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Relativistic Polygonal and Circular Paths: Solution for the Twins' Paradox Reviewed and Extended

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Abstract This article follows a previous one, which addressed the famous problem of Twins' Paradox: if one of two twins remains on Earth while the other gets on a rocket, travels at a speed close to the speed of light and returns, the traveler should find his brother older than himself. This is because, according to Special Relativity (SR), a moving clock runs slower than a 'stationary' one. The paradox lies in the fact that movement is relative, thus breaking the equivalence between frames of coordinates, a crucial principle at the heart of SR theory. The cited article highlighted the error in the premises that leads to the paradox, reanalyzing it and proving its non-existence either for a one-way or a round trip, despite the phenomenon of time dilation. An important extension of this conclusion is provided here. If one of the twins leaves the other, following a closed polygonal or circular path, in the end they find themselves at the same age. Thus, once again, the fundamental equivalency between frames of coordinates is fully respected. This – as before – is achieved by means of a new dilation factor, progressively "stronger" than Lorentz factor γ , but perfectly reflective. Issues concerning the geometry of space-time and apparently strange outcomes are also analyzed. In addition to strengthening theoretical consistency, restoring confidence in SR and allowing further developments, this study has possible experimental relevance in particle accelerators. Finally, one discovers the existence of a threshold speed for the shortest possible time to complete a closed path.

Keywords Special Relativity, Twin's Paradox, Time Dilation, Relativistic Asynchrony, Euclidean Geometry

1. Introduction

The so-called "twin's paradox" is the popular version, formulated by Paul Langevin in 1911 [1], of a serious issue caused by the relativistic time dilation phenomenon. In short: one of two twins remains on Earth while the other gets on a rocket, travels with a velocity near to c and gets back; since the theory of Special Relativity (SR) consistently sustains that a moving clock runs slower than a 'stationary' one, the voyager should return younger than his twin, when they meet again. The well-known paradox comes from the fact that, also consistently, SR takes all inertial frames as equivalent. So, from the point of view of the travelling twin, it is the other who is moving and the discrepancy on ages should be reverted. This is, of course, an absurd.

The basis for the paradox lies in a "typical consequence" of SR, stated long ago by Albert Einstein himself in his founding paper on relativity [2]:

> *"Consider in the points A and B of a coordinate frame K two clocks at rest, supposing they work in synchrony to whomever observe them in the frame at rest. Let us imagine now that we communicate to the clock in A a movement with velocity v along the straight line that lies both points, in the sense of B. When this clock arrives at B synchrony no longer exists: compared to the one which remained in B, the clock that has been moved presents a delay of $1/2\ tv^2/c^2$ (up to quantities of the fourth and higher orders), if we represent by t the duration of the displacement. We see at once that this result still*

holds if the clock moves from A to B along any arbitrary polygonal line, and this even when the points A and B coincide."

Einstein admits that *"the deduced result for a polygonal line remains valid for a continuous curve"*, even in the case of closed curves. All these conclusions are then expected to be verified in the modern particle accelerators; in fact, they appear to be… but not exactly in the form Einstein predicted. This somewhat unadvised extension is analyzed ahead.

In a former article [3], from which the preceding paragraphs were extracted, it has been proved that **there is no "twin's paradox" at all**, either in the case of a single voyage (the moving observer A' goes from point $\mathbf{0}$ to point $\mathbf{1}$, where he meets one of his twin brothers, B) or of a round trip (A' returns to the starting point, where he finds again his other twin, A).

As analyzed in that paper, the paradox results from inadvertent errors in the premises of the reasoning. The key to understand and finally resolve this puzzling issue is relativistic asynchrony, including permutations between past and future. The first case is resolved by noting that, according to Lorentz transformations, the coordinates in S' (the proper frame of A') of the observer B corresponding to the instant ${t'}_0 = 0$, when A' is at the starting point, are

$$\begin{cases} {t'}_{B0} = -\frac{\beta^2 t}{\sqrt{1-\beta^2}} \\ {x'}_{B0} = \frac{x}{\sqrt{1-\beta^2}}, \end{cases} \tag{1}$$

where t is the time required in the rest frame S for A' to cover the distance $x_B = x = vt$ between the immobile twins A and B (this is, between points $\mathbf{0}$ and $\mathbf{1}$).

Worth of notice is the fact that, at this very moment, for A' *the* observer B *is in the past*. For the instant t, when A' meets B, we get

$$\begin{cases} {t'}_B = t\frac{1-\beta^2}{\sqrt{1-\beta^2}} = t\sqrt{1-\beta^2} = t' \\ {x'}_B = \frac{x-vt}{\sqrt{1-\beta^2}} = 0, \end{cases} \tag{2}$$

this meaning that, despite the time dilation phenomenon, both twins have aged the same: t'. It is the negative time lag, at the moment of departure, that counterbalance exactly the longer time t for A' to reach B, in the proper frame of any of them, allowing us to understand the necessary – because of the equivalence between the coordinate frames of A' and B – physical result of the experience.

The case of A' meeting A is reviewed in the following paragraph. One comes to the conclusion that, in this round trip, the same time is required for both twins:

$$\mathbf{T}' = \mathbf{T} = t\left(\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}\right) \tag{3}$$

This is very satisfying because it resolves a false paradox, motivated by common sense, and – as the result above – demonstrates once again the amazing consistency of SR.

However, in the sequence of this cited paper, I was intrigued by the problem of the observer A' following polygonal or circular closed paths, suggested by Einstein. A strong belief in SR theory leaved me to the conclusion that, in the end, once again both twins should meet at the same age. Then, by a sort of logical imperative, I formulated the conjecture that the total time, for a regular polygon of n sides, should be given by the following equation:

$$\mathbf{T}' = \mathbf{T} = \frac{n}{2}\ t\left(\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}\right) = \delta_0\, nt, \tag{4}$$

thus preserving what I call the ***delta zero factor*** (related to time (t) measured in the frame at rest):

$$\delta_0 = \frac{1}{2}\left(\gamma + \frac{1}{\gamma}\right), \tag{5}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ is the well-known Lorentz factor. One may also write

$$\delta_0 = \frac{1-\beta^2/2}{\sqrt{1-\beta^2}}. \tag{6}$$

As we will see in the next paragraph, this conjecture appears to be true.

2. Polygonal Path

2.1. Lorentz Transformations

Consider an observer A in the origin of a supposed immobile coordinates frame S. Another observer, A', in the origin of an inertial frame S', is moving with velocity v along the x-axis, in the positive sense. According to Lorentz standard transformations,

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{7}$$

or, symbolically,

$$\mathrm{X}' = \mathbf{\Lambda}'\mathrm{X}, \tag{8}$$

where $\mathbf{\Lambda}'$ is the (inverse) Lorentz matrix.

Quite useful for our purposes is the case of a boost in an arbitrary direction $(\beta_x\,;\ \beta_y\,;\ \beta_z)$; Lorentz matrix presents then its generic form [4]:

$$\mathbf{\Lambda}' = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta} \end{pmatrix}. \tag{9}$$

We will generalize the round trip to a regular polygonal path of n sides inscribed in a circle, supposed at rest in the frame S. We will consider two constant basic data, β and r (the radius of the circle), and the basic variable n; this gives:

- an angle at the center, $\theta = \frac{360°}{n}$;
- a length of each side, $l = 2r\sin\frac{\theta}{2}$;
- a constant time, $t = l/v$, to cover the distance between two consecutive vertices.

We will also number vertices consecutively, from $\mathbf{0}$ to $\mathbf{n-1}$ (the vertex $\mathbf{n}$ being coincident with $\mathbf{0}$). For the first side of the path, from $\mathbf{0}$ to $\mathbf{1}$, the velocity $\mathbf{v}$ is aligned with the x-axis.

Finally, with no loss of generality, we will consider that the polygon exists in the plane xy (on the positive side of the y-axis); this cancels out the z coordinate, simplifying equations and calculations, as well as the Lorentz matrix itself by eliminating the 4th row and the 4th column. It is easy to conclude that (see Figure 1), when A' reaches $\mathbf{1}$, he begins moving in a direction that makes an angle $\varphi_1 = \theta$ with the x-axis. This is a cumulative process, in such a way that, for the trajectory beginning at the vertex $\mathbf{k}$, we get

$$\varphi_k = k\,\theta \quad \Rightarrow \quad \begin{cases} \beta_x = \beta\cos\varphi_k \\ \beta_y = \beta\sin\varphi_k. \end{cases} \tag{10}$$

Consequently, the 3×3 Lorentz matrix for this k^{th} leg of the path assumes the form

$$\mathbf{\Lambda}'_k = \begin{pmatrix} \gamma & -\gamma\beta\cos\varphi_k & -\gamma\beta\sin\varphi_k \\ -\gamma\beta\cos\varphi_k & 1+(\gamma-1)\cos^2\varphi_k & (\gamma-1)\sin\varphi_k\cos\varphi_k \\ -\gamma\beta\sin\varphi_k & (\gamma-1)\sin\varphi_k\cos\varphi_k & 1+(\gamma-1)\sin^2\varphi_k \end{pmatrix}. \tag{11}$$

2.2. The Linear Round Trip

As in [3], we will study the problem of linear (and also polygonal) round trip mainly from the point of view of the proper frame S' of the “moving” observer A'. In the end, all the reasoning and conclusions apply as well to the frame S by simply reversing velocities.

The linear round trip may be seen as a “two-sided” regular polygon. This means that

- $\theta = 180°$;
- $l = 2r$;

In the first leg of the route, the observer A' goes from $\mathbf{0}$ to $\mathbf{1}$. We obtain $\varphi_0 = 0°$ and the standard three-dimensional Lorentz matrix:

$$\mathbf{\Lambda}'_0 = \begin{pmatrix} \gamma & -\gamma\beta & 0 \\ -\gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We begin with the coordinates of $\mathbf{0}$ in time t, in the frame $S_0 \equiv S$, whose matrix is symbolized by $\mathrm{x}^0_{0(t)}$ [read “x 0 0 t”]:

$$\mathrm{x}^0_{0(t)} = \begin{pmatrix} ct \\ 0 \\ 0 \end{pmatrix}$$

and whose transformation gives the coordinates in $S'_0 \equiv S'$ of $\mathbf{0}$ relatively to $\mathbf{1}$ when A' reaches this last point:

$$\mathrm{x}'^0_{0(t)} = \mathbf{\Lambda}'_0\,\mathrm{x}^0_{0(t)} = \begin{pmatrix} \gamma ct \\ -\gamma l \\ 0 \end{pmatrix}.$$

Then, theoretically, the observer A' instantly reverts his movement. This corresponds to $\varphi_1 = 180°$ and

$$\mathbf{\Lambda}'_1 = \begin{pmatrix} \gamma & \gamma\beta & 0 \\ \gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix is relative to **new frames** S_1 and S'_1, this is, the former S_0 reset and centered in $\mathbf{1}$, as well as the former S'_0, which begins moving in the opposite direction of the x-axis, with velocity $-\vec{v}$, S'_1 being coincident with S_1 at the instant $t'_0 = t_0 = 0$. At this instant, the coordinates of $\mathbf{0}$ relatively to $\mathbf{1}$ are obviously those given by $\mathrm{x}'^0_{0(t)}$ and, so, we write

$$\mathbf{x}'^1_0 = \mathbf{\Lambda}'_0\,\mathrm{x}^0_{0(t)} = \begin{pmatrix} \gamma ct \\ -\gamma l \\ 0 \end{pmatrix},$$

the $\mathbf{x}$ in bold following a convention we will establish below.

The time lapse t is constant for each leg of the path; for the spatial coordinates of $\mathbf{0}$ in S_1, we have $x = -l\cos\varphi_0 = -l$ and $y = -l\sin\varphi_0 = 0$. Therefore, we obtain for the coordinates of $\mathbf{0}$ relatively to $\mathbf{2} \equiv \mathbf{0}$:

$$\mathbf{x}'^2_0 = \mathbf{\Lambda}'_1\,\mathrm{x}^1_{0(t)} = \begin{pmatrix} \gamma & \gamma\beta & 0 \\ \gamma\beta & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ -l \\ 0 \end{pmatrix} = \begin{pmatrix} ct\sqrt{1-\beta^2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} ct' \\ 0 \\ 0 \end{pmatrix}.$$

The null spatial coordinates represent the fact that at the end, A' returns to the starting point, as it should be.

As for the final coordinates (this is, from the beginning) – we will call them **summative coordinates**, their matrix being symbolized by $\mathbf{X}'^2_0$ –, we obtain them by adding the two partial coordinates:

$$\mathbf{X}'^2_0 = \mathbf{x}'^2_0 + \mathbf{x}'^1_0. \tag{12}$$

The result is:

$$\mathbf{X}'^2_0 = \begin{pmatrix} ct\left(\gamma + \frac{1}{\gamma}\right) \\ -\gamma l \\ 0 \end{pmatrix}.$$

Concerning time, making $\mathbf{T}' = \mathbf{T}'^2_0$, it comes that

$$\mathbf{T}' = t\left(\sqrt{1-\beta^2} + \frac{1}{\sqrt{1-\beta^2}}\right) = t\left(\gamma + \frac{1}{\gamma}\right) = 2\,\delta_0\,t. \tag{13}$$

The conclusion is clear: at the end of the round trip, A' meets A after a time lapse given by $\mathbf{T}' = \mathbf{T}$, which is exactly the same for both; this is because the result incorporates both dilation and contraction factors over the time variable t. Besides, we may also express $\mathbf{T}' = \mathbf{T}$ as a function of the proper time t' of the moving observer:

$$\mathbf{T}' = \mathbf{T} = 2\ \delta\, t', \quad \text{where } \delta = \gamma\delta_0 = \frac{1}{2}(\gamma^2 + 1) = \frac{1-\beta^2/2}{1-\beta^2} \tag{14}$$

is the ***delta factor***. For $|\beta| > 0$, one can easily prove that $\delta_0 > 1$ and, therefore, $\delta > 1$. As a matter of fact, suppose that

$$\delta_0 = \frac{2-\beta^2}{2\sqrt{1-\beta^2}} > 1 \quad \Rightarrow \quad 2-\beta^2 > 2\sqrt{1-\beta^2};$$

Squaring both members, we get

$$4 - 4\beta^2 + \beta^4 > 4 - 4\beta^2 \quad \Rightarrow \quad \beta^4 > 0,$$

which is obviously true.

This means that $\mathbf{T}' = \delta_0(2\,t)$ represents a **time dilation** over the classical measure of time, $2\,t$, for the round trip.

On the other hand,

$$\begin{cases} \lim\limits_{\beta\to 0}\delta_0 = \lim\limits_{\beta\to 0}\delta = 1\ ; \\ \lim\limits_{\beta\to 1}\delta_0 = \lim\limits_{\beta\to 1}\delta = \infty. \end{cases}$$

Remark that, contrary to time, which is really summative, the linear coordinate $\mathrm{X}' = -\gamma l$ does not express the total length covered by the moving observer, A or A', as measured by the other. It is enough to consider the correspondent Galilean transformation to understand this. The total length is the relativistic 'perimeter' of the "two-sided" polygonal line, given by

$$\mathrm{p}'_R = \mathrm{p}_R = v\mathbf{T}' = \delta_0(2\,l).$$

For $\beta \to 0$, $\mathrm{p}_R \to \mathrm{p} = 2\,l$, which is the Euclidean perimeter of this "two-sided" polygonal line. Similar observations remain valid for any closed polygonal path. We will come back to this issue.

Meanwhile, the previous analysis reveals something quite amazing: from the point of view of the 'still' observer, the 'moving' one takes longer to go than to get back:

$$\mathbf{t}'^1_0 = \gamma\,t \ > \ \mathbf{t}'^2_0 = t/\gamma.$$

These and further results, somehow strange, deserve reflection. For now, let us ascertain that Lorentz transformations translate the fact that any relative inertial motion of observers induces distortions in *measurements* either in time or in space. 'Standard' length contraction and time dilation are two aspects of these distortions. But if velocity changes (either in modulus or in direction), like it happens in the turning point $\mathbf{1}$, distortions increase. Keeping the modulus, the 'turning back' ($\varphi = 180°$) is the most radical change of velocity. In any case, the result is that the *total time*, from the beginning, and the *total distance* traveled by the observer are not $2\,t$ and $2\,l$, as one would think. But, in the end, in closed trajectories, this strangeness is absolutely coherent, avoiding paradoxes like that of twins.

2.3. Some Conventions and Definitions

Before proceeding, I must precise that I often use the notation x_B^A [read *"x B A"*] to represent the geometric coordinates of (point or observer) B relatively to A. The rule

$$x_B^A = x_B^C + x_C^A \tag{15}$$

translates the geometrical composition of vectors. Naturally, $x_B^B = x_B^A + x_A^B = 0$, and so $x_A^B = -x_B^A$. We may write, then, $x_B^A = x_B^O - x_A^O$ or, if the origin of the coordinates O is presupposed,

$$x_B^A = x_B - x_A.$$

The letters may be replaced by numbers, as we will do in this paper, with the same meaning. However, in Physics there is also the variable t, which makes the four-matrix x_A^A (or the correspondent four-vector) not necessarily null and, therefore, the above composition rule not necessarily valid. We may write, in SR,

$$\mathrm{x}_{B(t)}^A = \begin{pmatrix} ct_B^A \\ x_B^A \\ y_B^A \\ z_B^A \end{pmatrix} \tag{16}$$

as the matrix of coordinates of B relatively to A in time t.

Continuing the strategy and methodology of the previous section, we will deal with several coordinate frames, S_k and S'_k, all of them parallel to each other, consecutively reset and centered in each vertex k of the polygon. However, we will restrict the variable time, in each leg of the path, to the extremes: $t_0 = 0$ or $t = l/v$. So, to simplify notations, we will establish the convention that, for **the leg k** of the path (this is, between points $\mathbf{k-1}$ and $\mathbf{k}$):

- x_i^k or x'^k_i, written in normal lowercase, represent the matrix of coordinates of (point) i relatively to (point) k in time $t = l/v$;
- $\mathbf{x}_i^k$ or $\mathbf{x}'^k_i$, written in bold lowercase, represent the matrix of coordinates of i relatively to k in time $t_0 = 0$.

Moreover, we will represent a matrix of *geometric coordinates* (that is, excluding time) as $[x]$, keeping the form of right brackets for numeric matrices.

The main objective of this paper is to determine, for each vertex $\mathbf{k}$, the matrix of the so-called **summative coordinates** of vertex $\mathbf{0}$ in relation to $\mathbf{k}$, which is defined as follows:

$$\mathbf{X}'^k_0 = \mathbf{x}'^k_0 + \mathbf{X}'^{k-1}_0 \quad \text{along with } \mathbf{X}'^0_0 = 0\,,$$
$$\text{that is, } \mathbf{X}'^1_0 = \mathbf{x}'^1_0. \tag{17}$$

This recursive definition means that we sum the 'partial' coordinates of the origin $\mathbf{0}$ relatively to the initial point of the segment $[\mathbf{k}\quad \mathbf{k+1}]$, this is $\mathbf{x}'^k_0$, to all the $k-1$ precedent sums. The ultimate objective, the most relevant, is the final summative coordinates given by $\mathbf{X}'^n_0$.

2.4. Equilateral Triangle Path

After the round trip, the simplest route is that of a path in an equilateral triangle, supposed immobile in the reference frame S of observer A.

We will be primarily interested in calculating the *geometric coordinates* of vertex **0** relatively to each of the other vertices, because the time required to cover each leg is always the same: $t = l/v$. Examining Figure 1, we find out that the easiest way to do it begins with the understanding that relatively to the center of the polygon, **Q**, the coordinates of **0** are given by

$$[\mathrm{x}]_0^Q = r\begin{bmatrix}\cos\alpha_0\\ \sin\alpha_0\end{bmatrix}, \quad where \quad \alpha_0 = -\left(90^\circ + \frac{\theta}{2}\right). \quad (18)$$

Besides, the correspondent equation for a generic vertex **k** is quite simple:

$$[\mathrm{x}]_k^Q = r\begin{bmatrix}\cos(\alpha_0 + k\,\theta)\\ \sin(\alpha_0 + k\,\theta)\end{bmatrix} \quad (19)$$

But then, following (15), we get for the geometric coordinates of **0** relatively to **k**:

$$[\mathrm{x}]_0^k = [\mathrm{x}]_0^Q - [\mathrm{x}]_k^Q = r\begin{bmatrix}\cos\alpha_0 - \cos(\alpha_0 + k\,\theta)\\ \sin\alpha_0 - \sin(\alpha_0 + k\,\theta)\end{bmatrix}, \quad (20)$$

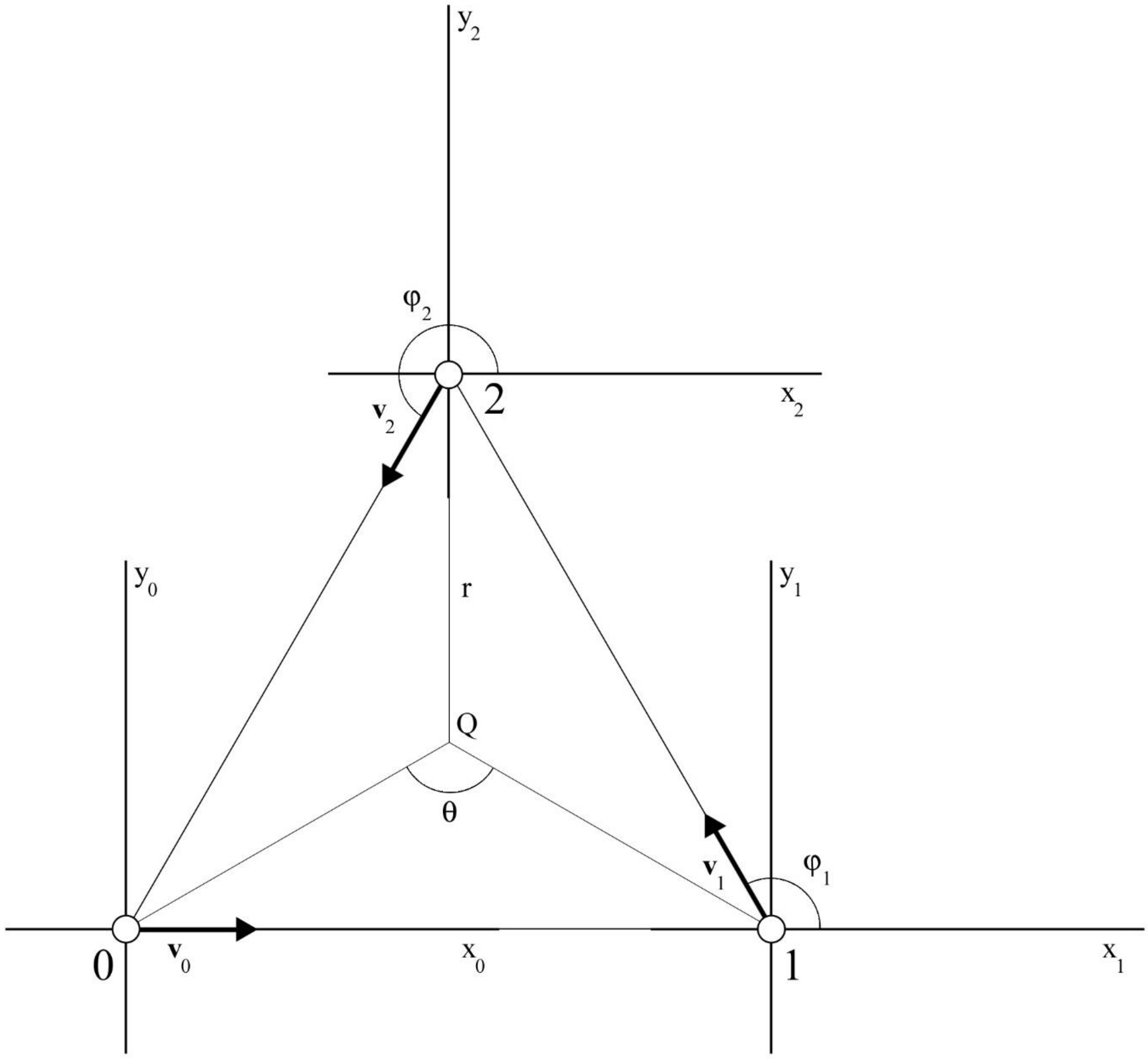

Figure 1. An equilateral triangle path, with the three frames of coordinates S_0, S_1 and S_2, one at each vertex (coincident with S'_0, S'_1 and S'_2, not represented, for $t = t' = 0$), and the velocity vectors at the beginning of each leg of the path. For simplicity, the circumference is also not represented.

Therefore, its coordinates including time are expressed by

$$\mathrm{x}_0^k = \begin{pmatrix} ct \\ r\big(\cos\alpha_0 - \cos(\alpha_0 + k\,\theta)\big) \\ r\big(\sin\alpha_0 - \sin(\alpha_0 + k\,\theta)\big) \end{pmatrix}; \qquad (21)$$

or, alternatively,

$$\mathrm{x}_0^k = \begin{pmatrix} ct \\ -r\left(\sin\frac{\theta}{2} + \sin\frac{2k-1}{2}\theta\right) \\ -r\left(\cos\frac{\theta}{2} - \cos\frac{2k-1}{2}\theta\right) \end{pmatrix}. \qquad (22)$$

Let us assign numeric values to variables, starting with the two basic data,

$$\begin{cases} \beta = 0.20 \quad \Rightarrow \quad \gamma = 1.0206 \quad \text{and} \quad \delta_0 = 1.0002\,, \\ r = 1000\ m, \end{cases}$$

and organize the successive phases of the route, for $n = 3$, following the methodology and the notations proposed above. We get

$$\begin{cases} l = 1732.0508\ m \\ t = 2.8887 \times 10^{-5}\ s \quad \text{and} \quad ct = 8660.2540\ m \\ \theta = 120°; \quad \alpha_0 = -150°. \end{cases}$$

In calculations, for simplicity, we will ignore metric units.

- First stretch, $\mathbf{0} \to \mathbf{1}$ $(k = 0)$:

$$\mathbf{x'}_0^1 = \mathbf{\Lambda'}_0\, \mathrm{x}_0^0$$
$$= \begin{pmatrix} 1.0206 & -0.2041 & 0.0000 \\ -0.2041 & 1.0206 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{pmatrix} \begin{pmatrix} 8660.2540 \\ 0 \\ 0 \end{pmatrix}$$
$$= \begin{pmatrix} 8838.8348 \\ -1767.7670 \\ 0 \end{pmatrix} = \mathbf{X'}_0^1.$$

- Second stretch, $\mathbf{1} \to \mathbf{2}$ $(k = 1)$:

According to (21):

$$\mathrm{x}_0^1 = \begin{pmatrix} ct \\ r\big(\cos(-150°) - \cos(-30°)\big) \\ r\big(\sin(-150°) - \sin(-30°)\big) \end{pmatrix}$$
$$= \begin{pmatrix} 8660.2540 \\ -1732.0508 \\ 0.0000 \end{pmatrix};$$

and so,

$$\mathbf{x'}_0^2 = \mathbf{\Lambda'}_1\, \mathrm{x}_0^1$$
$$= \begin{pmatrix} 1.0206 & 0.1021 & -0.1768 \\ 0.1021 & 1.0052 & -0.0089 \\ -0.1768 & -0.0089 & 1.0155 \end{pmatrix} \begin{pmatrix} 8660.2540 \\ -1732.0508 \\ 0.0000 \end{pmatrix}$$
$$= \begin{pmatrix} 8662.0581 \\ -857.0964 \\ -1515.4655 \end{pmatrix}.$$

We get the summative coordinates by

$$\mathbf{X'}_0^2 = \mathbf{x'}_0^2 + \mathbf{X'}_0^1 \quad \Rightarrow \quad \mathbf{X'}_0^2 = \begin{pmatrix} 17500.8928 \\ -2624.8634 \\ -1515.4655 \end{pmatrix}.$$

- Third stretch, $\mathbf{2} \to \mathbf{3}$ $(k = 2)$:

Now, we have,

$$\mathrm{x}_0^2 = \begin{pmatrix} ct \\ r\big(\cos(-150°) - \cos(90°)\big) \\ r\big(\sin(-150°) - \sin(90°)\big) \end{pmatrix} = \begin{pmatrix} 8660.2540 \\ -866.0254 \\ -1500.0000 \end{pmatrix};$$

So,

$$\mathbf{x'}_0^3 = \mathbf{\Lambda'}_2\, \mathrm{x}_0^2$$
$$= \begin{pmatrix} 1.0206 & 0.1021 & 0.1768 \\ 0.1021 & 1.0052 & 0.0089 \\ 0.1768 & 0.0089 & 1.0155 \end{pmatrix} \begin{pmatrix} 8660.2540 \\ -866.0254 \\ -1500.0000 \end{pmatrix}$$
$$= \begin{pmatrix} 8485.2814 \\ 0.0000 \\ 0.0000 \end{pmatrix}.$$

and, finally,

$$\mathbf{X'}_0^3 = \mathbf{x'}_0^3 + \mathbf{X'}_0^2 \quad \Rightarrow \quad \mathbf{X'}_0^3 = \begin{pmatrix} 25986.1742 \\ -2624.8634 \\ -1515.4655 \end{pmatrix},$$

which gives $\mathbf{T'}_0^3 = 8.6681 \times 10^{-5}\ s$. This is exactly the result given by (4):

$$\mathbf{T'}_0^3 = \frac{3}{2}\ 2.0004 \times 2.8887 \times 10^{-5}\ s$$
$$= 8.6681 \times 10^{-5}\ s,$$

thus confirming the proposed conjecture for n=3

2.5. Generic Polygonal Path

Take n as the number of sides of the polygon. Remark that the recursive definition established in (17) implies , for the final matrix of summative coordinates:

$$\mathbf{X'}_0^n = (\mathbf{x'}_0^n + \mathbf{x'}_0^{n-1} + \ldots + \mathbf{x'}_0^1) = \textstyle\sum_{k=1}^n \mathbf{x'}_0^k. \qquad (23)$$

So, to obtain this matrix, we do not really need to study each leg of the path, it is enough to previously calculate the n matrices x_0^0 to x_0^{n-1}, applying (21), and then calculate all the correspondent $\mathbf{x'}_0^k$ given by

$$\mathbf{x'}_0^k = \mathbf{\Lambda'}_{k-1}\ \mathrm{x}_0^{k-1}. \qquad (24)$$

The matrix $\mathbf{X'}_0^n$ follows, according to (23).

I made the calculations, for the current basic data ($\beta = 0{,}20$; $r = 1000\,m$), for n up to 10. Here are the final results:

- For $n = 4$:

$$\mathbf{X'}_0^4 = \begin{pmatrix} 28290.1632 \\ -2857.5893 \\ -2857.5893 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T'}_0^4 = 9.4366 \times 10^{-5}\ s$$

- For $n = 5$:

$$\mathbf{X'}_0^5 = \begin{pmatrix} 29395.3847 \\ -2969.2276 \\ -4086.7912 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T'}_0^5 = 9.8052 \times 10^{-5}\ s$$

- For $n = 6$:

$$\mathbf{X'}_0^6 = \begin{pmatrix} 30006.2493 \\ -3030.9310 \\ -5249.7266 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T'}_0^6 = 1.0009 \times 10^{-4}\ s$$

- For $= 7$:

$$\mathbf{X}'^{7}_{0} = \begin{pmatrix} 30378.1886 \\ -3068.5007 \\ -6371.8073 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T}'^{7}_{0} = 1.0133 \times 10^{-4} \ s$$

- For $= 8$:

$$\mathbf{X}'^{8}_{0} = \begin{pmatrix} 30621.0520 \\ -3093.0323 \\ -7467.2406 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T}'^{8}_{0} = 1.0214 \times 10^{-4} \ s$$

- For $= 9$:

$$\mathbf{X}'^{9}_{0} = \begin{pmatrix} 30788.2251 \\ -3109.9185 \\ -8544.4308 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T}'^{9}_{0} = 1.0270 \times 10^{-4} \ s$$

- For $= 10$:

$$\mathbf{X}'^{10}_{0} = \begin{pmatrix} 30908.1366 \\ -3122.0307 \\ -9608.6226 \end{pmatrix} \quad \Rightarrow \quad \mathbf{T}'^{10}_{0} = 1.0310 \times 10^{-4} \ s.$$

All these results match exactly (4); in fact, I verified it also for $n = 20$. For instance, for $n = 10$:

$$\begin{aligned} \mathbf{T}'^{10}_{0} &= 5 \times 2.0004 \times 1.0308 \times 10^{-5} \ s \\ &= 1.0310 \times 10^{-4} \ s. \end{aligned}$$

It is surely no accident; so, the proposed conjecture seems to be validated. And, if so, simply noting the final summative time by $\mathbf{T}$ or $\mathbf{T}'$, we may write

$$\mathbf{T} = \mathbf{T}' = \quad t\, \delta_0 = \frac{nl}{v}\, \delta_0 \quad \Rightarrow \quad \mathbf{T} = \mathbf{T}' = \frac{\mathrm{p}}{v}\, \delta_0. \qquad (25)$$

where $\mathrm{p} = nl$ is the Euclidean perimeter of the polygonal line. But then, the correspondent *relativistic perimeter*, $\mathrm{p}_R = \mathrm{p}'_R$, is given by

$$\mathrm{p}_R = v\mathbf{T} \quad \Rightarrow \quad \mathrm{p}_R = \delta_0 \mathrm{p}. \qquad (26)$$

3. Circular Path

If we accept the validity of the conjecture formalized by (4), then we may extend it to the circumference. Remark that the derived equations (25) and (26) establish $\mathbf{T}$ and p_R as independent from the number of sides of the polygon. So, they must remain valid for the limit $n \to \infty$, that is, when the polygon turns into a circle, yielding

$$\begin{cases} \mathbf{T} = \delta_0 \frac{2\pi r}{v} \\ \mathrm{p}_R = \delta_0\, 2\pi r \end{cases} \qquad (27)$$

For $\beta > 0$, we get $\mathrm{p}_R > 2\pi r$ and the direct relation between the perimeter and the radius of the circumference is no longer constant. Einstein had already understood this (equivalently, for a rotating frame S'), by arguing that there is a length contraction – from the point of view of immobile S – for a moving point along the circumference, but not for its radius, which is always perpendicular to the velocity vector [5]. But how must we interpret here the results above?

Take the observer A' moving along the circumference, leaving A behind him, until they meet again. They will agree on the time $\mathbf{T}$ that A' took in his journey. We can imagine (this is a thought experience) that A' had the opportunity to measure his course, directly obtaining p_R; or, both know the magnitude of the velocity v and obtain p_R by the first formula in (27). One way or the other, would both conclude that the proportion between p_R and the radius does not respect Euclidean Geometry?

This is somewhat disturbing, mainly because, even for the observer A, which remains still at one point in the circumference, the quotient "perimeter/radius" is no longer equal to 2π, instead depends on the state of movement of another observer. Moreover, he may not ascribe a definite magnitude to the perimeter of the circumference (the same is true for A' himself)!

In a way, this is true, for example, in relation to quantum particles in accelerators: it is known that their lifetime is significantly prolonged by their very high speed. This is usually attributed to the time dilation $\gamma\Delta t'$. But, after all, this means that, not only "for the particle" but also for the outsider observers, its path appears accordingly prolonged. On the one hand, this is no wonder: in fact, SR puts a strong emphasis on the relativity of measures. In order to measure the circumference, it needs to go through it, and this, of course, with a certain velocity; the slower the movement, the more p_R corresponds to Euclidian p.

On the other hand, to compare the perimeter measurement with the length of the radius, one needs to measure this length as well. And to achieve this, it is necessary to resort to a similar procedure: A' goes from A to the center of the circumference and returns to the starting point. But this is just the linear round trip analyzed before; the result of the measure, for the same velocity v, is, as we have seen,

$$\mathrm{r}_R = \ \delta_0\, r, \qquad (28)$$

the *relativistic radius* of the circumference, where r is the Euclidian radius.

Of course, because of the symmetry of the problem (and, as we will see ahead, under these circumstances, a circumference *remains* a circumference), this applies to any point in the circumference from which the observer departs toward the center. So, after all, there is a relativistic dilation in the radius length despite its perpendicularity towards the movement of A'. Now, applying $r = \mathrm{r}_R/\delta_0$ in the second equation of (27) we obtain

$$\mathrm{p}_R = 2\pi\, \mathrm{r}_R\,, \qquad (29)$$

an equality that is independent from the velocity of A', thus, preserving the Euclidean proportion. Here is something very satisfying!

Anyway, does all this mean that there is no "objective", say "absolute", measures either for the radius or the perimeter of a circumference? The answer must be negative. In whatever scale, there are "absolute" measures and relations: they correspond to Euclidean geometry, the simplest of all geometries. Remark that this geometry is at the basis of all the precedent reasoning and calculations, concerning, for instance, polygons and trigonometric relations. All the subsequent results structure themselves over the essential validity of this geometry. In a way, Euclidean geometry establishes an “absolute” – and this is consistent with the statement I proposed in another paper [6], that Minkowski’s pseudo-Euclidean geometry corresponds to the fundamental structure of the world.

4. Further Meditations

4.1. Dilation Factors

Let us establish first the relationship between the new dilation factors, δ_0 and δ, with the standard γ. Regarding δ_0, it stems directly from (6):

$$\delta_0 = \left(1 - \frac{\beta^2}{2}\right)\gamma. \qquad (30)$$

This implies that

$$\frac{1}{2} \leq \frac{\delta_0}{\gamma} < 1. \qquad (31)$$

But the most relevant is the relationship between factors δ and γ because they both operate on the *proper time* t' of the moving observer (or particle). From (14), we get

$$\delta = \delta_0\gamma \quad \Rightarrow \quad \delta = \left(1 - \frac{\beta^2}{2}\right)\gamma^2. \qquad (32)$$

Remark that, assuming $|\beta| > 0$, since $\delta_0 > 1$ (as we have seen in paragraph 2.2), it comes that $\delta > \gamma$. The Table 1 relates the values of δ and γ to increasing velocity values. This prevision may settle the question of what should be the *effective time dilation experienced in particle accelerators*, validating or not the present theory.

Table 1. Dilation factors γ and δ, as functions of β, and their proportion

β	γ	δ	δ/γ
0.00	1.0000	1.0000	100.00%
0.10	1.0050	1.0051	100.00%
0.20	1.0206	1.0208	100.02%
0.30	1.0483	1.0495	100.11%
0.40	1.0911	1.0952	100.38%
0.50	1.1547	1.1667	101.04%
0.60	1.2500	1.2813	102.50%
0.70	1.4003	1.4804	105.72%
0.80	1.6667	1.8889	113.33%
0.90	2.2942	3.1316	136.50%
0.95	3.2026	5.6282	175.741%

The corresponding chart of Figure 2 also relates γ and δ to increasing velocity values, but using gaps of 0.05 for β.

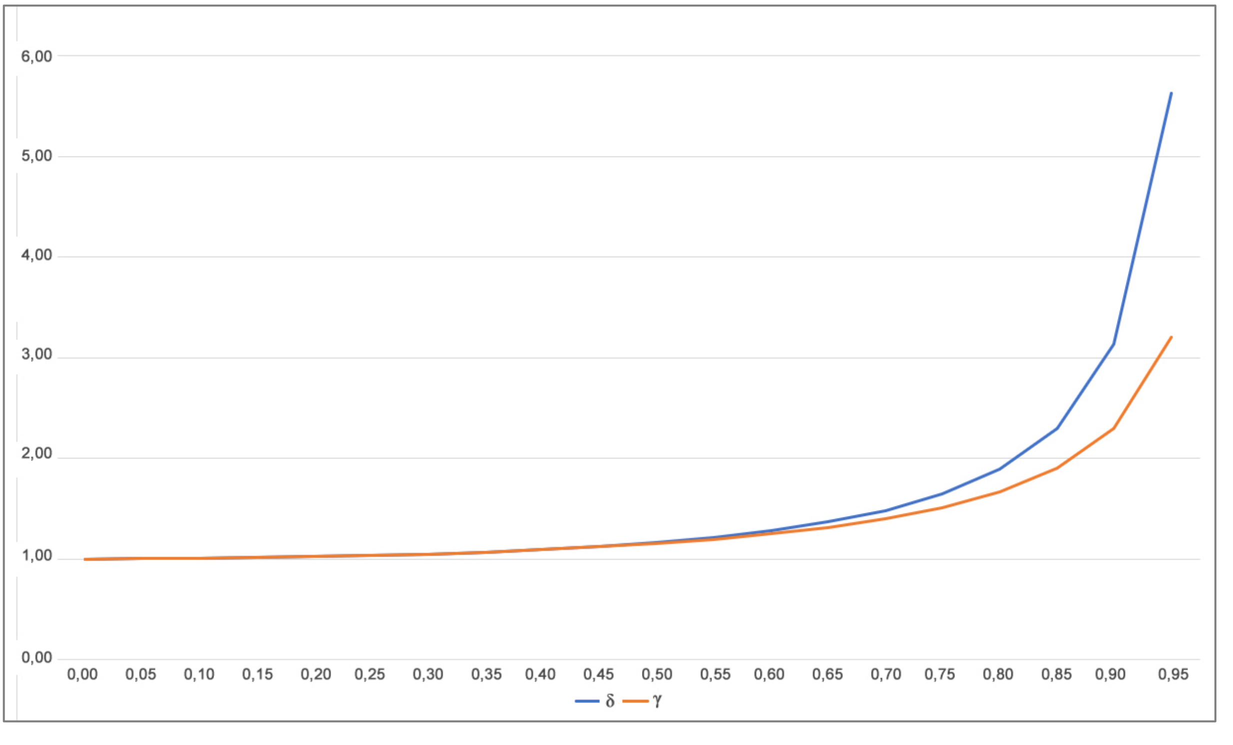

Figure 2. Dilation factors δ and γ, as functions of β

4.2. A Mysterious Sequence

In an attempt to prove the conjecture proposed by (4), rewritten as

$$\mathbf{T}' = \mathbf{T} = \frac{n}{2}\left(\gamma\, t + \frac{t}{\gamma}\right),$$

I noticed that this equation resembles the formula to calculate the sum of the first n terms of an arithmetic progression

$$a_1\ , \qquad a_2 = a_1 + q\ , \dots\ a_k = a_1 + (k-1)q\ , \dots\ a_n = a_1 + (n-1)q, \qquad \dots$$

where a_1 and q are respectively the initial term and the common difference of the progression; the well-known formula is

$$\sum_{i=1}^{n} a_i = \frac{n}{2}(a_1 + a_n).$$

Now, according to (23),

$$\mathbf{X}'^{n}_{0} = \sum_{k=1}^{n} \mathbf{x}'^{k}_{0} \quad \Rightarrow \quad \mathbf{T}'^{n}_{0} = \sum_{k=1}^{n} \mathbf{t}'^{k}_{0}.$$

Would this represent an arithmetic series? I also noticed that

$$\mathbf{x}'^{1}_{0} = \mathbf{\Lambda}'_0\, \mathrm{x}^{0}_{0} = \begin{pmatrix} \gamma ct \\ -\gamma l \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{t}'^{1}_{0} = \gamma\, t,$$

regardless of the value of n. Moreover, concerning the last term in the sum for $\mathbf{X}'^{n}_{0}$, I verified in the calculations for the results presented in section 2.5 – though I am not able to prove it – that the last term corresponds indeed to $\mathbf{t}'^{n}_{0} = \frac{t}{\gamma}$.

So, associating the sequence with $a_k = \mathbf{t}'^{k}_{0}$ as the k^{th} term to a **decreasing arithmetic progression**, with first and last terms given respectively by $\mathbf{t}'^{1}_{0} = \gamma t$ and $\mathbf{t}'^{\mathrm{n}}_{0} = \mathrm{t}/\gamma$, was a tempting idea [remember that $\mathbf{t}'^{k}_{0}$ is the time coordinate of vertex 0 relatively to vertex k in time $t_0 = 0$, corresponding to the first coordinate in $\mathbf{x}'^{k}_{0}$, given by (24), divided by c]. The common difference would be given by

$$q = -\frac{\gamma\, t\beta^2}{n-1}.$$

We have seen this negative numerator, $-\gamma\, t\beta^2$, in (1) for t'_{B0}. It is also, in the round trip, the coordinate t'^{1}_{0}, this is, of the observer A in the new frame S'_1 at the very instant t'_0 when A' reverts his movement [refer to [3], (12)].

These were strong indications in favour of the mentioned tempting idea, justifying a deeper examination. But it doesn't fit! As a matter of fact, except for n=3 (which fits into the arithmetic progression because the intermediate term is necessarily the average of the extremes), the sequence

$$\mathbf{t}'^{1}_{0}\ , \qquad \mathbf{t}'^{2}_{0}\ , \qquad \dots \qquad , \mathbf{t}'^{n}_{0}$$

obeys a logic somehow mysterious and capricious, as one can see in Table 2 (for each n, the sequence is arranged vertically; each number must be multiplied by $10^{-5}\, s$; each line corresponds to a k, displayed on the left).

Table 2. Sequences of $\mathbf{t}'^{k}_{0}$ for $1 < k < n$; values in the scale $1:\ 10^{-5}\, s$

k \ n	2	3	4	5	6	7	8	9	10
1	3.4044	2.9483	2.4072	2.0011	1.7022	1.4771	1.3028	1.1644	1.0520
2	3.2682	2.8894	2.4073	2.0258	1.7363	1.5140	1.3397	1.2001	1.0861
3		2.8304	2.3110	1.9610	1.7022	1.5008	1.3397	1.2081	1.0991
4			2.3110	1.8963	1.6341	1.4476	1.3028	1.1849	1.0861
5				1.9210	1.6001	1.3943	1.2507	1.1411	1.0520
6					1.6341	1.3812	1.2139	1.0973	1.0099
7						1.4180	1.2139	1.0740	0.9759
8							1.2507	1.0821	0.9629
9								1.1178	0.9759
10									1.0099

Sequences, instead of always decreasing, eventually become increasing (this always happens at the end, for $n \geq 5$; for $n = 4$, when the polygon is a square, $\mathbf{t'}_0^1 = \mathbf{t'}_0^2$ and $\mathbf{t'}_0^3 = \mathbf{t'}_0^4$) and sometimes repeat values.

However, Nature is wise! The chart in Figure 3 represents, for $n = 10$, a comparative evolution concerning the time sequence and the arithmetic progression. Looking at it, the strange behavior of $\mathbf{t'}_0^k$ variation finally makes some sense! There is a radial symmetry around the center (n=5.5). Two equidistant points $\mathbf{t'}_0^k$ from the extremes, each of its own, balance perfectly in relation to the corresponding points a_k of the arithmetic progression:

$$\mathbf{t'}_0^{1+b} - a_{1+b} = -(\mathbf{t'}_0^{n-b} - a_{n-b}).$$

It is precisely because of this that both sums give the same result:

$$\mathbf{T'}_0^n = \sum_{k=1}^{n} \mathbf{t'}_0^k = \sum_{i=1}^{n} a_i.$$

4.3. The Inner Sequence and Geometry

The "inner sequence" is the name I give to the sequence of $\mathbf{X'}_{k-1}^k$; that is, of the summative coordinates of each vertex respectively to the following one in the polygon, for the respective k^{th} leg of the path. We are mainly interested in the time sequence, $\mathbf{T'}_{k-1}^k$.

Remark that, because the time component ct is always the same, we may write

$$\mathbf{X'}_{k-1}^k = \mathbf{X'}_0^k - \mathbf{X'}_0^{k-1};$$

but, according to (17), $\mathbf{X'}_0^k = \mathbf{x'}_0^k + \mathbf{X'}_0^{k-1}$, and so:

$$\mathbf{X'}_{k-1}^k = \mathbf{x'}_0^k \quad \Rightarrow \quad \mathbf{T'}_{k-1}^k = \mathbf{t'}_0^k. \tag{33}$$

This means that we obtain exactly the same sequences as before. For instance (in scale $1 : 10^{-5}\, s$):

- For $n = 2$: initial $\mathbf{T'}_0^1 = 3.4044$; $\mathbf{T'}_1^2 = 3.2682$;
- For $n = 3$: $\mathbf{T'}_0^1 = 2.9483$; $\mathbf{T'}_1^2 = 2.8894$ and $\mathbf{T'}_2^3 = 2.8304$;
- For $n = 10$: $\mathbf{T'}_0^1 = \mathbf{T'}_4^5 = 1.0520$; $\mathbf{T'}_2^3 = 1.0991$ and $\mathbf{T'}_5^6 = \mathbf{T'}_9^{10} = 1.0099$.

From (33) we obtain the relativistic length of the correspondent k^{th} side of the polygon, $\mathbf{l'}^k$:

$$\mathbf{l'}^k = \mathbf{l'}_{k-1}^k = v\, \mathbf{T'}_{k-1}^k \quad \Rightarrow \quad \mathbf{l'}^k = v\, \mathbf{t'}_0^k. \tag{34}$$

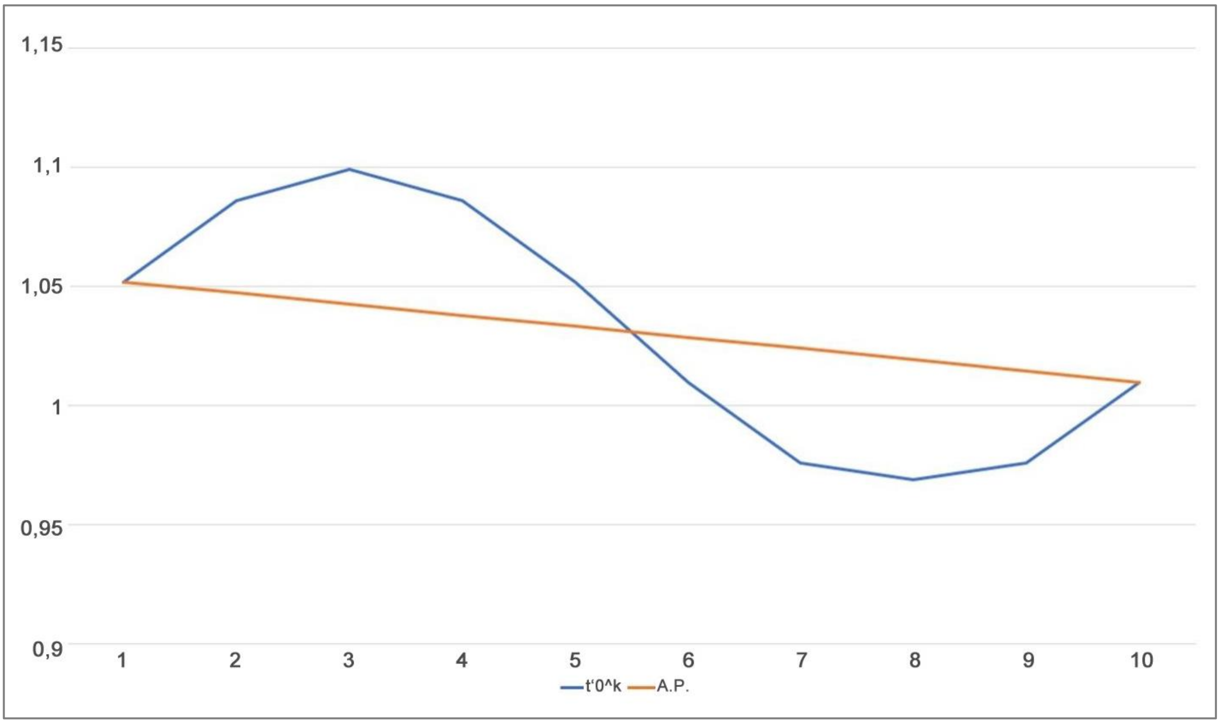

Figure 3. Comparative evolution concerning the time sequence $\mathbf{t'}_0^k$ and the arithmetic progression for $n = 10$

Due to the variation in the values of $\mathbf{t}'^{k}_{0}$, a proportional variation for l'^{k} results, as shown in Table 3 (note that, for instance, the constant Euclidean length for $n = 10$ is equal to 618,03 m). Consequently, the correspondent chart (see Figure 4), for the comparative evolution concerning the side lengths sequence and the arithmetic progression (with $\mathbf{l}'^{1}$ and $\mathbf{l}'^{n}$ as the first and last terms), is identical to Figure 3.

We conclude then that the polygon 'constructed' by the movement of A' – over a regular polygon in an 'immobile' *Euclidean substratum* – is **irregular**. But, in the end, it comes from (23) that

$$\sum_{k=1}^{n} \mathbf{l}'^{k} = v \sum_{k=1}^{n} \mathbf{t}'^{k}_{0} = v\, \mathbf{T}'^{n}_{0} = \mathrm{p}_R,$$

that is, the *relativistic perimeter* of the polygon.

In the case of $n = 10$, for instance, we obtain $\mathrm{p}_R = 6181.63$, either summing all the side lengths shown in Table 3 either applying formula (26): $\mathrm{p}_R = \delta\, \mathrm{p} = 1{,}0002 \times 6180.34 = 6181.63$.

Table 3. Sequences of $\mathbf{l}'^{k}$ for $1 < k < n$; values in the scale $1 : 1\, m$

k \ n	2	3	4	5	6	7	8	9	10
1	2041.24	1767.77	1443.38	1199.81	1020.62	885.66	781.15	698.15	630.78
2	1959.59	1732.41	1443.38	1214.64	1041.03	907.75	803.24	719.54	651.19
3		1697.06	1385.64	1175.82	1020.62	899.87	803,24	724.39	658.99
4			1385.64	1136.99	979.80	867.95	781.15	710.42	651.19
5				1151.82	959.38	836.03	749.90	684.18	630.78
6					979.80	828.15	727.81	657.94	605.55
7						850.24	727.81	643.98	585.13
8							749.90	648.83	577.34
9								670.22	585.13
10									605.55

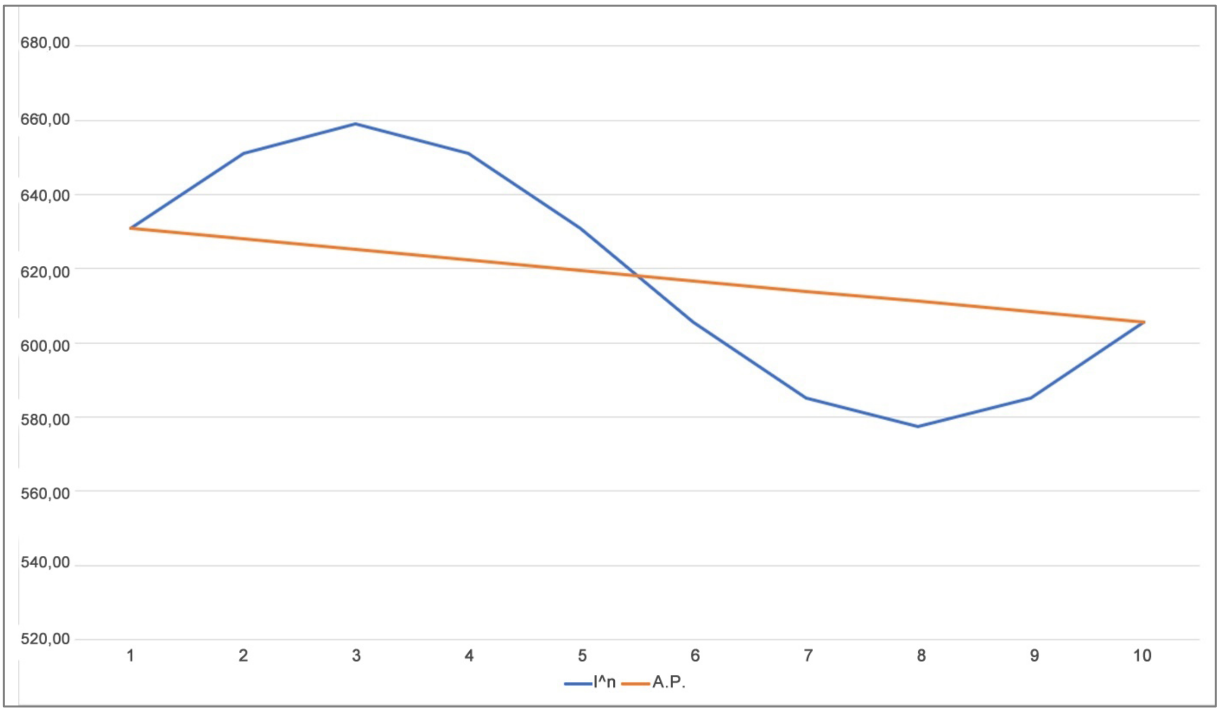

Figure 4. Comparative evolution concerning the side lengths sequence l'^{k}_{0} and the arithmetic progression for $n = 10$

The irregularity results from the change of trajectory at each vertex, that is, from the angular difference $\theta = 360°/n$, which, as pointed out, imply new distortions in Lorentz transformations. But as the number of sides increases, the angular difference decreases, which certainly implies a progressive smoothing of the irregularity until its disappearance at the limit, that is, in the case of the circumference. Moreover, the chart of the side length sequence tends to the straight line representing the correspondent arithmetic progression (see Figure 5); and, since, for the side length sequence,

$$a_1 - a_n = r\sin\left(\frac{180°}{n}\right)\frac{\beta^2}{\sqrt{1-\beta^2}},$$

we get

$$\lim_{n\to\infty}[a_1 - a_n] = 0.$$

This means that the straight tends itself to the horizontal. For instance,

$$\begin{cases} For\ \ n = 10: & \sin 18° = 0.30902\ ; & l(a_1 - a_n) = 25.2311\, m \\ For\ \ n = 20: & \sin 9° = 0.15643\ ; & l(a_1 - a_n) = 12.7728\, m \\ For\ \ n = 1000: & \sin 0.18° = 0.00314\ ; & l(a_1 - a_n) = 0.2565\, m \\ For\ \ n = 100000: & \sin 0.0018° = 0.00003\ ; & l(a_1 - a_n) = 0.0026\, m. \end{cases}$$

Finally, as n increases, both a_1 and a_n tend to zero. But remember that the total length, the relativistic perimeter, is given by $\mathrm{p}_R = \frac{n}{2}(a_1 + a_n)$. Note that, for the last number of vertexes, $n = 100000$, this gives $\mathrm{p}_R = 6284.4942\ m$, which is almost indistinguishable from the relativistic perimeter of the **circumference**, obtained from $\mathrm{p}_R = \delta\ 2\pi r$ (there is no difference until the sixth decimal case).

This all means that, as has already been observed, under these relativity conditions, a circumference remains a circumference. It also means that the arc length, for a circumference, remains proportional to the angle at the center (measured in radians):

$$\begin{cases} \mathrm{l} = \theta\ \mathrm{r} \\ \mathrm{l}_R = \theta\ \mathrm{r}_R \end{cases} \quad \text{(as it happens for } \theta = 2\pi) \quad \Rightarrow \quad \theta = \frac{\mathrm{l}_R}{\delta\ r}. \tag{35}$$

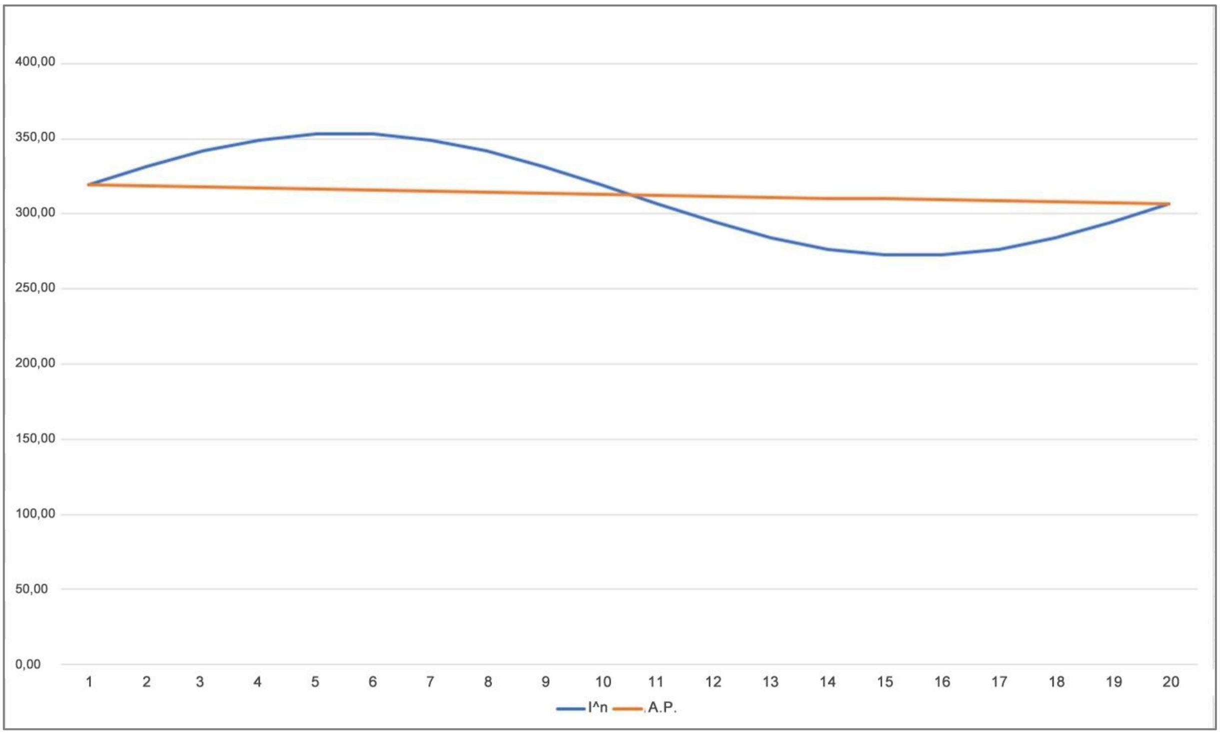

Figure 5. Comparative evolution concerning the side lengths sequence $\mathrm{l'}_0^k$ and the arithmetic progression for $= 20$; remark that both tend to a horizontal line.

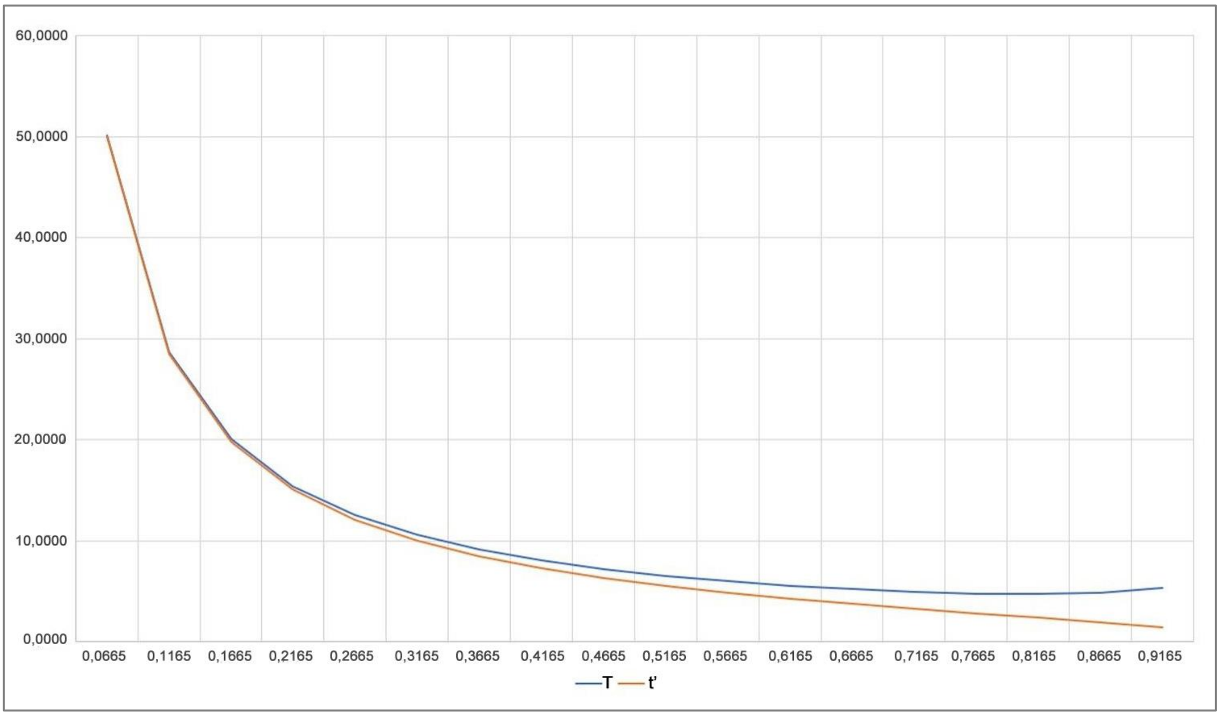

Figure 6. $\mathbf{T}$ and $\mathbf{t}'$ as functions of velocity factor β for given perimeter ($\mathrm{p} = 10^9\ m$)

4.4. The Behavior of Total Time T

Imagine a material particle traveling along a closed polygonal or circular path, with Euclidean perimeter p, and returning to the starting point. Two equations apply to the end of the process:

$$\begin{cases} \mathbf{t}' = \dfrac{\mathbf{t}}{\gamma} = \dfrac{\mathrm{p}}{c\beta\gamma} & \text{for the proper time of the moving particle; and} \\ \mathbf{T} = \delta\mathbf{t}' = \dfrac{\mathrm{p}}{c}\dfrac{1}{\beta}\,\delta_0 & \text{for the relativistic time in the rest frame of the path.} \end{cases}$$

The goal here is to study the behavior of $\mathbf{t}'$ and $\mathbf{T}$ as functions of velocity factor β. Making, for instance, $\mathrm{p} = 10^9\ m$, the resulting chart is shown in Figure 6. It jumps out that:

- As usual in SR, for small β (in this case, until about 20% of the speed of light), $\mathbf{T}$ and $\mathbf{t}'$ are hardly distinguishable.
- Curiously enough, there is a **minimum** for $\mathbf{T}$.

Let us search for this minimum, $\mathbf{T}_{min}$. We start by determining the derivative of the function $\mathbf{T} = f(\beta)$; resolving the problem, we obtain:

$$\frac{d\mathbf{T}}{d\beta} = \frac{\mathrm{p}}{2c}\,\frac{3\beta^2-2}{\beta^2(1-\beta^2)^{3/2}}. \tag{36}$$

Therefore, the minimum corresponds to

$$\frac{d\mathbf{T}}{d\beta} = 0 \quad \Rightarrow \quad \beta_\circ = \sqrt{\frac{2}{3}}\ . \tag{37}$$

Remark that this value of $\beta_\circ$ is independent from the perimeter p. Besides and the most important, *it is useless to accelerate the particle above the speed* $v_\circ = c\beta_\circ$; it will not reach the goal sooner, quite the contrary.

Finally, applying $\beta_\circ$ in $\mathbf{T} = \frac{\mathrm{p}}{c}\frac{1}{2\beta}\frac{2-\beta^2}{\sqrt{1-\beta^2}}$, we get

$$\mathbf{T}_{min} = \frac{\mathrm{p}}{c}\sqrt{2}. \tag{38}$$

In our example, $\mathbf{T}_{min} = 4.7173\ s$. Note that $\beta_\circ \approx 0.8165$; the chart in Figure 5 has then been constructed using β gaps of 0.05 and starting at $\beta = 0.0665$, so that the sixteenth term is $\beta_\circ$.

5. Conclusions

Using successive inertial reference frames, we theoretically studied the movement of an observer along a closed regular polygonal line. It is proved here that, when meeting another observer, left immobile behind him, both agree on the time elapsed since the departure: that is, they have aged the same. It is a dilated time, but not following the classical *Lorentz factor*, instead a bigger one, the so-called *delta factor*, which implies a dilated measured distance covered by the traveler: the relativistic perimeter of the polygonal line.

A fundamental conjecture, regarding this final time, is fully verified for polygons with up to twenty number of sides. This cannot be due to chance, allowing its extension to circular paths. Considerations about Euclidean geometry were presented, particularly regarding the ratio between perimeter and radius of a circumference, which keeps validity in SR, relating immobile and moving observers over the circumference, since one understands the correspondent relativistic perimeter and radius.

From an experimental point of view, in particle accelerators, the *delta dilation of time* should apply to the increased time life of the running unstable particles. Finally, the new time dilation factor δ has been analyzed, bringing forward a *threshold speed* for the shortest possible time to complete a closed path. The experimental verification of these two proposals will be a crucial test of the theory proposed here.

REFERENCES

[1] Paul Langevin. "L'Évolution de l'espace et du temps", Scientia X: 31-54.

[2] Hendrik A. Lorentz. Albert Einstein e Hermann Minkowski. The principle of relativity - Portuguese translation. Calouste Gulbenkian Foundation, Lisbon, 3rd edition. 1983; 55-62, §4. The translation from Portuguese to English is mine. I also replaced V for the modern notation c. An alternative translation may be found in Doc. 23, p. 153 of https://einsteinpapers.press.princeton.edu/vol2-trans/167.

[3] Luís Dias Ferreira. Criticism to the Twin's Paradox, Universal Journal of Physics and Application, Vol. 15, No. 1, pp. 1 - 7, 2021. DOI: 10.13189/ujpa.2021.150101.

[4] Wikipedia, Transformação de Lorentz, Online available from https://pt.wikipedia.org/wiki/Transformação_de_Lorentz (accessed April, 2022).

[5] Hendrik A. Lorentz. Albert Einstein e Hermann Minkowski. The principle of relativity; Chapter: The Foundations of the Theory of General Relativity (1916) - Portuguese translation. Calouste Gulbenkian Foundation, Lisbon, 3rd edition. 1983; 147.

[6] Luís Dias Ferreira. Einstein's Mistake: The Non-Objectivity of Spacetime Distortion, Sumerianz Journal of Scientific Research, 2019, Vol.2, No. 3, 32–48. Online available from https://www.sumerianz.com/pdf-files/sjsr2(3)32-48.pdf (accessed May 7, 2022).