Mathematics and Statistics 10(5): 995-1004, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100511

The Exact Solutions of the Space and Time Fractional Telegraph Equations by the Double Sadik Transform Method

Prapart Pue-on

Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham 44150, Thailand

Received June 8, 2022; Revised August 22, 2022; Accepted August 30, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] Prapart Pue-on, "The Exact Solutions of the Space and Time Fractional Telegraph Equations by the Double Sadik Transform Method," Mathematics and Statistics, Vol.10, No.5, pp. 995-1004, 2022. DOI: 10.13189/ms.2022.100511*

***(b)**: Prapart Pue-on, (2022). The Exact Solutions of the Space and Time Fractional Telegraph Equations by the Double Sadik Transform Method. Mathematics and Statistics, 10(5), 995-1004. DOI: 10.13189/ms.2022.100511*

Abstract The double integral transform is a robust implementation that is important in handling scientific and engineering problems. Besides its simplicity of use and straightforward application to the issue, the ability to reduce the problems to an algebraic equation that can be easily solved is a substantial advantage of the tool. Among the several integral transforms, the double Sadik transform is acknowledged to be one of the most frequently used in solving differential and integral equations. This work deals with investigating a generalized double integral transform called the double Sadik transform. The proof of the double Sadik transforms for partial fractional derivatives in the Caputo sense is displayed, and the double Sadik transforms method is introduced. The method has been applied to solve the initial boundary value problems for linear space and time-fractional telegraph equations. Moreover, the suggested strategy can be used on non-linear problems via an iterative method and a decomposition concept. Some known-solution questions are evaluated with relatively minimal computational cost. The results are represented by utilizing the Mittag-Leffler function and covering the solution of a classical telegraph equation. The obtained exact solutions not only show the accuracy and efficiency of the technique, but also reveal reliability when compared to those obtained using other methods.

Keywords Fractional Telegraph Equations, Exact Solution, Caputo Fractional Derivatives, Double Integral Transform, Sadik Transform

1 Introduction

A telegraph equation is a type of hyperbolic partial differential equation that was put forward by Oliver Heaviside in 1880 for describing the transmission line model. Since then, this class of equations has been discovered in many processes, such as signal analysis for electrical signal transmission and propagation, simulating reaction diffusion, and the optimization of a guided communication system [1, 2, 3]. If classical derivatives in the telegraph equation are replaced by fractional derivatives, this equation is well known as a space and time fractional telegraph equation. This sort of equation is crucial in several disciplines, including fluid mechanics, mathematical biology, electrochemistry, and physics, and has attracted the attention of scholars for over a decade. Despite the fact that space-time fractional telegraph equations can be found in a wide range of situations, the exact solution of the equations does seem to be unidentified. The complexity in addressing the solution of space and time fractional telegraph equations is a challenging problem that has piqued the attention of researchers. Recently, numerous techniques for finding solutions to this kind of equation have been developed, such as He's variational iteration method [4], Adomian decomposition method [5], generalized differential transform [6], Laplace variational iteration method [7], double Laplace transform method [8], perturbation theory and the Laplace transformation [9], Fourier transform [10, 11], and method of separation of variables [12].

The Sadik transform is a powerful mathematical tool that can be used in a range of different fields of engineering and science. It was originally introduced by Sadik in 2018 [13] and is regarded as a generalization of many integral transforms, such as the Laplace transform, the Elzaki transform, the Aboodh transform, the Kamal transform, and the Tarig transform, etc., because of adjusting the parameters. As a result, the progressive investigation of such transformations will encompass all the others. Researchers have been investigating developments and applications of the Sadik transform for several years, some of which are mentioned below. Redhwana et al [14] provided some properties of Sadik transform and its applications of fractional-order dynamical systems in control theory. Pue-on [15] has devised the Sadik decomposition method to overcome a system of nonlinear fractional Volterra integro-differential equations of the convolution type. Aggarwal and Bhatnagar [16] have applied the Sadik transform to population growth and decay problems. Ganesh and Nitin [17] obtained the solution to linear partial integro-differential equation by using Sadik Transform and Shivaji and Nitin [18] have applied the Sadik transform to solve a linear Volterra integral equation of convolution type. However, the single Sadik transform has limitations when applied to the kinds of problems that involve fractional partial differential equations with multi-derivative terms. Hence, a new implementation called the double Sadik transform is defined. The double Sadik transforms are two consecutive single Sadik transforms that were introduced by Singh in 2019 [19]. Singh not only proved some theories of transformation but also applied them to a one-dimensional wave equation. Although the double Sadik transform for partial differential equations has been developed, the application of the double Sadik transform to fractional differential equations has not been investigated. This work is concerned with the use of the double Sadik transform to solve the fractional telegraph equation. It indicates an enhanced tool for solving fractional partial differential equations and covers the achievements of Dhunde and Waghmare [8].

The purpose of this study is to give the exact solution to the initial boundary value problem for the space-time fractional telegraph equation,

$$\begin{aligned} \frac{\partial^{\gamma_1} u(x,t)}{\partial x^{\gamma_1}} &= a\frac{\partial^{\gamma_2} u(x,t)}{\partial t^{\gamma_2}} + b\frac{\partial^{\gamma_3} u(x,t)}{\partial t^{\gamma_3}} + cu(x,t) \\ &+ Nu(x,t) + g(x,t), \qquad (1) \\ & \quad 1 < \gamma_1, \gamma_2 \leq 2,\ 0 < \gamma_3 \leq 1,\ x,t \geq 0, \end{aligned}$$

with initial conditions

$$u(x,0) = h_1(x),\ u_t(x,0) = h_2(x), \qquad (2)$$

and boundary conditions

$$u(0,t) = f_1(t),\ u_x(0,t) = f_2(t), \qquad (3)$$

where a, b, c are constants, $Nu(x,t)$ is a nonlinear function and $g(x,t)$ is given function by employing the double Sadik transform. All the above fractional derivatives are regarded in the Caputo sense.

The rest of the paper is managed as follows. Section 2 presents a basic definition of fractional integrals, fractional Caputo derivatives, and the double Sadik transform. The proof of the double Sadik transform of fractional partial derivatives and the application of the double Sadik transform to the space and time-fractional telegraph equation are shown in Section 3. Finally, illustrative examples are shown in the last section.

2 Preliminaries

The essential notion of fractional calculus is presented in this section. The Riemann-Liouville fractional integral and the Caputo fractional derivative are provided. In addition, the basic idea and facts about the double Sadik transform are discussed.

2.1 Fractional Calculus

Definition 2.1 *[20] The Riemann-Liouville fractional integral operator of order $\gamma \geq 0$ is defined by*

$$J_a^\gamma \phi(x) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_a^x \frac{\phi(\tau)}{(x-\tau)^{1-\gamma}} d\tau, \gamma > 0,\ x > 0, \\ \phi(x), \gamma = 0, \end{cases}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

One can note that J_a^γ is linear operators, that is for any constant $c_1,\ c_2$

$$J_a^\gamma(c_1\phi(x) + c_2\psi(x)) = c_1 J_a^\gamma \phi(x) + c_2 J_a^\gamma \psi(x).$$

Moreover, the following properties can be proved for the Riemann-Liouville fractional integral:

1. $J_a^{\gamma_1} J_a^{\gamma_2} \phi(x) = J_a^{\gamma_2} J_a^{\gamma_1} \phi(x)$
2. $J_a^{\gamma_1} J_a^{\gamma_2} \phi(x) = J_a^{\gamma_1+\gamma_2} \phi(x)$.
3. $J_a^\gamma C = \frac{C}{\Gamma(\gamma+1)}(x-a)^\gamma$ for any constant $C \in \mathbb{R}$.
4. $J_a^\gamma (x-a)^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\gamma+1)}(x-a)^{\gamma+\mu},\ \ \mu > -1$.

Definition 2.2 *[20] For $n-1 < \gamma < n, n \in \mathbb{N}$. The Caputo fractional derivative operator of order γ is defined by*

$$D_a^\gamma \phi(x) = \frac{1}{\Gamma(n-\gamma)} \int_a^x (x-\tau)^{-\gamma+n-1} \phi^{(n)}(\tau) d\tau,\ \gamma > 0,\ x > 0,$$

where the function $\phi(x)$ has absolutely continuous derivatives up to order $n-1$. Specially, if $\gamma = n \in \mathbb{N}$, $D_a^\gamma \phi(x) = \phi^{(n)}(x)$.

The operators D_a^γ satisfies the following properties:

1. D_a^γ is a linear operator.
2. $D_a^\gamma C = 0$ for any constant $C \in \mathbb{R}$.
3. $D_a^\gamma (x-a)^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\gamma)}(x-a)^{p-\gamma}$, for $n-1 < \gamma < n, p > n-1$, and it is equal to zero otherwise.
4. $(J_a^\gamma D_a^\gamma \phi)(x) = \phi(x) - \sum_{k=0}^{\infty} \frac{\phi^{(k)}(a)}{k!}(x-a)^k$ for $\phi \in C^n[a,b]$ and $n-1 < \gamma < n$, with $n \in \mathbb{N}$.
5. If $\gamma \geq 0, \phi \in C[a,b]$, then $D^\gamma J^\gamma \phi(x) = \phi(x)$.

Definition 2.3 *The Caputo fractional derivative of function $u(x,t)$ is defined in [7] as*

$$\begin{aligned} \frac{\partial^{\gamma_1} u(x,t)}{\partial x^{\gamma_1}} &= \frac{1}{\Gamma(m-\gamma_1)} \int_0^x (x-\xi)^{m-\gamma_1-1} \frac{\partial^m u(\xi,t)}{\partial \xi^m} d\xi, \\ & \quad m-1 < \gamma_1 \leq m, m \in \mathbb{N}, \\ \frac{\partial^{\gamma_2} u(x,t)}{\partial t^{\gamma_2}} &= \frac{1}{\Gamma(n-\gamma_2)} \int_0^t (t-\tau)^{n-\gamma_2-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, \\ & \quad n-1 < \gamma_2 \leq n, n \in \mathbb{N}. \end{aligned}$$

2.2 The Double Sadik Transform

Before proceeding on to the double Sadik transform, let us review the definitions of a one-dimensional Sadik transform as follows.

Definition 2.4 *[13] If $f(t)$ is piecewise continuous function on the interval $0 \le t \le A$ for any $A > 0$ and $|f(t)| \le Ke^{Bt}$ when $t \ge M$, for any real constant B and some positive constant K and M. Then Sadik transform of $f(t)$ is defined by*

$$\mathcal{S}\left[f(t)\right] = \frac{1}{v^\beta}\int_0^\infty f(t)e^{-tv^\alpha}dt = F(v^\alpha, \beta)$$

where v is complex variable, α is any non zero real number, and β is any real number. Here $\mathcal{S}$ is called the Sadik transform operator.

Note 2.1 *The Sadik transform, which is a generalization of several integral transforms, can be turned to others by altering the values of α and β, as shown below.*

- *If $\alpha = 1, \beta = 0$, the Laplace transform is derived.*
- *If $\alpha = 1, \beta = 1$, the Aboodh Transform is obtained.*
- *If $\alpha = 1, \beta = -1$, the Laplace Carson transform is displayed.*
- *If $\alpha = -1, \beta = 0$, the Kamal Transform is shown.*
- *If $\alpha = -1, \beta = 1$, the Sumudu Transform is derived.*
- *If $\alpha = -1, \beta = -1$, the Elzaki Transform is found.*
- *If $\alpha = -2, \beta = 1$, the Tarig Transform is reached.*

Using the preceding concept, we can now extend the Sadik transform to the function of several variables, as shown below.

Definition 2.5 *Let $f(x,t)$ be a function of two variables x and t defined in the positive quadrant of the $xt-$plane. The Sadik transform of $f(x,t)$ with respect to x is defined by*

$$\mathcal{S}_x[f(x,t)] = F(w,t:\alpha,\beta) = \frac{1}{w^\beta}\int_0^\infty e^{-xw^\alpha}f(x,t)dx$$

and the Sadik transform of $f(x,t)$ with respect to t is defined by

$$\mathcal{S}_t[f(x,t)] = F(x,v:\alpha,\beta) = \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}f(x,t)dt.$$

Definition 2.6 *[19] Let $f(x,t)$ be a function that can be expressed as a convergent infinite series and $(x,t) \in \mathbb{R}^2$, the double Sadik transform is denoted by $\mathcal{S}_2[f(x,t)] = F(v,w:\alpha,\beta)$ and defined by*

$$\begin{aligned}\mathcal{S}_2[f(x,t)] &= F(w,v:\alpha,\beta) \\ &= \frac{1}{v^\beta w^\beta}\int_0^\infty\int_0^\infty e^{-(tv^\alpha+xw^\alpha)}f(x,t)dxdt\end{aligned}$$

where $x,t > 0$ and v,w are transform variables for t and x respectively, α is any non-zero real number and β is any real number, whenever the double improper integral is convergent. Here $\mathcal{S}_2$ is called the double Sadik transform operator.

One can deduce from the aforementioned definition that

1. $\mathcal{S}_2[f(x,t)] = \mathcal{S}_t\mathcal{S}_x[f(x,t)] = \mathcal{S}_x\mathcal{S}_t[f(x,t)]$.
2. If $f(x,t) = h(x)g(t)$ then
$$\mathcal{S}_2[f(x,t)] = \mathcal{S}_x\mathcal{S}_t[f(x,t)] = \mathcal{S}_x[h(x)]\mathcal{S}_t[g(t)].$$
3. The double Sadik transform operator $\mathcal{S}_2$ is a linear operation, i.e., for any constants c_1, c_2,
$$\mathcal{S}_2[c_1f(x,t) + c_2g(x,t)] = c_1\mathcal{S}_2[f(x,t)] + c_2\mathcal{S}_2[g(x,t)].$$

Theorem 2.1 *[19] Let $\mathcal{S}_2[f(x,t)] = F(w,v:\alpha,\beta)$ then the double Sadik transform for the partial derivatives of an arbitrary integer order are*

$$\begin{aligned}\mathcal{S}_2[\frac{\partial^m f(x,t)}{\partial x^m}] &= w^{m\alpha}F(w,v:\alpha,\beta) \\ &\quad -\sum_{k=0}^{m-1} w^{(m-1-k)\alpha-\beta}\mathcal{S}_t\left[\frac{\partial^k f(0,t)}{\partial x^k}\right], \\ \mathcal{S}_2[\frac{\partial^n f(x,t)}{\partial t^n}] &= v^{n\alpha}F(w,v:\alpha,\beta) \\ &\quad -\sum_{k=0}^{n-1} v^{(n-1-k)\alpha-\beta}\mathcal{S}_x\left[\frac{\partial^k f(x,0)}{\partial t^k}\right].\end{aligned}$$

Definition 2.7 *[14] The Mittag-Leffler function is defined by*

$$E_{p,q}(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(pk+q)},\ \ t,q \in \mathbb{C},\ \Re(p) > 0,\ \Re(q) > 0.$$

It is important to remember that when $q = 1$, the particular case of the Mittag-Leffler function is

$$E_{p,1}(t) = E_p(t) = \sum_{k=0}^\infty \frac{t^k}{\Gamma(pk+1)},\ \ t,p \in \mathbb{C},\ \Re(p) > 0.$$

Furthermore, one can show that

1. $E_{1,1}(t) = E_1(t) = e^t$.
2. $E_p(t) = E_{2p}(t^2) + tE_{2p,1+p}(t^2)$.
3. $E_{p,q}(t) = tE_{p,p+q}(t) + \frac{1}{\Gamma(q)}$.

Theorem 2.2 *[14] Let $f(t) = t^{pm+q-1}E_{p,q}(\pm at^p)$. The Sadik transform of f is given by:*

$$\frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}t^{pm+q-1}E^{(m)}_{p,q}(\pm at^p)dt = \frac{m!v^{\alpha p-(\alpha q+\beta)}}{(v^{\alpha p}\mp a)^{m+1}}$$

where $p,q \in \mathbb{C}$, $\Re(p) > 0, \Re(q) > 0, \Re(v) > |a|^{\frac{1}{\Re(\alpha p)}}$ and $E^{(m)}_{p,q}(z) = \frac{d^m}{dz^m}E_{p,q}(z)$.

3 Main Results

3.1 The Double Sadik Transform for Fractional Caputo Derivative

Theorem 3.1 *Let $\mathcal{S}_2[f(x,t)] = F(w,v:\alpha,\beta)$ then the double Sadik transform for the partial fractional Caputo derivatives are*

$$\begin{aligned}
\mathcal{S}_2[\frac{\partial^{\gamma_1} f(x,t)}{\partial x^{\gamma_1}}] &= w^{\gamma_1\alpha}F(w,v:\alpha,\beta) \\
&\quad -\sum_{k=0}^{m-1} w^{(\gamma_1-1-k)\alpha-\beta}\mathcal{S}_t[\frac{\partial^k f(0,t)}{\partial x^k}], \\
&\quad m-1<\gamma_1\leq m, m\in\mathbb{N}, \\
\mathcal{S}_2[\frac{\partial^{\gamma_2} f(x,t)}{\partial t^{\gamma_2}}] &= v^{\gamma_2\alpha}F(w,v:\alpha,\beta) \\
&\quad -\sum_{k=0}^{n-1} v^{(\gamma_2-1-k)\alpha-\beta}\mathcal{S}_x[\frac{\partial^k f(x,0)}{\partial t^k}], \\
&\quad n-1<\gamma_2\leq n, n\in\mathbb{N}.
\end{aligned}$$

Proof. In view of definitions 2.3 and 2.6, as well as Theorem 2.1, one can show that

$$\begin{aligned}
&\mathcal{S}_2[\frac{\partial^{\gamma_1} f(x,t)}{\partial x^{\gamma_1}}] \\
&= \frac{1}{v^\beta w^\beta}\int_0^\infty\int_0^\infty e^{-(tv^\alpha+xw^\alpha)}\frac{\partial^{\gamma_1} f(x,t)}{\partial x^{\gamma_1}}dxdt \\
&= \frac{1}{v^\beta w^\beta}\int_0^\infty\int_0^\infty e^{-(tv^\alpha+xw^\alpha)}\Big[\frac{1}{\Gamma(m-\gamma_1)} \\
&\quad \times\int_0^x (x-\xi)^{m-\gamma_1-1}\frac{\partial^m f(\xi,t)}{\partial \xi^m}d\xi\Big]dxdt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\Big[\frac{1}{\Gamma(m-\gamma_1)}\frac{1}{w^\beta}\int_0^\infty e^{-xw^\alpha} \\
&\quad \times\int_0^x (x-\xi)^{m-\gamma_1-1}\frac{\partial^m f(\xi,t)}{\partial \xi^m}d\xi dx\Big]dt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\Big[\frac{1}{\Gamma(m-\gamma_1)}\frac{1}{w^\beta}\int_0^\infty\int_\xi^\infty (x-\xi)^{m-\gamma_1-1} \\
&\quad \times e^{-xw^\alpha}\frac{\partial^m f(\xi,t)}{\partial \xi^m}dxd\xi\Big]dt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\Big[\frac{1}{\Gamma(m-\gamma_1)}\frac{1}{w^\beta}\int_0^\infty\int_0^\infty \tau^{m-\gamma_1-1}e^{-(\tau+\xi)w^\alpha} \\
&\quad \times\frac{\partial^m f(\xi,t)}{\partial \xi^m}d\tau d\xi\Big]dt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\Big[\frac{1}{\Gamma(m-\gamma_1)}\frac{1}{w^\beta}\int_0^\infty e^{-\xi w^\alpha}\frac{\partial^m f(\xi,t)}{\partial \xi^m} \\
&\quad \times\int_0^\infty \tau^{m-\gamma_1-1}e^{-\tau w^\alpha}d\tau d\xi\Big]dt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\Big[\frac{1}{\Gamma(m-\gamma_1)}\int_0^\infty e^{-\xi w^\alpha}\frac{\partial^m f(\xi,t)}{\partial \xi^m} \\
&\quad \times\Big(\frac{1}{w^\beta}\int_0^\infty e^{-\tau w^\alpha}\tau^{m-\gamma_1-1}d\tau\Big)d\xi\Big]dt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\Big[\frac{1}{\Gamma(m-\gamma_1)}\int_0^\infty e^{-\xi w^\alpha}\frac{\partial^m f(\xi,t)}{\partial \xi^m} \\
&\quad \times\mathcal{S}_x\Big(x^{m-\gamma_1-1}\Big)d\xi\Big]dt \\
&= \frac{1}{v^\beta}\int_0^\infty e^{-tv^\alpha}\Big[\frac{1}{\Gamma(m-\gamma_1)}\int_0^\infty e^{-\xi w^\alpha}\frac{\partial^m f(\xi,t)}{\partial \xi^m} \\
&\quad \times\Big(\frac{\Gamma(m-\gamma_1)}{w^{(m-\gamma_1)\alpha+\beta}}\Big)d\xi\Big]dt \\
&\mathcal{S}_2[\frac{\partial^{\gamma_1} f(x,t)}{\partial x^{\gamma_1}}] \\
&= \frac{1}{w^{(m-\gamma_1)\alpha}}\Big[\frac{1}{v^\beta w^\beta}\int_0^\infty\int_0^\infty e^{-(tv^\alpha+xw^\alpha)}\frac{\partial^m f(x,t)}{\partial x^m}dxdt\Big] \\
&= \frac{1}{w^{(m-\gamma_1)\alpha}}\mathcal{S}_2\Big[\frac{\partial^m f(x,t)}{\partial x^m}\Big] \\
&= \frac{1}{w^{(m-\gamma_1)\alpha}}\Big[w^{m\alpha}F(w,v:\alpha,\beta) \\
&\quad -\sum_{k=0}^{m-1} w^{(m-1-k)\alpha-\beta}\mathcal{S}_t[\frac{\partial^k f(0,t)}{\partial x^k}]\Big] \\
&= w^{\gamma_1\alpha}F(w,v:\alpha,\beta)-\sum_{k=0}^{m-1} w^{(\gamma_1-1-k)\alpha-\beta}\mathcal{S}_t[\frac{\partial^k f(0,t)}{\partial x^k}].
\end{aligned}$$

The remainder can be proved using the same strategy.

3.2 Application of Double Sadik Transform Method to Space and Time Fractional Telegraph Equations

Applying the double Sadik transform on both sides of the fractional telegraph equation in space and time (1) and using the linearity property as well as theorem 3.1 leads to

$$\begin{aligned}&w^{\gamma_1\alpha}\mathcal{S}_2[u(x,t)]-w^{(\gamma_1-1)\alpha-\beta}\mathcal{S}_t[u(0,t)]\\&-w^{(\gamma_1-2)\alpha-\beta}\mathcal{S}_t[u_x(0,t)]\\&=a\Big[v^{\gamma_2\alpha}\mathcal{S}_2[u(x,t)]-v^{(\gamma_2-1)\alpha-\beta}\mathcal{S}_x[u(x,0)]\\&-v^{(\gamma_2-2)\alpha-\beta}\mathcal{S}_x[u_t(x,0)]\Big]\\&+b\Big[v^{\gamma_3\alpha}\mathcal{S}_2[u(x,t)]-v^{(\gamma_3-1)\alpha-\beta}\mathcal{S}_x[u(x,0)]\Big]\\&+c\mathcal{S}_2[u(x,t)]+\mathcal{S}_2[Nu(x,t)]+\mathcal{S}_2[g(x,t)].\end{aligned}\tag{4}$$

Suppose the single Sadik transform of the initial and boundary conditions (2)- (3) are

$$\begin{aligned}\mathcal{S}_t[u(0,t)]&=\mathcal{S}_t[f_1(t)]=F_1(v:\alpha,\beta),\\\mathcal{S}_t[u_x(0,t)]&=\mathcal{S}_t[f_2(t)]=F_2(v:\alpha,\beta),\\\mathcal{S}_x[u(x,0)]&=\mathcal{S}_x[h_1(x)]=H_1(w:\alpha,\beta),\\\mathcal{S}_x[u_t(x,0)]&=\mathcal{S}_x[h_2(x)]=H_2(w:\alpha,\beta).\end{aligned}$$

Substituting these functions into (4), one find that

$$\begin{aligned}&w^{\gamma_1\alpha}\mathcal{S}_2[u(x,t)]-w^{(\gamma_1-1)\alpha-\beta}F_1(v:\alpha,\beta)\\&-w^{(\gamma_1-2)\alpha-\beta}F_2(v:\alpha,\beta)\\&=a\Big[v^{\gamma_2\alpha}\mathcal{S}_2[u(x,t)]-v^{(\gamma_2-1)\alpha-\beta}H_1(w:\alpha,\beta)\\&-v^{(\gamma_2-2)\alpha-\beta}H_2(w:\alpha,\beta)\Big]\\&+b\Big[v^{\gamma_3\alpha}\mathcal{S}_2[u(x,t)]-v^{(\gamma_3-1)\alpha-\beta}H_1(w:\alpha,\beta)\Big]\\&+c\mathcal{S}_2[u(x,t)]+\mathcal{S}_2[Nu(x,t)]+G(w,v:\alpha,\beta).\end{aligned}$$

where $\mathcal{S}_2[g(x,t)]=G(w,v:\alpha,\beta)$. By simplifying the equation, we arrive at

$$\begin{aligned}&\mathcal{S}_2[u(x,t)]=\frac{1}{(w^{\gamma_1\alpha}-av^{\gamma_2\alpha}-bv^{\gamma_3\alpha}-c)}\\&\times\Big[w^{(\gamma_1-1)\alpha-\beta}F_1(v:\alpha,\beta)+w^{(\gamma_1-2)\alpha-\beta}F_2(v:\alpha,\beta)\\&-av^{(\gamma_2-1)\alpha-\beta}H_1(w:\alpha,\beta)-av^{(\gamma_2-2)\alpha-\beta}H_2(w:\alpha,\beta)\\&-bv^{(\gamma_3-1)\alpha-\beta}H_1(w:\alpha,\beta)+\mathcal{S}_2[Nu(x,t)]\\&+G(w,v:\alpha,\beta)\Big]\end{aligned}\tag{5}$$

If $Nu(x,t)=0$, one can take the inverse double Sadik transform to both sides of the equation yields the solution to (1),

$$\begin{aligned}u(x,t)=\mathcal{S}_2^{-1}\Big[&\frac{1}{(w^{\gamma_1\alpha}-av^{\gamma_2\alpha}-bv^{\gamma_3\alpha}-c)}\\&\times\Big[w^{(\gamma_1-1)\alpha-\beta}F_1(v:\alpha,\beta)+w^{(\gamma_1-2)\alpha-\beta}F_2(v:\alpha,\beta)\\&-av^{(\gamma_2-1)\alpha-\beta}H_1(w:\alpha,\beta)-av^{(\gamma_2-2)\alpha-\beta}H_2(w:\alpha,\beta)\\&-bv^{(\gamma_3-1)\alpha-\beta}H_1(w:\alpha,\beta)+G(w,v:\alpha,\beta)\Big]\Big].\end{aligned}$$

Here, assume that the inverse double Sadik transform exists for each of the terms on the right side.
If $Nu(x,t)\neq 0$, the iterative stragety is required. Assume the solution is expressed in the form

$$u(x,t)=\sum_{k=0}^{\infty}u_k(x,t).\tag{6}$$

Substitute (6) in equation (5), the result is

$$\begin{aligned}&\mathcal{S}_2\Big[\sum_{k=0}^{\infty}u_k(x,t)\Big]=\frac{1}{(w^{\gamma_1\alpha}-av^{\gamma_2\alpha}-bv^{\gamma_3\alpha}-c)}\\&\times\Big[w^{(\gamma_1-1)\alpha-\beta}F_1(v:\alpha,\beta)+w^{(\gamma_1-2)\alpha-\beta}F_2(v:\alpha,\beta)\\&-av^{(\gamma_2-1)\alpha-\beta}H_1(w:\alpha,\beta)-av^{(\gamma_2-2)\alpha-\beta}H_2(w:\alpha,\beta)\\&-bv^{(\gamma_3-1)\alpha-\beta}H_1(w:\alpha,\beta)\\&+\mathcal{S}_2\Big[N\Big(\sum_{k=0}^{\infty}u_k(x,t)\Big)\Big]+G(w,v:\alpha,\beta)\Big].\end{aligned}\tag{7}$$

The nonlinear term Nu is decomposed as

$$\begin{aligned}N\Big(\sum_{k=0}^{\infty}u_k(x,t)\Big)=N(u_0(x,t))+\sum_{i=1}^{\infty}\Big[N\Big(\sum_{k=0}^{i}u_k(x,t)\Big)\\-N\Big(\sum_{k=0}^{i-1}u_k(x,t)\Big)\Big]\end{aligned}$$

then equation (7) becomes

$$\begin{aligned}&\mathcal{S}_2\Big[\sum_{k=0}^{\infty}u_k(x,t)\Big]=\frac{1}{(w^{\gamma_1\alpha}-av^{\gamma_2\alpha}-bv^{\gamma_3\alpha}-c)}\\&\times\Big[w^{(\gamma_1-1)\alpha-\beta}F_1(v:\alpha,\beta)+w^{(\gamma_1-2)\alpha-\beta}F_2(v:\alpha,\beta)\\&-av^{(\gamma_2-1)\alpha-\beta}H_1(w:\alpha,\beta)-av^{(\gamma_2-2)\alpha-\beta}H_2(w:\alpha,\beta)\\&-bv^{(\gamma_3-1)\alpha-\beta}H_1(w:\alpha,\beta)+\mathcal{S}_2\Big[N(u_0(x,t))\\&+\sum_{i=1}^{\infty}\Big[N\Big(\sum_{k=0}^{i}u_k(x,t)\Big)-N\Big(\sum_{k=0}^{i-1}u_k(x,t)\Big)\Big]\Big]\\&+G(w,v:\alpha,\beta)\Big].\end{aligned}\tag{8}$$

Apply inverse double Sadik transform, one obtains

$$\begin{aligned}&\sum_{k=0}^{\infty}u_k(x,t)=\mathcal{S}_2^{-1}\Bigg[\frac{1}{(w^{\gamma_1\alpha}-av^{\gamma_2\alpha}-bv^{\gamma_3\alpha}-c)}\\&\times\Big[w^{(\gamma_1-1)\alpha-\beta}F_1(v:\alpha,\beta)+w^{(\gamma_1-2)\alpha-\beta}F_2(v:\alpha,\beta)\\&-av^{(\gamma_2-1)\alpha-\beta}H_1(w:\alpha,\beta)-av^{(\gamma_2-2)\alpha-\beta}H_2(w:\alpha,\beta)\\&-bv^{(\gamma_3-1)\alpha-\beta}H_1(w:\alpha,\beta)+\mathcal{S}_2\Big[N(u_0(x,t))\\&+\sum_{i=1}^{\infty}\Big[N\Big(\sum_{k=0}^{i}u_k(x,t)\Big)-N\Big(\sum_{k=0}^{i-1}u_k(x,t)\Big)\Big]\Big]\\&+G(w,v:\alpha,\beta)\Big]\Bigg].\end{aligned}\tag{9}$$

Define the recursive relation

$$u_0(x,t) = \mathcal{S}_2^{-1}\Bigg[\frac{1}{(w^{\gamma_1\alpha} - av^{\gamma_2\alpha} - bv^{\gamma_3\alpha} - c)} \times\Big[w^{(\gamma_1-1)\alpha-\beta}F_1(v:\alpha,\beta) + w^{(\gamma_1-2)\alpha-\beta}F_2(v:\alpha,\beta) - av^{(\gamma_2-1)\alpha-\beta}H_1(w:\alpha,\beta) - av^{(\gamma_2-2)\alpha-\beta}H_2(w:\alpha,\beta) - bv^{(\gamma_3-1)\alpha-\beta}H_1(w:\alpha,\beta) + G_1(w,v:\alpha,\beta)\Big]\Bigg],$$

$$u_1(x,t) = \mathcal{S}_2^{-1}\Bigg[\frac{1}{(w^{\gamma_1\alpha} - av^{\gamma_2\alpha} - bv^{\gamma_3\alpha} - c)} \times\mathcal{S}_2\Big[N(u_0(x,t)) + G_2(w,v:\alpha,\beta)\Big]\Bigg],$$

$$u_{n+1}(x,t) = \mathcal{S}_2^{-1}\Bigg[\frac{1}{(w^{\gamma_1\alpha} - av^{\gamma_2\alpha} - bv^{\gamma_3\alpha} - c)} \times\mathcal{S}_2\Big[N\Big(\sum_{k=0}^{n}u_k(x,t)\Big) - N\Big(\sum_{k=0}^{n-1}u_k(x,t)\Big)\Big]\Bigg], n \geq 1,$$

here $G(w,v:\alpha,\beta) = G_1(w,v:\alpha,\beta) + G_2(w,v:\alpha,\beta)$. Hence, the solution of nonlinear fractional telegraph equation is shown

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots. \tag{10}$$

4 Illustrative Examples

In this section, we use the suggested method to solve various space and time fractional telegraph equations that have a known solution to test its efficiency and accuracy.

Example 4.1 *[7] Consider the following homogeneous space-fractional telegraph equation*

$$\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t),\ 1 < \gamma \leq 2, x, t \geq 0,$$

subject to the initial and boundary conditions

$$u(x,0) = E_\gamma(x^\gamma) + xE_{\gamma,2}(x^\gamma),$$
$$u_t(x,0) = -E_\gamma(x^\gamma) - xE_{\gamma,2}(x^\gamma),$$
$$u(0,t) = e^{-t},\ u_x(0,t) = e^{-t}.$$

The exact solution of this problem is

$$u(x,t) = e^{-t}[E_\gamma(x^\gamma) + xE_{\gamma,2}(x^\gamma)].$$

Here $a = b = c = 1, g(x,t) = 0, f_1(t) = f_2(t) = e^{-t}$, $h_1(x) = E_\gamma(x^\gamma) + xE_{\gamma,2}(x^\gamma)$ and $h_2(x) = -E_\gamma(x^\gamma) - xE_{\gamma,2}(x^\gamma)$. The single Sadik transform of the initial and boundary conditions are

$$F_1(v:\alpha,\beta) = F_2(v:\alpha,\beta) = \mathcal{S}_t[e^{-t}] = \frac{1}{v^\beta(v^\alpha+1)},$$
$$H_1(w:\alpha,\beta) = \mathcal{S}_x[E_\gamma(x^\gamma) + xE_{\gamma,2}(x^\gamma)] = \frac{w^{\gamma\alpha-\alpha-\beta}}{w^{\gamma\alpha}-1} + \frac{w^{\gamma\alpha-2\alpha-\beta}}{w^{\gamma\alpha}-1},$$
$$H_2(w:\alpha,\beta) = \mathcal{S}_x[-E_\gamma(x^\gamma) - xE_{\gamma,2}(x^\gamma)] = -\frac{w^{\gamma\alpha-\alpha-\beta}}{w^{\gamma\alpha}-1} - \frac{w^{\gamma\alpha-2\alpha-\beta}}{w^{\gamma\alpha}-1}.$$

After substituting into (5), one gets

$$\mathcal{S}_2[u(x,t)] = \frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^\alpha - 1}\Big[\frac{w^{\gamma\alpha-\alpha-\beta}}{v^\beta(v^\alpha+1)}(w^{-\alpha}+1) - \frac{w^{\gamma\alpha-\alpha-\beta}}{w^{\gamma\alpha}-1}(w^{-\alpha}+1)v^{\alpha-\beta}\Big].$$

By simplifying the above equation, we find that

$$\mathcal{S}_2[u(x,t)] = \frac{1}{v^\beta(v^\alpha+1)} \cdot \Big(\frac{w^{\gamma\alpha-\alpha-\beta}}{w^{\gamma\alpha}-1} + \frac{w^{\gamma\alpha-2\alpha-\beta}}{w^{\gamma\alpha}-1}\Big).$$

Applying the inverse double Sadik transform, the solution to this problem gives

$$u(x,t) = e^{-t}[E_\gamma(x^\gamma) + xE_{\gamma,2}(x^\gamma)].$$

which corresponds to the solution in [7] and [8]. If $\gamma = 2$, this result refers to the exact solution of the classical telegraph equation

$$u(x,t) = e^{x-t}.$$

Example 4.2 *[7] Now we consider the nonhomogeneous space-fractional telegraph equation*

$$\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) - x^2 - t + 1,\quad 1 < \gamma \leq 2, x, t \geq 0,$$

subject to the initial and boundary conditions

$$u(x,0) = x^2 - 2 - 2x^2E_{\gamma,3}(x^\gamma) + 2E_{\gamma,1}(x^\gamma),$$
$$u_t(x,0) = 1,\ u(0,t) = t,\ u_x(0,t) = 0.$$

The exact solution of the problem is

$$u(x) = t + x^2 - 2 - 2x^2E_{\gamma,3}(x^\gamma) + 2E_{\gamma,1}(x^\gamma).$$

Note that $a = b = c = 1, g(x,t) = 1 - t - x^2, f_1(t) = t$, $f_2(t) = 0, h_1(x) = x^2 - 2 - 2x^2E_{\gamma,3}(x^\gamma) + 2E_{\gamma,1}(x^\gamma)$ and $h_2(x) = 1$. The initial and boundary conditions are transformed to

$$F_1(v:\alpha,\beta) = \mathcal{S}_t[t] = \frac{1}{v^{2\alpha+\beta}},\ F_2(v:\alpha,\beta) = 0,$$
$$H_1(w:\alpha,\beta) = \mathcal{S}_x[x^2 - 2 - 2x^2E_{\gamma,3}(x^\gamma) + 2E_{\gamma,1}(x^\gamma)], = \frac{2}{w^{3\alpha+\beta}} - \frac{2}{w^{\alpha+\beta}} - \frac{2w^{\gamma\alpha-3\alpha-\beta}}{w^{\gamma\alpha}-1} + \frac{2w^{\gamma\alpha-\alpha-\beta}}{w^{\gamma\alpha}-1},$$
$$H_2(w:\alpha,\beta) = \mathcal{S}_x[1] = \frac{1}{w^{\alpha+\beta}},$$

whereas the double Sadik transform of nonhomogeneous term is

$$\begin{aligned} G(w, v:\alpha,\beta) &= \mathcal{S}_2[g(x,t)] \\ &= \frac{1}{v^{\alpha+\beta}w^{\alpha+\beta}} - \frac{1}{v^{2\alpha+\beta}w^{\alpha+\beta}} - \frac{2}{v^{\alpha+\beta}w^{3\alpha+\beta}}. \end{aligned}$$

Substituting these above transformed functions into (5), we get

$$\begin{aligned} \mathcal{S}_2[u(x,t)] = &\frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^{\alpha} - 1}\Big[\frac{w^{\gamma\alpha-\alpha-\beta}}{v^{2\alpha+\beta}} - \frac{v^{-\beta}}{w^{\alpha+\beta}} \\ &- \frac{1}{w^{\alpha+\beta}v^{2\alpha+\beta}} + \frac{1}{w^{\alpha+\beta}v^{\alpha+\beta}} - \frac{2}{w^{3\alpha+\beta}v^{\alpha+\beta}} \\ &- v^{\alpha-\beta}\big[\frac{2(1-w^{2\alpha})}{w^{3\alpha+\beta}} + \frac{2w^{\gamma\alpha-\alpha-\beta}(1-w^{-2\alpha})}{w^{\gamma\alpha}-1}\big] \\ &+ \frac{2v^{-\beta}(1-w^{2\alpha})}{w^{3\alpha+\beta}(w^{\gamma\alpha}-1)}\Big], \end{aligned}$$

and then by rearranging the equation, one obtains

$$\begin{aligned} \mathcal{S}_2[u(x,t)] = &\frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^{\alpha} - 1}\Big[\Big(\frac{w^{\gamma\alpha-\alpha-\beta}}{v^{2\alpha+\beta}} \\ &- \frac{v^{-\beta}}{w^{\alpha+\beta}} - \frac{1}{w^{\alpha+\beta}v^{2\alpha+\beta}} - \frac{1}{w^{\alpha+\beta}v^{\alpha+\beta}}\Big) \\ &+ \Big(\frac{2}{w^{\alpha+\beta}v^{\alpha+\beta}} - \frac{2}{w^{3\alpha+\beta}v^{\alpha+\beta}}\Big) + \frac{2v^{-\beta}(1-w^{2\alpha})}{w^{3\alpha+\beta}(w^{\gamma\alpha}-1)} \\ &- v^{\alpha-\beta}\big[\frac{2(1-w^{2\alpha})}{w^{3\alpha+\beta}} + \frac{2w^{\gamma\alpha}(w^{2\alpha}-1)}{w^{3\alpha+\beta}(w^{\gamma\alpha}-1)}\big]\Big]. \end{aligned}$$

Simplify the equation, it becomes

$$\begin{aligned} \mathcal{S}_2[u(x,t)] = &\frac{1}{w^{\alpha+\beta}v^{2\alpha+\beta}} + \frac{2}{v^{\alpha+\beta}w^{3\alpha+\beta}} - \frac{2}{v^{\alpha+\beta}w^{\alpha+\beta}} \\ &- \frac{2w^{\gamma\alpha-3\alpha-\beta}}{v^{\alpha+\beta}(w^{\gamma\alpha}-1)} + \frac{2w^{\gamma\alpha-\alpha-\beta}}{v^{\alpha+\beta}(w^{\gamma\alpha}-1)}. \end{aligned}$$

After applying the inverse double Sadik transform, one obtains the exact solution of the problem

$$u(x,t) = t + x^2 - 2 - 2x^2E_{\gamma,3}(x^{\gamma}) + 2E_{\gamma,1}(x^{\gamma}).$$

which relates to that same solution in [7] and [8]. One can note that if $\gamma = 2$, the exact solution is reduced to $u(x,t) = t + x^2$.

Example 4.3 *[21] Next, we consider the nonhomogeneous time fractional telegraph equation*

$$\begin{aligned} &\frac{\partial^{\gamma}u(x,t)}{\partial t^{\gamma}} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + 2t(x^2-x) \\ &\times\big(\frac{t^{1-\gamma}}{\Gamma(3-\gamma)} + 1\big) - 2t^2, 1 < \gamma \le 2, x, t \ge 0, \end{aligned}$$

with initial and boundary conditions

$$u(x,0) = 0,\ u_t(x,0) = 0,\ u(0,t) = 0,\ u_x(0,t) = -t^2.$$

The exact solution of the problem is $u(x) = (x^2-x)t^2$.

Here $a = b = 1, c = 0, g(x,t) = 2t^2 - 2t(x^2-x)\big(\frac{t^{1-\gamma}}{\Gamma(3-\gamma)} + 1\big), f_1(t) = 0, f_2(t) = -t^2, h_1(x) = 0$ and $h_2(x) = 0$. The single Sadik transform of the initial and boundary conditions are

$$\begin{aligned} &F_1(v:\alpha,\beta) = 0,\ F_2(v:\alpha,\beta) = -\frac{2}{v^{3\alpha+\beta}}, \\ &H_1(w:\alpha,\beta) = H_2(w:\alpha,\beta) = 0 \end{aligned}$$

and the double Sadik transform of $g(x,t)$ is

$$\begin{aligned} G(w,v:\alpha,\beta) = &\frac{4}{w^{\alpha+\beta}v^{3\alpha+\beta}} - \Big(\frac{2}{w^{3\alpha+\beta}} - \frac{1}{w^{2\alpha+\beta}}\Big) \\ &\times\Big(\frac{2}{v^{(3-\gamma)\alpha+\beta}} + \frac{2}{v^{2\alpha+\beta}}\Big). \end{aligned}$$

Substituting these functions into (5), we get

$$\begin{aligned} \mathcal{S}_2[u(x,t)] = &\frac{1}{(w^{2\alpha} - v^{\gamma\alpha} - v^{\alpha})}\Big[-\frac{2w^{-\beta}}{v^{3\alpha+\beta}} + \frac{4}{v^{3\alpha+\beta}w^{\alpha+\beta}} \\ &-(\frac{2}{w^{3\alpha+\beta}} - \frac{1}{w^{2\alpha+\beta}})(\frac{2}{v^{(3-\gamma)\alpha+\beta}} + \frac{2}{v^{2\alpha+\beta}})\Big]. \end{aligned}$$

Simplifying the equation, it becomes

$$\mathcal{S}_2[u(x,t)] = \frac{1}{v^{3\alpha+\beta}}(\frac{4}{w^{3\alpha+\beta}} - \frac{2}{w^{2\alpha+\beta}}).$$

After applying the inverse double Sadik transform, the exact solution of the problem is

$$u(x,t) = \frac{t^2}{2}(\frac{4x^2}{2} - 2x) = (x^2-x)t^2$$

which coincides to a solution in [21].

Example 4.4 *[22] Consider the homogeneous time-fractional telegraph equation*

$$\begin{aligned} &\frac{\partial^{\gamma_1}u(x,t)}{\partial t^{\gamma_1}} + \frac{\partial^{\gamma_2}u(x,t)}{\partial t^{\gamma_2}} + u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2}, \\ &1 < \gamma_1 \le 2,\ \frac{1}{2} < \gamma_2 \le 1,\ x,t \ge 0, \end{aligned}$$

with initial and boundary conditions

$$\begin{aligned} &u(x,0) = 0,\ u_t(x,0) = e^x, \\ &u(0,t) = u_x(0,t) = tE_{\gamma_1-\gamma_2,2}(-t^{\gamma_1-\gamma_2}). \end{aligned}$$

The exact solution of the problem is

$$u(x) = e^x tE_{\gamma_1-\gamma_2,2}(-t^{\gamma_1-\gamma_2}).$$

Note that $a = b = c = 1, g(x,t) = 0, f_1(t) = f_2(t) = tE_{\gamma_1-\gamma_2,2}(-t^{\gamma_1-\gamma_2}), h_1(x) = 0$ and $h_2(x) = e^x$. The single Sadik transform of the initial and boundary conditions are

$$\begin{aligned} &F_1(v:\alpha,\beta) = F_2(v:\alpha,\beta) = \frac{v^{(\gamma_1-\gamma_2)\alpha-2\alpha-\beta}}{v^{(\gamma_1-\gamma_2)\alpha}+1}, \\ &H_1(w:\alpha,\beta) = 0,\ H_2(w:\alpha,\beta) = \frac{1}{w^{\beta}(w^{\alpha}-1)}. \end{aligned}$$

Substituting into (5), one obtains

$$\begin{aligned} \mathcal{S}_2[u(x,t)] = &\frac{1}{(w^{2\alpha} - v^{\gamma_1\alpha} - v^{\gamma_2\alpha} - 1)}\Big[-\frac{v^{(\gamma_1-2)\alpha-\beta}}{w^{\beta}(w^{\alpha}-1)} \\ &+ \frac{(w^{\alpha-\beta} + w^{-\beta})v^{(\gamma_1-\gamma_2)\alpha-2\alpha-\beta}}{v^{(\gamma_1-\gamma_2)\alpha}+1}\Big]. \end{aligned}$$

After simplifying the equation, we find that

$$\mathcal{S}_2[u(x,t)] = \frac{1}{w^\beta(w^\alpha - 1)}\Big[\frac{v^{(\gamma_1-\gamma_2)\alpha-2\alpha-\beta}}{v^{(\gamma_1-\gamma_2)\alpha}+1}\Big].$$

Applying the inverse double Sadik transform, it provides the solution

$$u(x,t) = e^x t E_{\gamma_1-\gamma_2,2}(-t^{\gamma_1-\gamma_2})$$

which corresponds to the solution in [22].

Example 4.5 *[22] Next, we consider the nonhomogeneous time-fractional telegraph equation*

$$\frac{\partial^\gamma u(x,t)}{\partial t^\gamma} + \frac{\partial^{\gamma-1}u(x,t)}{\partial t^{\gamma-1}} + u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} + \sinh x \frac{t^n}{\Gamma(n+1)}, 1 < \gamma \le 2,\ x,t \ge 0,$$

with initial and boundary conditions

$$u(x,0) = 0,\ u_t(x,0) = 0,\ u(0,t) = 0,$$
$$u_x(0,t) = t^{n+\gamma}E_{1,n+\gamma+1}(-t).$$

The exact solution of the problem is

$$u(x) = (\sinh x)t^{n+\gamma}E_{1,n+\gamma+1}(-t).$$

Here $a = b = c = 1, g(x,t) = -\sinh x \cdot \frac{t^n}{\Gamma(n+1)}, f_1(t) = 0, f_2(t) = t^{n+\gamma}E_{1,n+\gamma+1}(-t)$, $h_1(x) = 0$ and $h_2(x) = 0$. The single Sadik transform of initial and boundary conditions are

$$F_1(v:\alpha,\beta) = 0, H_1(w:\alpha,\beta) = H_2(w:\alpha,\beta) = 0,$$
$$F_2(v:\alpha,\beta) = \frac{v^{-n\alpha-\gamma\alpha-\beta}}{v^\alpha+1}$$

and the double Sadik transform of $g(x,t)$ is

$$G(w,v:\alpha,\beta) = -\frac{1}{w^\beta(w^{2\alpha}-1)}\cdot\frac{1}{v^{(n+1)\alpha+\beta}}.$$

Substituting above functions into (5), one obtains

$$\mathcal{S}_2[u(x,t)] = \frac{1}{(w^{2\alpha} - v^{\gamma\alpha} - v^{\gamma\alpha-\alpha} - 1)}\Big[\frac{w^{-\beta}v^{-n\alpha-\gamma\alpha-\beta}}{v^\alpha+1} - \frac{1}{w^\beta(w^{2\alpha}-1)}\cdot\frac{1}{v^{(n+1)\gamma+\beta}}\Big].$$

The above equation can be simplified as

$$\mathcal{S}_2[u(x,t)] = \frac{1}{w^\beta(w^{2\alpha}-1)}.\frac{v^{-n\alpha-\gamma\alpha-\beta}}{v^\alpha+1}.$$

Applying the inverse double Sadik transform, the solution to this problem gives

$$u(x,t) = \sinh x \cdot t^{n+\gamma}E_{1,n+\gamma+1}(-t),$$

which relates toward the solution in [22].

Example 4.6 *[8] Consider the nonlinear nonhomogeneous space-fractional telegraph equation*

$$\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} = \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u^2(x,t) - e^{-2t}(x-x^2)^2 - 2e^{-t}\frac{x^{2-\gamma}}{\Gamma(3-\gamma)},\ 1 < \gamma \le 2,\ x,t \ge 0, \quad (11)$$

with initial and boundary conditions

$$u(x,0) = x - x^2,\ u_t(x,0) = x^2 - x,\ u(0,t) = 0,\ u_x(0,t) = e^{-t}.$$

The exact solution of the problem is $u(x) = e^{-t}(x-x^2)$.

Note that $a = b = 1, c = 0, f_1(t) = 0, f_2(t) = e^{-t}, h_1(x) = x - x^2$ and $h_2(x) = x^2 - x$. The single Sadik transform of these functions are

$$F_1(v:\alpha,\beta) = 0,\ F_2(v:\alpha,\beta) = \frac{1}{v^\beta(v^\alpha+1)},$$
$$H_1(w:\alpha,\beta) = \frac{1}{w^{2\alpha+\beta}} - \frac{2}{w^{3\alpha+\beta}},$$
$$H_2(w:\alpha,\beta) = \frac{2}{w^{3\alpha+\beta}} - \frac{1}{w^{2\alpha+\beta}},$$

and the double Sadik transform of $g(x,t) = 2e^{-t}\frac{x^{2-\gamma}}{\Gamma(3-\gamma)}$ is

$$\mathcal{S}_2[g(x,t)] = \frac{2}{v^\beta(v^\alpha+1)}\cdot\frac{1}{w^{(3-\gamma)\alpha+\beta}} = \frac{2w^{\gamma\alpha}}{v^\beta(v^\alpha+1)w^{3\alpha+\beta}}.$$

By applying the double Sadik transform on both sides of (11) and using the above conditions, one obtains

$$\mathcal{S}_2[u(x,t)] = \frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^\alpha}\Big[\frac{w^{\gamma\alpha-2\alpha-\beta}}{v^\beta(v^\alpha+1)} - v^{\alpha-\beta}\big[\frac{1}{w^{2\alpha+\beta}} - \frac{2}{w^{3\alpha+\beta}}\big] - \frac{2w^{\gamma\alpha}}{v^\beta(v^\alpha+1)w^{3\alpha+\beta}} + \mathcal{S}_2[u^2(x,t)] - \mathcal{S}_2[e^{-2t}(x-x^2)^2]\Big].$$

It can be simplified in the form

$$\mathcal{S}_2[u(x,t)] = \big[\frac{1}{w^{2\alpha+\beta}} - \frac{2}{w^{3\alpha+\beta}}\big]\frac{1}{v^\beta(v^\alpha+1)} + \Big[\frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^\alpha}\big[\mathcal{S}_2[u^2(x,t)] - \mathcal{S}_2[e^{-2t}(x-x^2)^2]\big]\Big].$$

Taking the inverse double Sadik transform, we get

$$u(x,t) = (x-x^2)e^{-t} + \mathcal{S}_2^{-1}\Big[\frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^\alpha}\big[\mathcal{S}_2[u^2(x,t)] - \mathcal{S}_2[e^{-2t}(x-x^2)^2]\big]\Big]. \quad (12)$$

In order to overcome the non-linearity of the problem, we write the solution as

$$u(x,t) = \sum_{i=0}^\infty u_i(x,t) \quad (13)$$

and the nonlinear term $N(u) = u^2$ can be decomposed as

$$\Big[\sum_{i=0}^{\infty} u_i(x,t)\Big]^2 = [u_0(x,t)]^2 + \sum_{i=1}^{\infty}\left[\Big[\sum_{k=0}^{i} u_k(x,t)\Big]^2 - \Big[\sum_{k=0}^{i-1} u_k(x,t)\Big]^2\right]. \tag{14}$$

Substituting (13) and (14) in (12), we obtain

$$\sum_{k=0}^{\infty} u_k(x,t)) = (x-x^2)e^{-t} + \mathcal{S}_2^{-1}\left[\frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^{\alpha}} \times \left[\mathcal{S}_2\Big[[u_0(x,t)]^2 + \sum_{i=1}^{\infty}\Big[\Big[\sum_{k=0}^{i} u_k(x,t)\Big]^2 - \Big[\sum_{k=0}^{i-1} u_k(x,t)\Big]^2\Big]\Big] - \mathcal{S}_2[e^{-2t}(x-x^2)^2]\right]\right].$$

By defining the recursive relation to this problem as

$$u_0(x,t) = (x-x^2)e^{-t},$$

$$u_1(x,t) = \mathcal{S}_2^{-1}\left[\frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^{\alpha}}\left[\mathcal{S}_2[u_0(x,t)]^2 - \mathcal{S}_2[e^{-2t}(x-x^2)^2]\right]\right],$$

$$u_{n+1}(x,t) = \mathcal{S}_2^{-1}\left[\frac{1}{w^{\gamma\alpha} - v^{2\alpha} - v^{\alpha}}\mathcal{S}_2\left[\Big[\sum_{k=0}^{n} u_k(x,t)\Big]^2 - \Big[\sum_{k=0}^{n-1} u_k(x,t)\Big]^2\right]\right], n \geq 1,$$

the components of solution $u_k(x,t)$ have shown that

$$u_0(x,t) = (x-x^2)e^{-t}, u_n(x,t) = 0,\ n = 1,2,3,\ldots.$$

Hence, the solution (13) of the problem is $u(x,t) = (x - x^2)e^{-t}$. The exact solution obtained is related to the solution in [8].

5 Conclusions

This study was successful in offering a dominant analytical strategy for resolving the space and time fractional telegraph equations. As in the preceding computation, the method delivered the exact solution of an initial boundary value problem for linear and nonlinear equations in an uncomplicated algorithm without requiring any discretization, linearization, or perturbation. The approach has the advantage of requiring less processing cost since it does not involve computing an Adomian polynomial, Lagrange multiplier, or He's polynomials. Furthermore, the obtained results are analogous to available techniques such as the double Laplace transform [8], double Elzaki transform, double Kamal transforms, and so on, and the advancement is viewed as a generalization of the other double integral transform methods. In conclusion, this methodology provides a compelling, accurate, and efficient framework for carrying out research efforts.

Acknowledgements

This research project was financially supported by Mahasarakham University. The author is deeply indebted to the reviewers for their valuable comments and suggestions.

REFERENCES

[1] Debnath L. , Nonlinear Partial Differential Equations for Scientists and Engineers, Birkhäuser, Boston, Mass, USA, 1997.

[2] Metaxas A. C. and Meredith R. J., Industrial Microwave Heating, Peter Peregrinus, London, UK, 1993.

[3] Campos D., Méndez V.,"Different microscopic interpretations of the reaction-telegrapher equation," Journal of Physics A: Mathematical and Theoretical, vol. 42, no. 7, pp. 1–13,2009. DOI:10.1088/1751-8113/42/7/075003

[4] Sevimlican A.,"An approximation to solution of space and time fractional telegraph equations by He's variational iteration method," Mathematical Problems in Engineering, vol. 2010, Article ID 290631, 10 pages, 2010. DOI:10.1155/2010/290631

[5] Garg M., Sharma A.,"Solution of space-time fractional telegraph equation by Adomian decomposition method," Journal of Inequalities and Special Functions, vol. 2, no. 1, pp. 1-7, 2011. www.ilirias.com/jiasf/repository/docs/JIASF2-1-1.pdf

[6] Garg M., Manohar P., Kalla S.L.,"Generalized differential transform method to space-time fractional telegraph equation," International Journal of Differential Equations, vol. 2011, Article ID 548982, 9 pages, 2011. DOI:10.1155/2011/548982

[7] Alawad F.A., Yousif E.A., Arbab A.I., "A new technique of Laplace variational iteration method for solving space-time fractional telegraph equations," International Journal of Differential Equations, vol. 2013, Article ID 256593, 10 pages, 2013. DOI:10.1155/2013/256593

[8] Dhunde R.R., Waghmare G.L., "Double Laplace transform method for solving space and time fractional telegraph equations," International Journal of Mathematics and Mathematical Sciences, vol. 2016, Article ID 1414595, 7 pages, 2016. DOI:10.1155/2016/1414595

[9] Khan Y., Diblik J., Faraz N., Smarda Z., "An efficient new perturbation Laplace method for space-time fractional telegraph equations," Advances in Difference Equations, vol. 2012, no. 204, pp. 1–9, 2012. DOI:10.1186/1687-1847-2012-204

[10] Orsingher E., Xuelei Z., "The space-fractional telegraph equation and the related fractional telegraph process," Chinese Annals of Mathematics Series B , vol. 24, no. 1, pp. 45–56, 2003. DOI:10.1142/S0252959903000050

[11] Momani S., " Analytic and approximate solutions of the space- and time-fractional telegraph equations," Applied Mathematics and Computation, vol. 170, no. 2, pp. 1126–1134, 2005. DOI:10.1016/j.amc.2005.01.009

[12] Fino A.Z., Ibrahim H.,"Analytical solution for a generalized space-time fractional telegraph equation, " Mathematical Methods in the Applied Sciences, vol. 36, no. 14, pp. 1813–1824, 2013. DOI:10.1002/mma.2727

[13] Shaikh S.L., "Introducing a new integral transform Sadik transform," American International Journal of Research in Science, Technology, Engineering & Mathematics, vol. 22, no. 1, pp. 100–102, 2018. DOI:10.13140/RG.2.2.25805.08161

[14] Redhwana S.S., Shaikh S.L., Abdo M.S.,"Some properties of Sadik transform and its applications of fractional-order dynamical systems in control theory," Advances in the Theory of Nonlinear Analysis and its Applications, vol. 4, no. 1, pp. 51–66, 2019. DOI:10.31197/atnaa.647503

[15] Pue-on P.,"The modified Sadik decomposition method to solve a system of nonlinear fractional Volterra integro-differential equations of convolution type," WSEAS Transactions on Mathematics, vol. 20, no. 34, pp. 335–343. 2021. DOI:10.37394/23206.2021.20.34

[16] Aggarwal S., Bhatnagar K.,"Sadik Transform for Handling Population Growth and Decay Problems," Journal of Applied Science and Computations, vol. 6, no. 6, pp. 1212–1221, 2019.`www.j-asc.com/gallery/152-june-3126.pdf`

[17] Ganesh S.K. and Nitin S.A.,"Solution to linear partial integro-differential equation by using Sadik Transform," Journal of Applied Science and Computations, vol. 6, no. 4, pp. 639–645, 2019. `www.j-asc.com/gallery/81-april-2443.pdf`

[18] Shivaji S.P. and Nitin S.A.,"Applications of Sadik Transform for solving Bessels function and linear Volterra integral equation of convolution type," CIKITUSI Journal for Multidisciplinary Research, vol. 6, no. 3, pp. 85–91, 2019. DOI:16.10089.CJMR.2019.V6I3.18.2780

[19] Singh Y., "On some theorems and applications of double Sadik transform," Compliance Engineering Journal, vol. 10, no. 10, pp. 164-174, 2019. DOI:16.10089.CEJ.2019.V10I7.285311.2730

[20] Kilbas A., Srivastava H., Trujillo J., Theory and Applications of Fractional Differential Equations, 1st ed., Elsevier, New York, 2006.

[21] Sarwar S., Rashidi M.M., "Approximate solution of two-term fractional-order diffusion, wave-diffusion, and telegraph models arising in mathematical physics using optimal homotopy asymptotic method," Waves in Random and Complex Media, vol. 26, no. 3, pp. 365–382, 2016. DOI:10.1080/17455030.2016.1158436

[22] Nirmala R.J., Balachandran K., "Analysis of solutions of time fractional telegraph equation," Journal of the Korean Society for Industrial and Applied Mathematics, vol. 18, no. 3, pp. 209–224, 2014. DOI:10.12941/jksiam.2014.18.209