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Solutions of Nonlinear Fractional Differential Equations with Nondifferentiable Terms

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Abstract In this research, we employ a newly developed strategy based on a modified version of the Adomian decomposition method (ADM) to solve nonlinear fractional differential equations (FDE) with both differential and nondifferential variables. FDE have disturbed the interest of many researchers. This is due to the development of both the theory and applications of fractional calculus. This track from various areas of fractional differential equations can be used to model various fields of science and engineering such as fluid flows, viscoelasticity, electrochemistry, control, electromagnetic, and many others. Several fractional derivative definitions have been presented, including Riemann–Liouville, Caputo,and Caputo– Fabrizio fractional derivative. We just need to calculate the first Adomain polynomial in this technique avoiding the hurdles in the nondifferentiable nonlinear terms' remaining polynomials. Furthermore, the proposed technique is easy to programme and produces the desired output with minimal work and time on the same processor. When compared to the exact solution, this method has the advantage of reducing calculation steps, while producing accurate results. The supporting evidence proves that modified Adomian decomposition has an advantage over traditional Adomian decomposition method which can be explained very clear with nonlinear fractional differential equations. Our computational examples with difficult issues are used to prove the new algorithm's efficiency. The results show that the modified ADM is powerful, which has a faster convergence solution than the original one. Convergence analysis is discussed, also the uniqueness is explained.

Keywords Modified Adomian Decomposition, Nondifferentiable Terms, Nonlinear Fractional Differential Equations

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1 Introduction

Currently, fractional differential equations (FDE) have much attention for many mathematicians and physicians researchers as they provide better mathematical modeling of many world phenomena than the classical order derivatives, see [1, 2, 3, 4, 16]. In addition, complex physical phenomena in many branches can be modeled with more sensitivity via such equations as biomedical engineering, signal processing also image and video processing that provides more importance to FDE. It is well known that many fractional problems are solved by researchers who struggled to find analytical solutions for such problems in powerful ways, so they have applied approximate methods in many papers, such as Variational Iteration Method (VIM) [14], Laplace Decomposition Method (LDM)[15], Homotopy Perturbation Method (HPM) and Adomian Decomposition Method (ADM). Among these methods, Adomian decomposition method [5, 6, 7, 8, 9, 10, 11, 12, 13] is a famous and powerful technique to solve all kinds of equations. It is effective not only for linear problems but also for nonlinear ones. Its efficiency is to provide a rapid convergent solution as an infinite series and only a few iterations reach the desired accuracy. Many attempts are provided to improve the method by several modifications to get more accuracy than the traditional one.

The goal of this study is to use an unique feature using the Modified Adomian Decomposition Method to calculate closed-form solutions for equations with differential and nondifferentiable nonlinear variables that were previously impossible to solve due to method restrictions. We acquire a solution herein directly due to the calculation of the only first term of the Adomian polynomial, which leads to a decrease in computations, allowing us to get the solution quickly.

2 Description of the technique

Consider the nonlinear FDE of the form

$$\begin{cases} {}_0D_t^{\alpha} y\,(t) + g\,(t)\, f\,(y(t)) = x(t), \\ {}_0D_t^{\alpha-k} y\,(t)\big|_{t=0} = 0,\ k = 1, 2, ..., n. \end{cases} \tag{1}$$

Applying with I^{α} to both side of Eq. (1), as defined for Riemann-Liouville Fractional Derivative:

$$I^{\alpha}\, y\,(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{2}$$

By using initial condition, we have

$$y\,(t) - 0 = I^{\alpha} x(t) + I^{\alpha} g\,(t)\, f\,(y(t))\,. \tag{3}$$

The solution can be written in an infinite series as given below

$$y\,(t) = \sum_{i=0}^{\infty} y_i\,(t)\,. \tag{4}$$

Also, the nonlinear term can be decomposed as

$$N\,(y) = \sum_{n=0}^{\infty} A_n, \tag{5}$$

or with the formula of El-kalla Adomian polynomials $\tilde{A}_n$ where the partial sum $S_n = y_0 + y_1 + ... + y_n = \sum_{i=0}^{n} y_{i(t)}$ follows [?]

$$\tilde{A}_n = f\,(S_n) - \sum_{i=0}^{n-1} \tilde{A}_i \quad n \geq 1. \tag{6}$$

So that, the algorithm of substituting Eq. (4) and (5) to both side of Eq. (3) gives

$$\sum_{i=0}^{\infty} y_i\,(t) = 0 + I^{\alpha} \left\{ x(t) + \sum_{n=0}^{\infty} A_n \right\}. \tag{7}$$

Then, adding the term $I^{\alpha}[\sum_{n=0}^{\infty} a_n t^n] - pI^{\alpha}[\sum_{n=0}^{\infty} a_n t^n]$ to the right hand side of Eq. (7), we obtain, see [11]

$$\begin{aligned} \sum_{i=0}^{\infty} y_i\,(t) &= 0 + I^{\alpha}\,[x(t)] + I^{\alpha} \left[\sum_{n=0}^{\infty} a_n t^n \right] \\ &\quad -pI^{\alpha} \left[\sum_{n=0}^{\infty} a_n t^n \right] + I^{\alpha} \left[\sum_{n=0}^{\infty} A_n \right], \end{aligned} \tag{8}$$

where p is an artificial parameter that we can assume $p = 1$ and a_n, $n = 0, 1, 2, ...$, are unknown coefficients. Therefore, the relation become

$$\begin{cases} y_0\,(t) = 0 + I^{\alpha}\,[x(t)] + I^{\alpha}\,[\sum_{n=0}^{\infty} a_n t^n]\,; \\ y_1\,(t) = -pI^{\alpha}\,[\sum_{n=0}^{\infty} a_n t^n] + I^{\alpha}\,[A_0]\,. \end{cases} \tag{9}$$

Avoiding calculation of Adomian terms A_n, $n = 0, 1, 2, ...$, we let $y_1 = 0$ that result $y_k = 0$ for $k = 2, 3, ...$. Putting $p = 1$, we establish the solution as

$$y\,(t) = 0 + I^{\alpha} \left[\sum_{n=0}^{\infty} a_n t^n \right]. \tag{10}$$

3 Convergence analysis:

3.1 Existence and uniqueness

Define a mapping $F: E \to E$ where $E = (C[J], \|\cdot\|)$ is a Banach space of all continuous functions on J with the norm $\|x\| = \max_{t\epsilon J} x(t)$. Assume that $x(t)$ is bounded $\forall t \in J = [0,T]$, $T \in R^{+}$ and $N(y)$ is Lipschitz continuos with Lipschitz constant L such as,

$$|N(y) - N(z)| \leq L|y - z|.$$

Theorem: If $0 < \phi < 1$ where $\phi = \frac{LT^{\alpha}}{\Gamma(\alpha+1)}$, then the series (4) is the solution of the problem (1) and this solution is unique [17]

Proof : First, we define the mapping $F: E \to E$ as

$$Fy = 0 + I^{\alpha}x(t) + I^{\alpha}N(y(t)). \tag{11}$$

Let y and $z \in E$ are two different solutions of Eq. (1). and using Eq.(11)

$$\begin{aligned} Fy - Fz &= I^{\alpha}x(t) + I^{\alpha}N(y(t)) - I^{\alpha}x(t) - I^{\alpha}N(z(t)) \\ &= I^{\alpha}[N(y(t)) - N(z(t)], \end{aligned} \tag{12}$$

which implies that

$$|Fy - Fz| = |I^{\alpha}[N(y(t)) - N(z(t))]|$$

using the norm

$$\begin{aligned} \|Fy - Fz\| &= \max_{t\epsilon J} |I^{\alpha}[N(y(t)) - N(z(t))]| \\ &\leq L\max_{t\epsilon J} |y(t) - z(t)| \ \ |I^{\alpha}(1)| \\ &\leq L\max_{t\epsilon J} |y - z| \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} d\tau \right| \\ &\leq \frac{LT^{\alpha}}{\Gamma(\alpha+1)} \|y - z\| \\ &\leq \phi \|y - z\|, \end{aligned}$$

under the condition $0 < \phi < 1$, the mapping F is contraction and hence there exists a unique solution $x \in C[J]$ for the problem (1) and this completes the proof.

3.2 Proof of convergence:

Theorem: The solution of the problem (1) using ADM converges if $0 < \phi < 1$, $\phi = \frac{LT^{\alpha}}{\Gamma(1+\alpha)}$.

Proof: Let S_p, S_q be two arbitrary sums with, $p \geqslant q$. Now, we are going to prove that $\{S_p\}$ is a Caushy sequence in this Banach space [18, 19]. We have

$$S_p - S_q = I^{\alpha} \sum_{i=0}^{n} a_i t^i - I^{\alpha} \sum_{i=0}^{m} a_i t^i = I^{\alpha} \sum_{i=m+1}^{n} a_i t^i, \tag{13}$$

Then [20]

$$\begin{aligned} \|S_p - S_q\| &= \max_{t\epsilon J} |S_p - S_q| = \max_{t\epsilon J} \left| I^{\alpha} \sum_{i=m+1}^{n} a_i t^i \right| \\ &= \max_{t\epsilon J} \left| \sum_{i=m+1}^{n} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} [S_p - S_q] \, d\tau \right| \\ &\leq \frac{1}{\Gamma(\alpha)} \max_{t\epsilon J} |S_p - S_q| \left| \int_0^t (t-\tau)^{\alpha-1} d\tau \right| \\ &\leq \frac{LT^{\alpha}}{\Gamma(\alpha+1)} \|S_p - S_q\| \\ &\leq \phi \|S_p - S_q\|. \end{aligned}$$

Since $0 < \phi < 1, p \geqslant q$ but $q \to \infty$ then $\|S_p - S_q\| \to 0$ and hence, $\{S_p\}$ is a Caushy sequence in this Banach space the solution is converge.

4 Numerical Examples

In this section, we show the effect of the new technique with examples.

Example 4.1. *Consider the following nonlinear fractional equation*

$$D^{1/2}y(t) - y^2 = \Gamma\left(\frac{3}{2}\right) - t,\ y(0) = 0. \tag{14}$$

Appling $I^\alpha(\cdot)$ to both sides of Eq. (14), and using initial condition, we have

$$y(t) = y(0) + I^{1/2}\left[\Gamma\left(\frac{3}{2}\right) - t\right] + I^{1/2}\left[y^2\right]. \tag{15}$$

From (4) and (5), Eq. (15) becomes

$$\sum_{i=0}^{\infty} y_i(t) = y(0) + I^{1/2}\left[\Gamma\left(\frac{3}{2}\right) - t\right] + I^{1/2}\left[\sum_{i=0}^{\infty} y_i^2(t)\right].$$

Adding the term $I^\alpha\left[\sum_{n=0}^{\infty} a_n t^n\right] - pI^\alpha\left[\sum_{n=0}^{\infty} a_n t^n\right]$, the recursive relation become

$$\begin{aligned} y_0 &= y(0) + I^{1/2}\left[\Gamma\left(\frac{3}{2}\right) - t\right] + I^{1/2}\left[\sum_{n=0}^{\infty} a_n t^n\right] \\ y_{n+1} &= I^{1/2}\left[\sum_{i=0}^{\infty} y_i^2(t)\right] - pI^{1/2}\left[\sum_{n=0}^{\infty} a_n t^n\right]. \end{aligned}$$

Appling formula (9), we find

$$\begin{aligned} y_0 &= I^{1/2}\left[\Gamma\left(\frac{3}{2}\right) - t\right] + I^{1/2}\left[a_0t^0 + a_1t^1 + a_2t^2 + a_3t^3 + ...\right]; \\ y_1 &= I^{1/2}\left[y_0^2\right] - pI^{1/2}\left[a_0t^0 + a_1t^1 + a_2t^2 + a_3t^3 + ...\right]. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} y_0 &= t^{1/2} + \frac{2a_0}{\sqrt{\pi}}t^{1/2} - \frac{4}{3\sqrt{\pi}}t^{3/2} + \frac{4a_1}{3\sqrt{\pi}}t^{3/2} + \frac{16a_2}{15\sqrt{\pi}}t^{5/2} + \frac{32a_3}{35\sqrt{\pi}}t^{7/2} + ..; \\ y_1 &= \frac{-2a_0p}{\sqrt{\pi}}t^{1/2} + \left[\frac{16a_0^2}{3\pi^{3/2}} + \frac{16a_0}{3\sqrt{\pi}} + \frac{4}{3\sqrt{\pi}} - \frac{4a_1p}{3\sqrt{\pi}}\right]t^{\frac{3}{2}} + ... \end{aligned}$$

Setting $y_1 = 0$, we obtain the coefficients $a_0, a_1, a_2, a_3, ...$

$$\frac{-2a_0p}{\sqrt{\pi}}t^{1/2} + \left[\frac{16a_0^2}{3\pi^{3/2}} + \frac{16a_0}{3\sqrt{\pi}} + \frac{4}{3\sqrt{\pi}} - \frac{4a_1p}{3\sqrt{\pi}}\right]t^{\frac{3}{2}} + ... = 0,$$

and so

$$a_0 = 0,\ a_1 = \frac{1}{p},\ \text{and } a_k = 0 \text{ for } k = 2, 3, ...$$

Substituting in Eq. (4), with initial condition and the value of $a_k, k = 0, 1, 2, ...$ we get

$$y(t) = t^{1/2} - \frac{4}{3\sqrt{\pi}}t^{3/2} + I^{1/2}\left[0 + \frac{1}{p}t^1 + 0 + 0 + ...\right].$$

Let $p = 1$ we get

$$\begin{aligned} y(t) &= t^{1/2} - \frac{4}{3\sqrt{\pi}}t^{3/2} + I^{1/2}[t] \\ &= t^{1/2} - \frac{4}{3\sqrt{\pi}}t^{3/2} + \frac{4}{3\sqrt{\pi}}t^{3/2} \\ &= t^{1/2}. \end{aligned}$$

where $y(t) = t^{1/2}$ is the exact solution of this problem.

Example 4.2. *Consider the following nonlinear fractional equation*

$$D^{1/2}y\left(t\right)+e^{1/y}=\frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}+e^{1/t},\ y\left(0\right)=0. \tag{16}$$

Appling $I^{\alpha}\left(\cdot\right)$ to both sides of Eq. (16), and using initial condition, we have

$$y\left(t\right)=y\left(0\right)+I^{1/2}\left[\frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}+e^{1/t}\right]-I^{1/2}\left[e^{1/y}\right]. \tag{17}$$

From (4) and (5), Eq. (17) becomes

$$\sum_{i=0}^{\infty}y_{i}\left(t\right)=y\left(0\right)+I^{1/2}\left[\frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}+e^{1/t}\right]-I^{1/2}\left[\sum_{i=0}^{\infty}e^{1/y_{i}(t)}\right].$$

Adding the term $I^{\alpha}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right]-pI^{\alpha}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right]$, the recursive relation become

$$\begin{aligned}y_{0}&=y\left(0\right)+I^{1/2}\left[\frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}+e^{1/t}\right]+I^{1/2}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right]\\ y_{n+1}&=-I^{1/2}\left[\sum_{i=0}^{\infty}e^{1/y_{i}(t)}\right]-pI^{1/2}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right].\end{aligned}$$

Using Taylor expansion for exponential function at $t=1$

$$\begin{aligned}y_{0}&\simeq y\left(0\right)+I^{1/2}\left[\frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}+2e-e\,t\right]+I^{1/2}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right]\\ y_{n+1}&\simeq -I^{1/2}\left[\sum_{i=0}^{\infty}2e-e\,y_{n}\right]-pI^{1/2}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right].\end{aligned}$$

Appling formula (9), we find

$$\begin{aligned}y_{0}&=I^{1/2}\left[\frac{1}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}+2e-e\,t\right]+I^{1/2}\left[a_{0}t^{0}+a_{1}t^{1}+a_{2}t^{2}+a_{3}t^{3}+...\right];\\ y_{1}&=-I^{1/2}\left[2e-e\,y_{0}\right]-pI^{1/2}\left[a_{0}t^{0}+a_{1}t^{1}+a_{2}t^{2}+a_{3}t^{3}+....\right].\end{aligned}$$

Hence, we obtain

$$\begin{aligned}y_{0}&=\frac{2a_{0}}{\sqrt{\pi}}t^{1/2}+\frac{4e}{\sqrt{\pi}}t^{1/2}+t+\left[\frac{4a_{1}}{3\sqrt{\pi}}-\frac{4e}{3\sqrt{\pi}}\right]t^{3/2}+\frac{16a_{2}}{15\sqrt{\pi}}t^{5/2}+\frac{32a_{3}}{35\sqrt{\pi}}t^{7/2}+..;\\ y_{1}&=\left[-\frac{4e}{\sqrt{\pi}}-\frac{2a_{0}p}{\sqrt{\pi}}\right]t^{1/2}+\left[a_{0}e+2e^{2}\right]t+\left[\frac{4e}{3\sqrt{\pi}}-\frac{4a_{1}p}{3\sqrt{\pi}}\right]t^{3/2}+....\end{aligned}$$

Setting $y_{1}=0$, we obtain the coefficients $a_{0},a_{1},a_{2},a_{3},...$

$$\left[-\frac{4e}{\sqrt{\pi}}-\frac{2a_{0}p}{\sqrt{\pi}}\right]t^{1/2}+\left[a_{0}e+2e^{2}\right]t+\left[\frac{4e}{3\sqrt{\pi}}-\frac{4a_{1}p}{3\sqrt{\pi}}\right]t^{3/2}+...=0,$$

and so

$$a_{0}=\frac{-2e}{p},\ a_{1}=\frac{e}{p},\ a_{2}=0,\ a_{3}=0,\ \textit{and}\ a_{k}=0\ \textit{for}\ k=2,3,....$$

Substituting in Eq. (4), with initial condition and the value of $a_{k},k=0,1,2,...$ we get

$$y\left(t\right)=\frac{4e}{\sqrt{\pi}}t^{1/2}-\frac{4e}{3\sqrt{\pi}}t^{3/2}+t+I^{1/2}\left[\frac{-2e}{p}t^{0}+\frac{e}{p}t^{1}+0+0+...\right].$$

Let $p=1$ we get [see fig(1)]

$$\begin{aligned}y\left(t\right)&=\frac{4e}{\sqrt{\pi}}t^{1/2}-\frac{4e}{3\sqrt{\pi}}t^{3/2}+t+I^{1/2}\left[-2e+et\right]\\ &=\frac{4e}{\sqrt{\pi}}t^{1/2}-\frac{4e}{3\sqrt{\pi}}t^{3/2}+t+\frac{2\left(-2e\right)}{\sqrt{\pi}}t^{1/2}+\frac{4e}{3\sqrt{\pi}}t^{3/2}.\end{aligned}$$

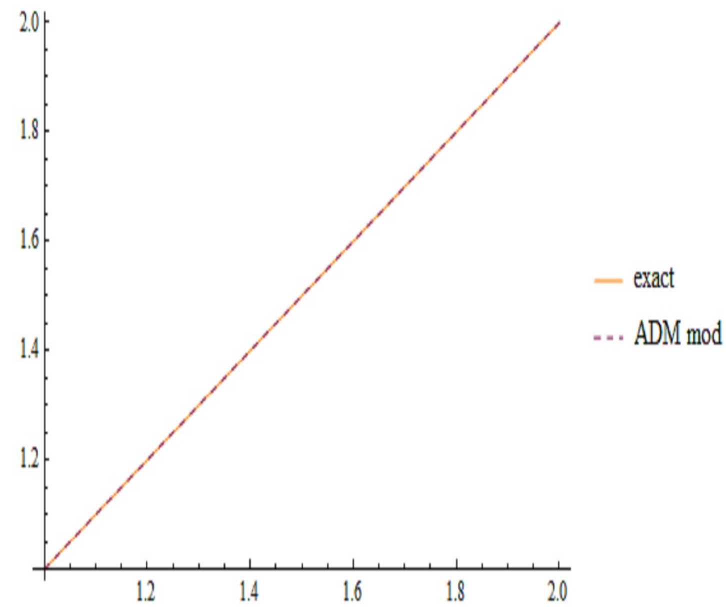

Figure 1. A comparison between the exact solution and the ADM modication

Example 4.3. *Consider the following fractional equation with a nonfifferentiable nonlinear term*

$$D^{\frac{3}{2}}y\left(t\right)+y\left|y\right|=t^{4}+\frac{2}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}},\ y\left(0\right)=0. \tag{18}$$

Appling $D^{-\alpha}\left(\cdot\right)$ to both sides of Eq. (18), and using initial condition,

$$y\left(t\right)=y\left(0\right)+I^{3/2}\left[t^{4}+\frac{2}{\Gamma(\frac{3}{2})}t^{1/2}\right]-I^{3/2}[y\left|y\right|]. \tag{19}$$

From (4) and (5), Eq. (19) becomes

$$\sum_{i=0}^{\infty}y_{i}\left(t\right)=y\left(0\right)+I^{3/2}\left[t^{4}+\frac{2}{\Gamma(\frac{3}{2})}t^{1/2}\right]-I^{3/2}\left[\sum_{i=0}^{\infty}y_{i}\left(t\right)\left|\sum_{i=0}^{\infty}y_{i}\left(t\right)\right|\right].$$

Then adding the term $I^{\alpha}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right]-pI^{\alpha}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right]$, the recursive relation become

$$\begin{aligned} y_{0} &= y\left(0\right)+I^{3/2}[t^{4}+\frac{2}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}]+I^{3/2}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right] \\ y_{n+1} &= -I^{3/2}\left[\sum_{i=0}^{\infty}y_{i}\left(t\right)\left|\sum_{i=0}^{\infty}y_{i}\left(t\right)\right|\right]-pI^{3/2}\left[\sum_{n=0}^{\infty}a_{n}t^{n}\right]. \end{aligned}$$

Appling formula (9), we find

$$\begin{aligned} y_{0} &= I^{3/2}[t^{4}+\frac{2}{\Gamma(\frac{3}{2})}t^{\frac{1}{2}}]+I^{3/2}[a_{0}t^{0}+a_{1}t^{1}+a_{2}t^{2}+a_{3}t^{3}+...] \\ y_{1} &= -I^{3/2}[y_{0}\left|y_{0}\right|]-pI^{3/2}[a_{0}t^{0}+a_{1}t^{1}+a_{2}t^{2}+a_{3}t^{3}+...]. \end{aligned} \tag{20}$$

Hence, we obtain

$$\begin{aligned} y_{0} &= \frac{4a_{0}}{3\sqrt{\pi}}t^{3/2}+t^{2}+\frac{8a_{1}}{15\sqrt{\pi}}t^{5/2}+\frac{32a_{2}}{105\sqrt{\pi}}t^{7/2}+\frac{64a_{3}}{315\sqrt{\pi}}t^{9/2}+..; \\ y_{1} &= \frac{-4a_{0}p}{3\sqrt{\pi}}t^{\frac{3}{2}}-\frac{8a_{1}p}{15\sqrt{\pi}}t^{\frac{5}{2}}-\frac{32a_{2}p}{105\sqrt{\pi}}t^{\frac{7}{2}}-\frac{1024a_{0}^{2}}{2835\pi^{3/2}}t^{\frac{9}{2}} \\ &\quad -\frac{64a_{3}p}{315\sqrt{\pi}}t^{\frac{9}{2}}-(\frac{512}{3465\sqrt{\pi}}+\frac{512a_{4}p}{3465\sqrt{\pi}})t^{\frac{11}{2}}+... \end{aligned}$$

Setting $y_1 = 0$, we obtain the coefficients $a_0, a_1, a_2, a_3, ...$

$$\frac{-4a_0p}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{8a_1p}{15\sqrt{\pi}}t^{\frac{5}{2}} - \frac{32a_2p}{105\sqrt{\pi}}t^{\frac{7}{2}} - \frac{1024a_0^2}{2835\pi^{3/2}}t^{\frac{9}{2}} - \frac{64a_3p}{315\sqrt{\pi}}t^{\frac{9}{2}} - (\frac{512}{3465\sqrt{\pi}} + \frac{512a_4p}{3465\sqrt{\pi}})t^{\frac{11}{2}} + ...$$

and so

$$a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ a_4 = \frac{-1}{p},\ \textit{and } a_k = 0 \textit{ for } k = 5, 6, ...$$

Substituting in Eq. (4), we get

$$y\left(t\right) = t^2 + \frac{512}{3465\sqrt{\pi}}t^{11/2} + I^{1/2}\left[0 + 0 + 0 + 0 - \frac{1}{p}t^4 ...\right].$$

Let $p = 1$ we get

$$\begin{aligned} y\left(t\right) &= t^2 + \frac{512}{3465\sqrt{\pi}}t^{11/2} + I^{1/2}\left[-\frac{1}{p}t^4\right] \\ &= t^2 + \frac{512}{3465\sqrt{\pi}}t^{11/2} - \frac{512}{3465\sqrt{\pi}}t^{11/2} \\ &= t^2. \end{aligned}$$

where $y\left(t\right) = t^2$ is the exact solution of this problem.

Example 4.4. *Consider the following fractional equation with a nonfifferentiable nonlinear term [**?**]*

$$D^{0.1}D^{0.5}y\left(t\right) = \frac{t}{1+t} + \frac{|y|^{0.5}}{1+t},\ y\left(0\right) = 1, D_0^{0.5}y\left(0\right) = 1. \tag{21}$$

Frist Appling $I^{0.1}\left(\cdot\right)$ to both sides of Eq. (21) using initial condition, so we have

$$\begin{aligned} D^{0.5}y\left(t\right) &= y\left(0\right) + I^{0.1}\left[\frac{t}{1+t} + \frac{|y|^{0.5}}{1+t}\right] \\ &= 1 + I^{0.1}\left[\frac{t}{1+t} + \frac{|y|^{0.5}}{1+t}\right]. \end{aligned}$$

Second Appling $I^{0.5}\left(\cdot\right)$

$$\begin{aligned} y\left(t\right) &= D_0^{0.5}y\left(0\right) + I^{0.5}\left[1\right] + I^{0.5}\left[I^{0.1}\left[\frac{t}{1+t} + \frac{|y|^{0.5}}{1+t}\right]\right] \\ &= 1 + \frac{1}{\Gamma(\frac{3}{2})}t^{1/2} + I^{0.6}\left[\frac{t}{1+t} + \frac{|y|^{0.5}}{1+t}\right]. \end{aligned}$$

Adding the term $I^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^n\right] - pI^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^n\right]$ so the solution will be in the form

$$\begin{aligned} y_0 &= 1 + \frac{1}{\Gamma(\frac{3}{2})}t^{1/2} + I^{0.6}\left[\sum_{n=0}^{\infty} a_n t^n\right] \\ y_{n+1} &= I^{0.6}\left[\frac{t}{1+t} + \frac{\left|\sum_{i=0}^{\infty} y_i\left(t\right)\right|^{0.5}}{1+t}\right] - pI^{0.6}\left[\sum_{n=0}^{\infty} a_n t^n\right]. \end{aligned} \tag{22}$$

Appling formula (9), we find

$$\begin{aligned} y_0 &= 1 + \frac{1}{\Gamma(\frac{3}{2})}t^{1/2} + I^{0.6}[a_0t^0 + a_1t^1 + a_2t^2 + a_3t^3 + ...] \\ y_1 &= I^{0.6}\left[\frac{t}{1+t} + \frac{|y_0|^{0.5}}{1+t}\right] - pI^{0.6}\left[a_0t^0 + a_1t^1 + a_2t^2 + a_3t^3 + ...\right]. \end{aligned}$$

Hence, we obtain

$$y_0 = 1 + \frac{2}{\sqrt{\pi}} t^{1/2} + 1.119 a_0 t^{0.6} + 0.699 a_1 t^{1.6} + 0.538 a_2 t^{2.6} + ...;$$
$$y_1 = (1.119 - 1.119 a_0 p)\, t^{0.6} + (1.119 - 1.119 a_0 p - 0.699 a_1 p)\, t^{1.6} + ...$$

Setting $y_1 = 0$, *we obtain the coefficients* $a_0, a_1, a_2, a_3, ...$*Put* $y_1 = 0$ *then getting the coefficients* $a_0, a_1, a_2, a_3, ...$

$$(1.119 - 1.119 a_0 p)\, t^{0.6} + (1.119 - 1.119 a_0 p - 0.699 a_1 p)\, t^{1.6} + ... = 0$$

and so

$$a_0 = \frac{1}{p},\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,\ \textit{and}\ a_k = 0\, \textit{for}\ k = 4, 5, ...$$

Substituting in Eq. (4), we get

$$y\,(t) = 1 + \frac{2}{\sqrt{\pi}} t^{1/2} + I^{0.6}\left[\frac{1}{p} t^0 + 0 + 0 + ...\right].$$

Let $p = 1$ *we get [see fig(2)]*

$$y\,(t) = 1 + \frac{2}{\sqrt{\pi}} t^{1/2} + I^{0.6}\,[1]$$
$$= 1 + \frac{2}{\sqrt{\pi}} t^{1/2} + \frac{1}{\Gamma(\frac{8}{5})} t^{0.6}.$$

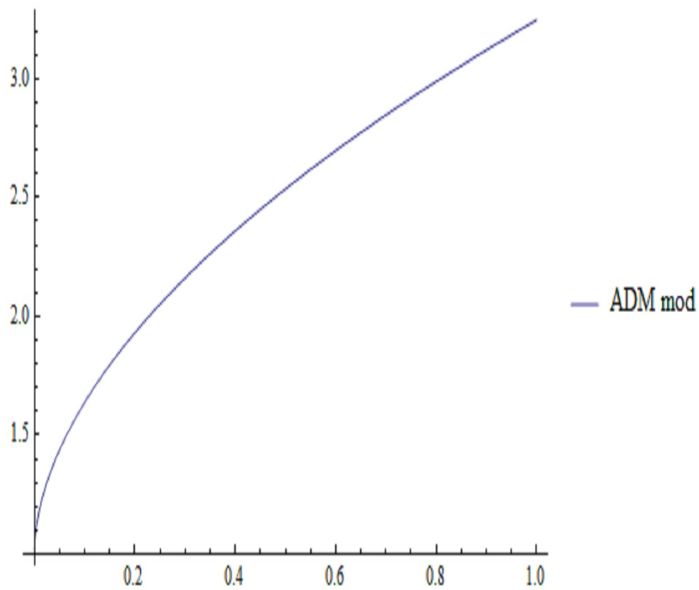

Figure 2. A graph of ADM mod

5 Conclusion

The method described here is used to solve nonlinear FDEs. This acceleration technique is based on the Adomian decomposition method with a novel definition of the Adomian polynomials, which only requires the calculation of the first component, allowing us to deal with a small number of terms and achieve a quick answer. This novel technique includes a feature that addresses the problem of nondifferentiable terms in problems. The efficiency of the proposed mechanism was proved using this technique, which is supported by nonlinear examples. In future work, we can use this method in different kinds of equations to simplify the solution, such as fractional quadratic integral equations and multidimensional FDEs.

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