}{dt} \leq -E_5\,(t) + \gamma, \tag{6.5}$$

where $E_4\,(t) = \mathcal{E}_\varepsilon\,(t) + \frac{1}{p+1}\int_0^l \lambda(x)\,|u\,(x,t)\,v\,(x,t)|^{p+1}dx$,

$$E_5\,(t) = E_2\,(t) - \ \eta_3\{\int_0^l |u_t\,(x,t)|^2dx + \int_0^l |v_t\,(x,t)|^2dx + \int_0^l |u_{xx}\,(x,t)|^2dx +$$

$$+\int_0^l |v_x\,(x,t)|^2dx\} + 2\varepsilon\int_0^l \lambda(x)\,|u\,(x,t)\,v\,(x,t)|^{p+1}dx,$$

$$\gamma = \frac{p_0^4+1}{4\eta_3 c_0^4}\int_0^l |h_1\,(x)|^2dx + \frac{p_0^2+1}{4\eta_3 p_0^2}\int_0^l |h_2\,(x)|^2dx$$

Lemma 6.2. *Assume that conditions* (2.8), (2.9) *are satisfied. Then there exist* $0 < c_7 < c_8$ *and* $0 < c_9 < c_{10}$ *such that*

$$c_7E_2(t) \leq E_4(t) \leq c_8E_2(t),\ \ t \geq 0\ \ , \tag{6.6}$$

$$c_9E_2(t) \leq E_5(t) \leq c_{10}E_2(t). \tag{6.7}$$

The proof of Lemma (6.2) is carried out in the same way as the proof of Lemma (4.1).

From (6.5)-(6.7) follows that

$$\frac{dE_4(t)}{dt} \leq -\frac{c_9}{c_8}E_4(t) + \gamma. \tag{6.8}$$

From here we obtain

$$E_5(t) \leq \frac{c_9\gamma}{c_8} + e^{-\frac{c_9}{c_8}t}[E_5(0) - \frac{c_9\gamma}{c_8}]. \tag{6.9}$$

Using Poincaré's inequality, it isn't difficult to prove that

$$\beta_1 \|\omega(\mathrm{t})\|_{\mathcal{H}} \leq E_4(t) \leq \beta_2 \|\omega(\mathrm{t})\|_{\mathcal{H}}. \tag{6.10}$$

From (6.9) and (6.10) follows that

$$\|\omega(\mathrm{t})\|_{\mathcal{H}} \leq \frac{c_9\gamma}{c_8\beta_1} + e^{-\frac{c_9}{c_8}t}[\frac{\beta_2}{\beta_1}\|\omega(0)\|_{\mathcal{H}} - \frac{c_9\gamma}{c_8\beta_1}]. \tag{6.11}$$

If we take $r = \frac{c_9\gamma}{c_8\beta_1} + 1$, it follows from (6.11) that $B_0 = \{\omega \in \mathcal{H} : \|\omega(\mathrm{t})\|_{\mathcal{H}} \leq r\}$ is the absorbing set. □

7 Asymptotic compactness and existence of a minimal global attractor of corresponding nonlinear semigroup

We express the nonlinear semigroup as follows:

$$W_t(w_0) = e^{tA_0}w_0 + V_t(w_0), \tag{7.1}$$

where $w_0 \in \mathcal{H}$, and

$$V_t(w_0) = \int_0^t V_{t-s}[A_1(w(s)) + F_1(w(s))]ds,$$

$$F_1(\omega) = (0,\ g_1(\cdot, u, v) + h_1(\cdot),\ 0,\ g_2(\cdot, u, v) + h_2(\cdot),\ 0,\ 0).$$

Let us introduce the notation

$$V_t(B) = \{y:\ \ y = V_t(w_0),\ \ w_0 \in B\ \}$$

,

where B is some set from $\mathcal{H}$.

By virtue of Remark, there exist constants $M > 0$ and $\omega > 0$ such that the following inequality holds:

$$\left\|e^{tA_0}w_0\right\|_{\mathcal{H}} \leq Me^{-t\omega}\|w_0\|_{\mathcal{H}}.$$

Theorem 7.1. *Suppose that conditions* (2.8), (2.9) *are satisfied. Then* $\overline{\bigcup_{t\geq 0}\overline{V_t(B)}}$ *is precompact in* $\mathcal{H}$ *for any bounded set* $B \subset \mathcal{H}$.

Firstly, let us give the proof of the following statement.

Lemma 7.2. . $D(A_0)$*is compactly embedded in the space* $\mathcal{H}$.

Proof. Let's assume that

$$\|A_0 y_n\|_{\mathcal{H}} \leq c \tag{7.2}$$

where $y_n = (y_{2n}, y_{2n}, y_{3n}, y_{4n}, z_{1n}, z_{2n}) \in D(A_0)$, and

$$A_0 y_n = (-y_{1n}, y_{2nxxxx} + \lambda_1 y_{1n} + \lambda_2 z_{1n}, -y_{3n}, -y_{4nxx} + \mu_1 y_{3n} + \mu_2 z_{2n}, \frac{1}{\tau_1}z_{1n}, \frac{1}{\tau_2}z_{2n}).$$

By definition of $\|\cdot\|_{\mathcal{H}}$ and (7.2), we have

$$\|A_0 y_n\|_{\mathcal{H}} = \|y_{2n}\|_{\widehat{\mathrm{H}}^2} + \|y_{1nxxxx} + \lambda_1 y_{2n} + \lambda_2 z_{1n}(\cdot,1)\|_{L_2(0,l)} + \|y_{3n}\|_{\widehat{\mathrm{H}}^1} +$$

$$+\|y_{4nxx} + \mu_1 y_{3n} + \mu_2 z_{2n}(\cdot,1)\|_{L_2(0,l)} + \sum_{i=1}^{2}\frac{1}{\tau_i}\|z_{in\rho}\|_{L_2((0,1)\times(0,l))} \leq c. \tag{7.3}$$

Then, from the sequences $\{y_{2n}\}, \{y_{4n}\}, \{z_{1n}\}$, and $\{z_{2n}\}$, one can select subsequence's $\{y_{2n_k}\}$, $\{y_{4n_k}\}$, $\{z_{1n_k}\}$, and $\{z_{2n_k}\}$, respectively, such that

$$y_{2n_k} \to y_2 \quad weakly\ in\ \widehat{\mathrm{H}}^2 \quad \text{as} \quad n_k \to \infty, \tag{7.4}$$

$$y_{4n_k} \to y_4 \quad weakly\ in \quad \widehat{\mathrm{H}}^1 \quad \text{as} \quad n_k \to \infty, \tag{7.5}$$

$$z_{in_k} \to z_i, z_{in_k\rho} \to z_{i\rho} \quad weakly\ in \quad L_2((0,1)\times(0,l))$$
$$\text{as} \quad n_k \to \infty, i = 1,2, \tag{7.6}$$

By virtue of the embedding theorem [18], we obtain from (7.6) that

$$z_{in_k}(\cdot,1) \to z_i(\cdot,1) \quad in \quad L_2(0,l)) \quad \text{as} \quad n_k \to \infty\ ,\ i = 1,2. \tag{7.7}$$

Taking into account (7.4) -(7.7), from (7.3) we get that

$$\|y_{1n_kxxxx}\|_{L_2(0,l)} \le c \tag{7.8}$$

$$\|y_{2n_kxx}\|_{L_2(0,l)} \le c \tag{7.9}$$

From the sequence $\{y_{1n_k}\}$ and $\{y_{2n_k}\}$ we can select a subsequence which will again be denoted by $\{y_{1n_k}\}$ and $\{y_{2n_k}\}$, respectively such that

$$y_{1n_kxxxx} \to y_{1xxxx} \quad weakly\ in \quad L_2((0,l)) \quad \text{as} \quad n_k \to \infty, \tag{7.10}$$

$$y_{2n_kxx} \to y_{2xx} \quad weakly\ in \quad L_2((0,l)) \quad \text{as} \quad n_k \to \infty. \tag{7.11}$$

By virtue of the embedding theorems(see [18]), from (7.4)-(7.11) we obtain

$$y_{n_k} \to y \quad in \quad \mathcal{H}\ .$$

Therefore, operator A_0 transforms the bounded set into the compact one in $\mathcal{H}$.

□

Proof of Theorem (7.1). According to (6.1), for any bounded set $B \subset \mathcal{H}$, there exists $t_B > 0$ such that $W_t(B) \subset B_0$ for $t > t_B$. It follows from here that if for any $t > 0$, $w(t)$ is the solution of the problem (3.15), then

$$\|w(t)\|_{\mathcal{H}} \le c(r), \quad t \ge 0. \tag{7.12}$$

It's obvious that the function $y = \int_0^t V_{t-s}[A_1(w(s)) + F_1(w(s))ds$ is the solution of the problem

$$y'(t) = A_0 y(t) + A_1(w(t)) + F_1(w(t)), \tag{7.13}$$

$$y(0) = 0. \tag{7.14}$$

If we denote $y_h(t) = \frac{1}{h}[y(t+h) - y(t)]$, $f_h(t) = \frac{1}{h}[f(t+h) - f(t)]$, $where$ $f(t) = A_1(w(t)) + F_1(w(t))$, then we have

$$y'_h(t) = A_0 y_h(t) + f_h(t), y_h(0) = \frac{1}{h} y(h).$$

From here we get that

$$y_h(t) = V_t(y_h(0)) + \int_0^t V_{t-\tau} f_h(\tau) d\tau. \tag{7.15}$$

It is clear that

$$\|f_h(t)\|_{\mathcal{H}} \le \frac{1}{|h|}\|A_1(w(t+h)) - A_1(w(t))\|_{\mathcal{H}} + \frac{1}{|h|}\|F_1(w(t+h)) - F_1(w(t))\|_{\mathcal{H}} \tag{7.16}$$

and by virtue of the definition of $A_1(w)$ we have

$$\frac{1}{|h|}\|A_1(w(t+h)) - A_1(w(t))\|_H \le \frac{2}{|h|}\left\|\lceil u(t+h) - v(t+h)\rceil_+ - \lceil u(t) - v(t)\rceil_+\right\|_H \le$$

$$\le \frac{2}{|h|}\|u(\cdot,t+h) - u(\cdot,t)\|_{L_2(0,l)} + \frac{2}{|h|}\|v(\cdot,t+h) - v(\cdot,t)\|_{L_2(0,l)}. \tag{7.17}$$

Passing to the limit as $h \to 0$, we get

$$\lim_{h\to 0} \frac{1}{|h|} \|A_1(w(t+h)) - A_1(w(t))\|_H \ \le 2[\| \, u_t(\cdot,t)\|_{L_2(0,l)} + \| \, v_t(\cdot,t)\|_{L_2(0,l)}] \le c(B), t \ge 0. \tag{7.18}$$

Similarly, it is easy to prove that

$$\lim_{h\to o} \frac{1}{|h|} \|F_1(w(t+h)) - F_1(w(t))\|_{\mathcal{H}} \le c(B), \ t \ge 0\,. \tag{7.19}$$

From the above estimates (7.16) - (7.19), we derive

$$\|y_h(t)\|_{\mathcal{H}} \le Me^{-\omega t}\|y_h(0)\|_{\mathcal{H}} + \frac{M}{\omega}(1 - e^{-\omega t})\, c(\|(w_0\|_{\mathcal{H}}) \le c(B), \ \ t \ge 0. \tag{7.20}$$

Since $y'(\cdot) \in \ C([0,\infty); \ \mathcal{H})$, therefore, from here we have

$$\|y'(t)\|_{\mathcal{H}} \le c(B), \quad t \ge 0. \tag{7.21}$$

Considering (7.12) in (7.21), we get

$$\|A_0 y(t)\|_{\mathcal{H}} \le c(B), \ \ t \ge 0. \tag{7.22}$$

Using Lemma (7.2), we get from here that $\overline{\{y(t), \ t \ge 0, w_0 \in B \ \}} \ = \overline{\bigcup_{t\ge 0} \overline{V_t(B)}}$ is a compact set.
This completes the proof of the Theorem (7.1).
Thus, a strongly continuous nonlinear semigroup W_t, by virtue of the Theorem (6.1), has an absorbing set, and by virtue of the Theorem (7.1), W_t is an asymptotically compact semigroup.
Further, using the standard method, we obtain the following theorem on existence of minimal global attractor(see [3, 6, 10])

Theorem 7.3. *Suppose that conditions* (2.8),(2.9) *and* (6.1) *are satisfied. Then the semigroup W_t generated by the problem* (2.1)-(2.7) *has a global minimal attractor, so that, it is connected, invariant in $\mathcal{H}$, and bounded set in $D(A_0)$.*

8 Conclusions

The constant delay of the linear aerodynamic resistance forces does not affect the dynamics of the oscillations of the suspension bridge. We have shown that the corresponding dynamical system forms a linear continuous semigroup in some function space. In the homogeneous case, this semigroup decreases exponentially. In the autonomous case, this non-linear semigroup has a minimal bounded global attractor.

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