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Evolution Equations of Pseudo Spherical Images for Timelike Curves in Minkowski 3-Space

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Abstract The pseudo spherical images of non-lightlike curves in Minkowski geometry are curves on the unit pseudo sphere, which are intimately related to the curvatures of the original ones. These images are obtained by means of Frenet-Serret frame vector fields associated with the curves. This classical topic is a well-known concept in Lorentzian geometry of curves. In this paper, we introduce the pseudo spherical images for a timelike curve in Minkowski 3-space. Our main purpose of the work is to obtain the time evolution equations of the orthonormal frame and curvatures of these images. The compatibility conditions for the evolutions are used. Finally, the theoretical results obtained through this study are given by some important theorems and explained in two computational examples with the corresponding graphs.

Keywords Evolution Equations, Timelike Curves, Pseudo Spherical Images, Serret-Frenet Frame

1 Introduction

The curves and their geometric properties play an important role in the field of differential geometry and in many branches of science such as mechanics and physics. They have some applications such as computer aided geometric design (CAGD) and mathematical modeling [1]. Further, curves are usually studied as subsets of an ambient space with a notion of equivalence. For example, one may study curves in the plane, the usual three dimensional space, the Minkowski space, curves on a sphere, etc.[2, 3]

Among these properties are the evolution equations for curves which is our concern, in which time is a fundamental element of the study. These equations have a clear effect in many physical and dynamic applications and also have a notable affect in several fields as in image processing and how it is useful in images and shapes recognizing [4, 5]. Besides, how the evolution of a curve related to the level sets as well as the massive change of studying the generalization of the local maximum and minimum which are known as ridges and ravines [6, 7, 8]. In addition, one can see the effect of the evolution equations to the Fluid study such as studying the surfaces evolution in Turbulence [9]. For studying these equations with respect to curves or surfaces in any space , there are some enormous ways such as equations of motion which describe the dynamical system of the curves via their frame field, via their velocity vector or even via their accelerations etc. For more details see [4, 7, 10].

In the Minkowski geometry, the pseudo spherical images of non-lightlike curves are curves on the unit pseudo sphere intimately related to the curvatures of the original curves. These images are obtained by means of Frenet-Serret frame vector fields associated to the meant curves, so this classical topic is a well-known concept in Lorentzian geometry of the curves, see [11, 12].

There are many studies of evolution equations that have been done in different spaces (see for instance, [7, 9, 10, 13]). Also, evolution equations for the elliptic partial differential equations and the magnetic geodesics equations have been obtained in [8].

As an extension of such studies, we interested here with the study of pseudo spherical images of a non-lightlike curve, especially timelike curve and derive their evolution of time equations attributed to the curvature and torsion of the considered timelike curve. The derivation of these equations is based on the numerical integration of the Frenet frame with the help of Mathematica package [14, 15].

2 Fundamental Concepts

In this section, we list some notions, formulas and conclusions for the theory of curves and Lorentzian vectors that we will use in this paper (see for more details, [2, 3, 16]).
Let Minkowski 3-space E_1^3 be the vector space E^3 provide with the Lorentzian inner product $\langle\, ,\, \rangle$ given by

$$\langle X, X \rangle = -x_1^2 + x_2^2 + x_3^2,$$

where $X = (x_1, x_2, x_3) \in E_1^3$. An arbitrary vector u in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike, timelike and lightlike (null) if $\langle u, u \rangle > 0$ or $u = 0$, $\langle u, u \rangle < 0$ and $\langle u, u \rangle = 0$ and $u \neq 0$, respectively. Similarly, a curve γ, locally parameterized by $\gamma = \gamma(s) : I \subset R \longrightarrow E_1^3$ where s is a pseudo arc length parameter, is called a spacelike curve if $\langle \gamma'(s), \gamma'(s)\rangle > 0$, timelike if $\langle \gamma'(s), \gamma'(s) \rangle < 0$ and lightlike if $\langle \gamma'(s), \gamma'(s) \rangle = 0$ and $\gamma'(s) \neq 0$ for all $s \in I$. The curve γ is said to be regular if $\gamma' \neq 0$ and it is parameterized by the pseudo arc length s.
The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in E_1^3$ are orthogonal if and only if $\langle X , Y \rangle = 0$.
The Lorentzian cross product is defined as

$$X \times_{E_1^3} Y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix},$$

and the norm of a vector $v \in E_1^3$ is given by $\|v\| = \sqrt{|\langle v, v \rangle|}$.
The unit pseudo sphere in the Minkowski E_1^3 is defined by:

$$S_1^2 = \{(x_1, x_2, x_3) \in E_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = 1\}.$$

We denote the Frenet trihedron of the curve γ by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in the Minkowski E_1^3. These vectors are respectively, the tangent, normal and binormal and their derivatives are given in the matrix form as follows

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \epsilon_{\mathbf{B}}\kappa(s) & 0 & \tau(s) \\ 0 & \epsilon_{\mathbf{T}}\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}, \quad (1)$$

where $(')$ denotes the derivative with respect to s. The curvature κ and the torsion τ of γ are determined by

$$\begin{aligned} \kappa(s) &= \epsilon_{\mathbf{N}} ||\Gamma''(s)||, \\ \tau(s) &= \epsilon_{\mathbf{B}} \frac{\det(\Gamma', \Gamma'', \Gamma''')}{\kappa^2(s)}, \end{aligned} \quad (2)$$

where

$$\langle \mathbf{T}, \mathbf{T}\rangle_{E_1^3} = \epsilon_{\mathbf{T}},\ \langle \mathbf{N}, \mathbf{N}\rangle_{E_1^3} = \epsilon_{\mathbf{N}},\ \langle \mathbf{B}, \mathbf{B}\rangle_{E_1^3} = -\epsilon_{\mathbf{T}}\epsilon_{\mathbf{N}} = \epsilon_{\mathbf{B}},$$

and

$$\mathbf{T} \times_{E_1^3} \mathbf{N} = \mathbf{B},\ \ \mathbf{N} \times_{E_1^3} \mathbf{B} = -\epsilon_{\mathbf{N}}\mathbf{T},\ \ \mathbf{B} \times_{E_1^3} \mathbf{T} = -\epsilon_{\mathbf{T}}\mathbf{N}.$$

The Darboux rotation vector $\mathbf{D}$ is defined in its general form as follows

$$\mathbf{D} = \epsilon_{\mathbf{T}} \frac{\tau}{\kappa} \mathbf{T} + \epsilon_{\mathbf{N}}\epsilon_{\mathbf{B}}\mathbf{B}, \quad (3)$$

noting that

$$\begin{cases} \mathbf{T}' = \mathbf{D} \times \mathbf{T}, \\ \mathbf{N}' = \mathbf{D} \times \mathbf{N}, \\ \mathbf{B}' = \mathbf{D} \times \mathbf{B}. \end{cases}$$

3 Evolution Equations of a Timelike Curve

In this section, we will give a short notion about the timelike curve in E_1^3, then we derive its evolution equations. Let $\Gamma = \Gamma(s)$ be a given timelike curve with pseudo arc length parameter s in E_1^3 and $\{\mathbf{T}(s), \mathbf{n}(s), \mathbf{p}(s)\}$ its Frenet frame, where $\mathbf{T}(s) = \Gamma'(s)$, $\mathbf{n}(s) = \frac{\Gamma''(s)}{||\Gamma''(s)||}$, $\mathbf{p}(s) = \mathbf{T}(s) \times \mathbf{n}(s)$ are the tangent, principal normal and binormal vector fields, respectively. Even with the orthogonality of the trihedron frame, they are needless to be positively oriented as in the Euclidean space [16]. Then, the Frenet formulas for the curve Γ are read

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{n}'(s) \\ \mathbf{p}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{n}(s) \\ \mathbf{p}(s) \end{bmatrix}, \quad (4)$$

where

$$\langle \mathbf{T}, \mathbf{T}\rangle_{E_1^3} = -1,\ \langle \mathbf{n}, \mathbf{n}\rangle_{E_1^3} = 1,\ \langle \mathbf{p}, \mathbf{p}\rangle_{E_1^3} = 1,$$

$$\mathbf{n} \times_{E_1^3} \mathbf{p} = -\mathbf{T},\ \ \mathbf{p} \times_{E_1^3} \mathbf{T} = \mathbf{n}.$$

The curvatures $\kappa(s)$ and $\tau(s)$ for Γ are expressed as

$$\begin{aligned} \kappa(s) &= ||\Gamma''(s)||, \\ |\tau(s)| &= \frac{\det(\Gamma', \Gamma'', \Gamma''')}{\kappa^2(s)}. \end{aligned} \quad (5)$$

In this regards, the Darboux vector has the following form

$$\mathbf{D} = -\frac{\tau}{\kappa}\mathbf{T} + \mathbf{p}. \quad (6)$$

Now, we will introduce the evolution equations for the curve Γ through the following theorem.

Theorem 3.1 *Let $\Gamma : I \subset R \longrightarrow E_1^3$ be a timelike curve which has Frenet frame $\{\mathbf{T}, \mathbf{n}, \mathbf{p}\}$ and curvatures κ and τ. The time evolution equations for the curvatures of Γ are given by*

$$\begin{cases} \kappa_t = (\tau\,\zeta_2 + \zeta_{3s}); \zeta_3 = (\frac{\kappa\zeta_1 + \zeta_{2s}}{\tau}), \\ \tau_t = \zeta_{1s} + \kappa\,\zeta_2. \end{cases} \quad (7)$$

where ζ_1, ζ_2, and ζ_3 are the velocities of the moving frame that describe the motion of Γ.

Proof 1 *Let the curve Γ move with time t and using Darboux equation (6), then one can write*

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{T}(s,t) \\ \mathbf{n}(s,t) \\ \mathbf{p}(s,t) \end{pmatrix} = \zeta \times \begin{pmatrix} \mathbf{T}(s,t) \\ \mathbf{n}(s,t) \\ \mathbf{p}(s,t) \end{pmatrix},$$

where $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ is the velocity vector. In the light of this, the velocity matrix of the curve can be written as:

$$\frac{\partial}{\partial t}\Omega = \mathbf{E}\,\Omega, \tag{8}$$

with noting that

$$\Omega = \begin{pmatrix} \mathbf{T}(s,t) \\ \mathbf{n}(s,t) \\ \mathbf{p}(s,t) \end{pmatrix}, \mathbf{E} = \begin{pmatrix} 0 & \zeta_3(s,t) & -\zeta_2(s,t) \\ \zeta_3(s,t) & 0 & \zeta_1(s,t) \\ -\zeta_2(s,t) & -\zeta_1(s,t) & 0 \end{pmatrix}. \tag{9}$$

On the other hand, Frenet equations (4) can be written as:

$$\frac{\partial \Omega}{\partial s} = \mathbf{F}\,\Omega, \tag{10}$$

where

$$\mathbf{F} = \begin{pmatrix} 0 & \kappa(s,t) & 0 \\ \kappa(s,t) & 0 & \tau(s,t) \\ 0 & -\tau(s,t) & 0 \end{pmatrix}, \tag{11}$$

then, the compatibility condition for Eqs. (8) and (10) and using Eqs. (9) and (11) is expressed as

$$\frac{\partial}{\partial t}\frac{\partial \Omega}{\partial s} = \frac{\partial}{\partial s}\frac{\partial \Omega}{\partial t}, \tag{12}$$

which reads

$$\frac{\partial \mathbf{F}}{\partial t} - \frac{\partial \mathbf{E}}{\partial s} + [\mathbf{F}, \mathbf{E}] = \mathbf{0}_{3\times 3}, \tag{13}$$

where

$$[\mathbf{F}, \mathbf{E}] = \mathbf{FE} - \mathbf{EF},$$

is the Lie bracket of $\mathbf{E}$ and $\mathbf{F}$.
In virtue of the above, the compatibility condition takes the next form

$$\begin{pmatrix} 0 & \kappa_t - \zeta_{3s} - \tau\,\zeta_2 & \kappa\,\zeta_1 + \zeta_{2s} - \tau\,\zeta_3 \\ \kappa_t - \zeta_{3s} - \tau\,\zeta_2 & 0 & \tau_t - \zeta_{1s} - \kappa\,\zeta_2 \\ \kappa\,\zeta_1 + \zeta_{2s} - \tau\,\zeta_3 & -\tau_t + \zeta_{1s} + \kappa\,\zeta_2 & 0 \end{pmatrix} = \mathbf{0}_{3\times 3}, \tag{14}$$

which gives the required evolution equations (7). Thus, the proof is completed.

4 Pseudo Spherical Images of a Timelike Curve and Evolution Equations

4.1 Tangent Pseudo Spherical Image

We recall that, the pseudo spherical indicatrices of a curve are known to be the image of this curve under a mapping from the curve onto the unit pseudo sphere S_1^2 by a unit vector (e.g. $\mathbf{T}$, $\mathbf{n}$ or $\mathbf{p}$) of this curve [3]. Then, the tangent pseudo spherical image $\Gamma_{\mathbf{T}}$ of the curve Γ is defined as the locus on the unit pseudo sphere traced by the tangent $\mathbf{T}(s)$ to that curve. Accordingly, it can be given by

$$\Gamma_{\mathbf{T}}(\phi_{\mathbf{T}}(s)) = \mathbf{T}(s), \tag{15}$$

where $\phi_{\mathbf{T}} : I \longrightarrow I_{\mathbf{T}}$, $\phi_{\mathbf{T}}(s) = \int \kappa(s)ds$ is a regular C^∞ function. The Frenet formulas for $\Gamma_{\mathbf{T}}$ can be introduced by

$$\begin{cases} \mathbf{T}_{\mathbf{T}} = \mathbf{n}, \\ \mathbf{n}_{\mathbf{T}} = \Sigma_1 \mathbf{T} + \Sigma_2 \mathbf{p}, \\ \mathbf{p}_{\mathbf{T}} = -\Sigma_2 \mathbf{T} - \Sigma_1 \mathbf{p}, \end{cases} \tag{16}$$

and its curvatures $\kappa_{\mathbf{T}}$ and $\tau_{\mathbf{T}}$ are expressed as

$$\begin{cases} \kappa_{\mathbf{T}} = \frac{1}{\Sigma_1}, \\ \tau_{\mathbf{T}} = \frac{\Sigma_1^2}{\kappa^3}\left(-\kappa'\tau + \tau'\kappa\right), \end{cases} \tag{17}$$

where $\Sigma_1 = \frac{\kappa}{\sqrt{|\tau^2-\kappa^2|}}$ and $\Sigma_2 = \frac{\tau}{\sqrt{|\tau^2-\kappa^2|}}$.

Theorem 4.1 *Consider $\Gamma_{\mathbf{T}}(\phi_{\mathbf{T}}(s)) = \mathbf{T}(s)$ be the tangent pseudo spherical image of the timelike curve Γ. Let $\{\mathbf{T}_{\mathbf{T}}, \mathbf{n}_{\mathbf{T}}, \mathbf{p}_{\mathbf{T}}\}$ be its moving Frenet frame, then the time evolution equations for $\Gamma_{\mathbf{T}}$ are determined by*

$$\begin{cases} \frac{\partial \mathbf{T}_{\mathbf{T}}}{\partial t} = \zeta_3 \mathbf{T} - \zeta_1 \mathbf{p}, \\ \frac{\partial \mathbf{n}_{\mathbf{T}}}{\partial t} = (\Sigma_1 + \zeta_2\,\Sigma_2)\,\mathbf{T} + (\zeta_3\,\Sigma_1 + \zeta_1\,\Sigma_2)\,\mathbf{n} \\ \qquad + (\zeta_2\,\Sigma_1 + \Sigma_2)\,\mathbf{p}, \\ \frac{\partial \mathbf{p}_{\mathbf{T}}}{\partial t} = -(\zeta_2\,\Sigma_1 + \Sigma_2)\,\mathbf{T} - (\zeta_1\,\Sigma_1 + \zeta_3\,\Sigma_2)\,\mathbf{n} \\ \qquad - (\Sigma_1 + \zeta_2\,\Sigma_2)\,\mathbf{p}. \end{cases} \tag{18}$$

and the evolution equations of its curvatures are:

$$\begin{cases} \frac{\partial \kappa_{\mathbf{T}}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\Sigma_1}\right), \\ \frac{\partial \tau_{\mathbf{T}}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\Sigma_1^2}{\kappa^3}\left(-\kappa'\tau + \tau'\kappa\right)\right), \end{cases} \tag{19}$$

where ζ_1, ζ_2 and ζ_3 are the velocities.

Proof 2 *From the above considerations, Eq. (16) can be reformulated to take the form*

$$\Omega_{\mathbf{T}} = \mathbf{Q}_{\mathbf{T}}\,\Omega, \tag{20}$$

where

$$\Omega_{\mathbf{T}} = \begin{pmatrix} \mathbf{T}_{\mathbf{T}} \\ \mathbf{n}_{\mathbf{T}} \\ \mathbf{p}_{\mathbf{T}} \end{pmatrix}, \quad \mathbf{Q}_{\mathbf{T}} = \begin{pmatrix} 0 & 1 & 0 \\ \Sigma_1 & 0 & \Sigma_2 \\ -\Sigma_2 & 0 & -\Sigma_1 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}. \tag{21}$$

In order to obtain the evolution equations (18), it is important to take the derivative of Eq. (20) with respect to t and using Eq. (21) to get

$$\frac{\partial \Omega_{\mathbf{T}}}{\partial t} = \frac{\partial \mathbf{Q}_{\mathbf{T}}}{\partial t}\Omega + \mathbf{Q}_{\mathbf{T}}\frac{\partial \Omega}{\partial t}. \tag{22}$$

Equating the coefficients of Eq. (22) after substituting (21), we get the required result.

In addition, Eq. (19) comes from the directly derivation of Eq. (17). It completes the proof.

4.2 Binormal Pseudo Spherical Image

In a similar way, one can consider the binormal pseudo spherical image of Γ as follows:

Let $\Gamma = \Gamma(s)$ be a timelike curve with pseudo arc length parameter s in E_1^3. Let $\{\mathbf{T}, \mathbf{n}, \mathbf{p}\}$ be its Frenet frame. Then, the binormal pseudo spherical image of Γ is described as the locus on the unit pseudo sphere traced by the binormal $\mathbf{p}(s)$ of Γ. Therefore, we can write

$$\Gamma_{\mathbf{p}}(\phi_{\mathbf{p}}(s)) = \mathbf{p}(s), \tag{23}$$

where $\phi_{\mathbf{p}} : I \longrightarrow I_{\mathbf{p}}$, $\phi_{\mathbf{p}}(s) = \int \tau(s)ds$ is a regular C^{∞} function. The Frenet matrix which describes the motion of the binormal pseudo spherical curve is

$$\begin{cases} \mathbf{T_p} = -\mathbf{n}, \\ \mathbf{n_p} = -\Sigma_1 \mathbf{T} - \Sigma_2 \mathbf{p}, \\ \mathbf{p_p} = -\Sigma_2 \mathbf{T} - \Sigma_1 \mathbf{p}, \end{cases} \tag{24}$$

whereas its curvatures are read

$$\begin{cases} \kappa_{\mathbf{p}} = \frac{1}{\Sigma_2^2}, \\ \tau_{\mathbf{p}} = \frac{\Sigma_2^2}{\tau^3}\left(\kappa'\tau - \tau'\kappa\right). \end{cases} \tag{25}$$

Theorem 4.2 *Suppose that $\Gamma_{\mathbf{p}}(\phi_{\mathbf{p}}(s)) = \mathbf{p}(s)$ is the binormal pseudo spherical image of the timelike curve Γ. Let $\{\mathbf{T_p}, \mathbf{n_p}, \mathbf{p_p}\}$ be the moving Frenet frame, then its time evolution equations are*

$$\begin{cases} \frac{\partial \mathbf{T_p}}{\partial t} = -\zeta_3 \mathbf{T} + \zeta_1 \mathbf{p}, \\ \frac{\partial \mathbf{n_p}}{\partial t} = -\left(\Sigma_1 + \zeta_2\, \Sigma_2\right) \mathbf{T} - \left(\zeta_3\, \Sigma_1 + \zeta_1\, \Sigma_2\right) \mathbf{n} \\ \qquad - \left(\zeta_2\, \Sigma_1 + \Sigma_2\right) \mathbf{p}, \\ \frac{\partial \mathbf{p_p}}{\partial t} = \left(-\zeta_2\, \Sigma_1 - \Sigma_2\right) \mathbf{T} + \left(-\zeta_3\, \Sigma_1 - \zeta_1\, \Sigma_2\right) \mathbf{n} \\ \qquad - \left(\Sigma_1 + \zeta_2\, \Sigma_2\right) \mathbf{p}, \end{cases} \tag{26}$$

and the evolution equations of its curvatures are

$$\begin{cases} \frac{\partial \kappa_{\mathbf{p}}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\Sigma_2}\right), \\ \frac{\partial \tau_{\mathbf{p}}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\Sigma_2^2}{\tau^3}\left(\kappa'\tau - \tau'\kappa\right)\right), \end{cases} \tag{27}$$

where Σ_1 and Σ_2 are defined as before and ζ_1, ζ_2 and ζ_3 are the velocities.

Proof 3 *Using Eq. (24) which can be replaced by*

$$\frac{\partial \Omega_{\mathbf{p}}}{\partial t} = \mathbf{Q_p}\, \Omega, \tag{28}$$

where

$$\Omega_{\mathbf{p}} = \begin{pmatrix} \mathbf{T_p} \\ \mathbf{n_p} \\ \mathbf{p_p} \end{pmatrix}, \mathbf{Q_p} = \begin{pmatrix} 0 & -1 & 0 \\ -\Sigma_1 & 0 & -\Sigma_2 \\ -\Sigma_2 & 0 & -\Sigma_1 \end{pmatrix} \text{ and } \Omega = \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}.$$

The evolution equations (26) can be obtained through the derivation of Eq. (28) with respect to t and equating both sides of the following resulting equation

$$\frac{\partial \Omega_{\mathbf{p}}}{\partial t} = \frac{\partial \mathbf{Q_p}}{\partial t}\Omega + \mathbf{Q_p}\, \frac{\partial \Omega}{\partial t},$$

taking into account $\Omega_{\mathbf{p}}$ and $\mathbf{Q_p}$.

On the other hand, the evolution equations (26) can be obtained.

4.3 Normal Pseudo Spherical Image

As we have done in the case of tangent and binormal pseudo spherical images of Γ, the normal pseudo spherical image of Γ can be defined as follows:

Consider $\Gamma = \Gamma(s)$ be a timelike curve with pseudo arc length parameter s in E_1^3. Let $\{\mathbf{T}, \mathbf{n}, \mathbf{p}\}$ be Frenet frame of Γ, then the normal pseudo spherical image of Γ is described as the locus on the unit pseudo sphere traced by the normal $\mathbf{n}(s)$ of the curve. Therefore, it can be written as

$$\Gamma_{\mathbf{n}}(\phi_{\mathbf{n}}(s)) = \mathbf{n}(s), \tag{29}$$

where $\phi_{\mathbf{n}} : I \longrightarrow I_{\mathbf{n}}$, $\phi_{\mathbf{n}}(s) = \int \sqrt{|\tau^2(s) - \kappa^2(s)|}ds$ is a regular C^{∞} function. Frenet equations which describe the motion of $\Gamma_{\mathbf{n}}$ are read as

$$\begin{cases} \mathbf{T_n} = \Sigma_1 \mathbf{T} + \Sigma_2 \mathbf{p}, \\ \mathbf{n_n} = \Sigma_3 \mathbf{T} + \Sigma_4 \mathbf{n} + \Sigma_5 \mathbf{p}, \\ \mathbf{p_n} = \Sigma_6 \mathbf{T} + \Sigma_7 \mathbf{n} + \Sigma_8 \mathbf{p}, \end{cases} \tag{30}$$

where

$$\begin{cases} \Sigma_1 = \frac{\kappa}{\sqrt{|\tau^2 - \kappa^2|}}, \quad \Sigma_2 = \frac{\tau}{\sqrt{|\tau^2 - \kappa^2|}}, \\ \Sigma_3 = -\frac{\tau(-\tau\,\kappa' + \tau'\,\kappa)}{\left(\tau^2 - \kappa^2\right)^{\frac{1}{2}}\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{\frac{1}{2}}}, \\ \Sigma_4 = -\frac{1}{\left(\tau^2 - \kappa^2\right)^{-\frac{3}{2}}\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{\frac{1}{2}}}, \\ \Sigma_5 = -\frac{\kappa(-\tau\,\kappa' + \tau'\,\kappa)}{\left(\tau^2 - \kappa^2\right)^{\frac{1}{2}}\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{\frac{1}{2}}}, \\ \Sigma_6 = \frac{\tau(|-\kappa^2 + \tau^2|)}{\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{\frac{1}{2}}} \\ \Sigma_7 = \frac{\tau\,\kappa' - \tau'\,\kappa}{\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{\frac{1}{2}}}, \\ \Sigma_8 = \frac{\kappa(|-\kappa^2 + \tau^2|)}{\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{\frac{1}{2}}}, \end{cases} \tag{31}$$

and the curvatures of $\Gamma_{\mathbf{n}}$ are determined by

$$\begin{cases} \kappa_{\mathbf{n}} = \frac{1}{\left(\tau^2 - \kappa^2\right)^{\frac{3}{2}}\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{-\frac{1}{2}}} \\ \tau_{\mathbf{n}} = -\left(\frac{(-\kappa'\tau + \tau'\kappa)(4\kappa^2 - \tau^2)(\kappa\kappa' - \tau\tau')}{-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'}\right) + \xi_{\mathbf{n}}, \end{cases} \tag{32}$$

where

$$\xi_{\mathbf{n}} = \left(\frac{\begin{array}{c}(-\kappa^2(3\tau^4 - \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 + \kappa'^2) + 2\kappa\kappa'\tau\tau')\times \\ (\tau^2(-3\kappa'\tau' + \tau\kappa'') - \kappa^2(3\kappa'\tau' + \tau\tau'' + \kappa^3\tau'' + 3\kappa\tau(\kappa'^2 + \\ \tau'^2) - \tau\tau'))\end{array}}{(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau')^2}\right)$$

Theorem 4.3 *Let $\Gamma_{\mathbf{n}}(\phi_{\mathbf{n}}(s)) = \mathbf{n}(s)$ be the normal pseudo spherical image of the timelike curve Γ, and $\{\mathbf{T_n}, \mathbf{n_n}, \mathbf{p_n}\}$ be its moving Frenet frame, then the time evolution equations for $\Gamma_{\mathbf{n}}$ can be formulated as*

$$\begin{cases} \frac{\partial \mathbf{T_n}}{\partial t} = \left(\Sigma_1 + \zeta_2 \Sigma_2\right) \mathbf{T} + \left(\zeta_3 \Sigma_1 + \zeta_1 \Sigma_2\right) \mathbf{n} + \left(\zeta_2 \Sigma_1 + \Sigma_2\right) \mathbf{p}, \\ \frac{\partial \mathbf{n_n}}{\partial t} = \left(\Sigma_3 + \zeta_3\, \Sigma_4 + \zeta_2 \Sigma_5\right) \mathbf{T} + \left(\zeta_3\, \Sigma_3 + \Sigma_4 + \zeta_1\, \Sigma_5\right) \mathbf{n} + \left(\zeta_2\, \Sigma_3 - \zeta_1 \Sigma_4 + \Sigma_5\right) \mathbf{p}, \\ \frac{\partial \mathbf{p_n}}{\partial t} = \left(\Sigma_6 + \zeta_3\, \Sigma_7 + \zeta_2 \Sigma_8\right) \mathbf{T} + \left(\zeta_3\, \Sigma_6 + \Sigma_7 + \zeta_1\, \Sigma_8\right) \mathbf{n} + \left(\zeta_2\, \Sigma_6 - \zeta_1 \Sigma_7 + \Sigma_8\right) \mathbf{p}, \end{cases} \tag{33}$$

and the evolution equations of its curvatures are

$$\begin{cases} \frac{\partial \kappa_{\mathbf{n}}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{\left(\tau^2 - \kappa^2\right)^{\frac{1}{2}} \times \left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^{-\frac{1}{2}}}\right), \\ \frac{\partial \tau_{\mathbf{n}}}{\partial t} = \frac{\partial}{\partial t}\left(-\frac{(-\kappa'\tau + \tau'\kappa)(4\kappa^2 - \tau^2)(\kappa\kappa' - \tau\tau')}{-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'} + \right. \\ \quad \left. \frac{\left(-\kappa^2(3\tau^4 - \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 + \kappa'^2) + 2\kappa\kappa'\tau\tau'\right) \times \left(\tau^2(-3\kappa'\tau' + \tau\kappa'') - \kappa^2(3\kappa'\tau' + \tau\tau'' + \kappa^3\tau'' + 3\kappa\tau(\kappa'^2 + \tau'^2) - \tau\tau')\right)}{\left(-\kappa^2(3\tau^4 + \tau'^2 + \kappa^4 - 3\kappa^2\,\tau^2) + \tau^2(\tau^4 - \kappa'^2) + 2\kappa\kappa'\tau\tau'\right)^2}\right). \end{cases} \tag{34}$$

Proof 4 *The same technique as we have done in the case of tangent and binormal pseudo spherical images is used for proving the evolution equations (33) of* $\Gamma_{\mathbf{n}}$. *After equating the coefficients which are introduced from the resulting equations*

$$\frac{\partial \Omega_{\mathbf{n}}}{\partial t} = \frac{\partial \mathbf{Q}_{\mathbf{n}}}{\partial t}\Omega + \mathbf{Q}_{\mathbf{n}}\frac{\partial \Omega}{\partial t},$$

where

$$\Omega_{\mathbf{n}} = \mathbf{Q}_{\mathbf{n}}\,\Omega, \tag{35}$$

$$\Omega_{\mathbf{n}} = \begin{pmatrix} \mathbf{T}_{\mathbf{n}} \\ \mathbf{n}_{\mathbf{n}} \\ \mathbf{p}_{\mathbf{n}} \end{pmatrix}, \quad \mathbf{Q}_{\mathbf{n}} = \begin{pmatrix} \Sigma_1 & 0 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 & \Sigma_5 \\ \Sigma_6 & \Sigma_7 & \Sigma_8 \end{pmatrix} \ \textit{and}\ \Omega = \begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}, \tag{36}$$

Eq. (33) is achieved.

Now, we can say that Eq. (34) can be obtained after some complicated computations of Eq. (32), which completes the proof.

4.4 Applications

In this section, we consider some computational examples as an application of how to get the tangent, binormal and normal pseudo spherical images of a given timelike curve in E_1^3. Then, we calculate their evolution equations for their frames as well as their curvatures.

Example 4.1 *Consider* $\Gamma(s)$ *be a timelike curve given by (see Fig. 1)*

$$\Gamma(s) = \left(\sinh\sqrt{2}s, \cosh\sqrt{2}s, s\right),$$

the tangent $\mathbf{T}$, the normal $\mathbf{n}$ and the binormal $\mathbf{p}$ of Γ are respectively, computed as follows:

$$\begin{cases} \mathbf{T} = \left(\sqrt{2}\cosh\sqrt{2}s, \sqrt{2}\sinh\sqrt{2}s, 1\right), \\ \mathbf{n} = \left(\sinh\sqrt{2}s, \cosh\sqrt{2}s, 0\right), \\ \mathbf{p} = \left(\cosh\sqrt{2}s, \sinh\sqrt{2}s, \sqrt{2}\right), \end{cases}$$

and the curvatures are calculated as:

$$\begin{cases} \kappa = 2, \\ \tau = -\sqrt{2}. \end{cases}$$

For the curve Γ, the pseudo spherical image of the tangent is given by

$$\Gamma_{\mathbf{T}}(\phi_{\mathbf{T}}(s) = \left(\sqrt{2}\cosh\sqrt{2}s, \sqrt{2}\sinh\sqrt{2}s, 1\right),$$

the Frenet apparatus of $\Gamma_{\mathbf{T}}(\phi_{\mathbf{T}}(s)$ is obtained as follows

$$\begin{cases} \mathbf{T}_{\mathbf{T}} = \mathbf{n} = \left(\sinh\sqrt{2}s, \cosh\sqrt{2}s, 0\right), \\ \mathbf{n}_{\mathbf{T}} = \sqrt{2}\mathbf{T} - \mathbf{p} = \left(\cosh\sqrt{2}s, \sinh\sqrt{2}s, 0\right), \\ \mathbf{p}_{\mathbf{T}} = \mathbf{T} - \sqrt{2}\mathbf{p} = (0, 0, -1), \\ \kappa_{\mathbf{T}} = \frac{1}{\sqrt{2}}, \\ \tau_{\mathbf{T}} = 0. \end{cases}$$

According to *Theorem 4.1*, the evolution equations of $\Gamma_{\mathbf{T}}(\phi_{\mathbf{T}}(s)$ can be introduced as

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{T}_{\mathbf{T}} \\ \mathbf{n}_{\mathbf{T}} \\ \mathbf{p}_{\mathbf{T}} \end{pmatrix} = \begin{pmatrix} \zeta_3 & 0 & -\zeta_1 \\ -\zeta_2 + \sqrt{2} & \sqrt{2}\zeta_3 - \zeta_1 & \sqrt{2}\zeta_2 - 1 \\ -\sqrt{2}\zeta_2 - 1 & -\zeta_3 - \sqrt{2}\zeta_1 & -\zeta_2 - \sqrt{2} \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}. \tag{37}$$

Also, the pseudo spherical image of the binormal is given by

$$\Gamma_{\mathbf{p}}(\phi_{\mathbf{p}}(s) = \left(\cosh\sqrt{2}s, \sinh\sqrt{2}s, \sqrt{2}\right),$$

the Frenet apparatus of $\Gamma_{\mathbf{p}}(\phi_{\mathbf{p}}(s)$ is obtained as

$$\begin{cases} \mathbf{T}_{\mathbf{p}} = -\mathbf{n} = \left(-\sinh\sqrt{2}s, -\cosh\sqrt{2}s, 0\right), \\ \mathbf{n}_{\mathbf{p}} = \sqrt{2}\mathbf{T} - \mathbf{p} = \left(\cosh\sqrt{2}s, \sinh\sqrt{2}s, 0\right), \\ \mathbf{p}_{\mathbf{p}} = \mathbf{T} - \sqrt{2}\mathbf{p} = (0, 0, -1), \\ \kappa_{\mathbf{p}} = -1, \\ \tau_{\mathbf{p}} = 0. \end{cases}$$

From Eq. (26), the evolution equations of $\Gamma_{\mathbf{p}}(\phi_{\mathbf{p}}(s)$ are given by

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{T}_{\mathbf{p}} \\ \mathbf{n}_{\mathbf{p}} \\ \mathbf{p}_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} -\zeta_3 & 0 & \zeta_1 \\ \zeta_2 - \sqrt{2} & -\sqrt{2}\zeta_3 + \zeta_1 & -\sqrt{2}\zeta_2 + 1 \\ -\sqrt{2}\zeta_2 + 1 & \zeta_3 - \sqrt{2}\zeta_1 & \zeta_2 - \sqrt{2} \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}. \tag{38}$$

Finally, the pseudo spherical image of the normal is

$$\Gamma_{\mathbf{n}}(\phi_{\mathbf{n}}(s) = \left(\sinh\sqrt{2}s, \cosh\sqrt{2}s, 0\right),$$

with Frenet apparatus

$$\begin{cases} \mathbf{T}_{\mathbf{n}} = \sqrt{2}\mathbf{T} - \mathbf{p} = \left(\cosh\sqrt{2}s, \sinh\sqrt{2}s, 0\right), \\ \mathbf{n}_{\mathbf{n}} = \mathbf{n} = \left(\sinh\sqrt{2}s, \cosh\sqrt{2}s, 0\right), \\ \mathbf{p}_{\mathbf{n}} = \mathbf{T} - \sqrt{2}\mathbf{p} = (0, 0, -1), \\ \kappa_{\mathbf{n}} = 1, \\ \tau_{\mathbf{n}} = 0. \end{cases}$$

From aforementioned data, the evolution equations of $\Gamma_{\mathbf{n}}(\phi_{\mathbf{n}}(s)$ are read

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{T}_{\mathbf{n}} \\ \mathbf{n}_{\mathbf{n}} \\ \mathbf{p}_{\mathbf{n}} \end{pmatrix} = \begin{pmatrix} -\zeta_2 + \sqrt{2} & \sqrt{2}\zeta_3 - \zeta_1 & \sqrt{2}\zeta_2 - 1 \\ \zeta_3 & 1 & -\zeta_1 \\ 1 - \sqrt{2}\zeta_2 & \zeta_3 - \sqrt{2}\zeta_1 & \zeta_2 - \sqrt{2} \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}. \tag{39}$$

Hereafter, one can see the pseudo spherical images and their evolutions through Figs. (2, 3), Figs. (4, 5) and Figs. (6, 7).

Example 4.2 *Assume the timelike curve is given by (see Fig. 7)*

$$\Upsilon(s) = \left(\sqrt{3}s, \sqrt{2}\cos s, \sqrt{2}\sin s\right).$$

Thus, the tangent $\mathbf{T}$, the normal $\mathbf{n}$ and the binormal $\mathbf{p}$ of Υ are respectively, computed as follows:

$$\begin{cases} \mathbf{T} = \left(\sqrt{3}, -\sqrt{2}\sin s, \sqrt{2}\cos s\right), \\ \mathbf{n} = (0, -\cos s, -\sin s), \\ \mathbf{p} = \left(-\sqrt{2}, \sqrt{3}\sin s, -\sqrt{3}\cos s\right), \end{cases}$$

and its curvatures are given by

$$\begin{cases} \kappa = \sqrt{2}, \\ \tau = \sqrt{3}. \end{cases}$$

For the curve Υ, the pseudo spherical image of the tangent is obtained by

$$\Upsilon_{\mathbf{T}}(\phi_{\mathbf{T}}(s) = \left(\sqrt{3}, -\sqrt{2}\sin s, \sqrt{2}\cos s\right),$$

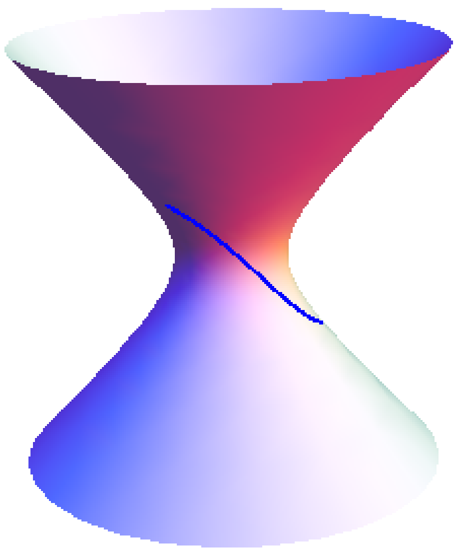

Figure 1. The timelike curve $\Gamma = \Gamma(s)$ on the unit pseudo sphere S_1^2.

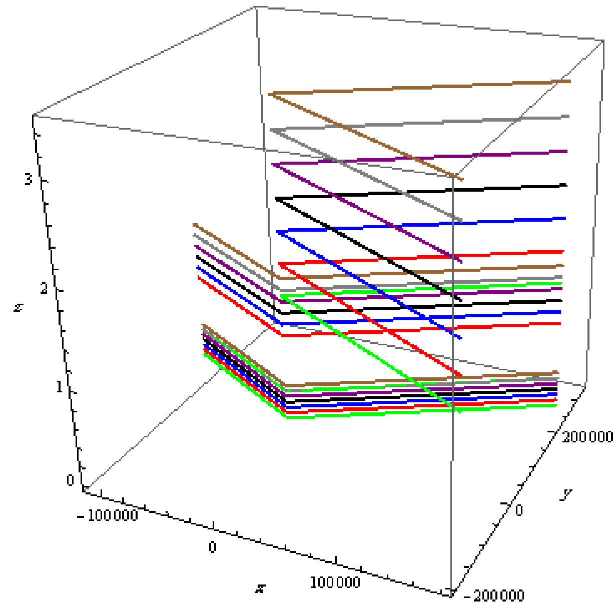

Figure 3. The time evolution of $\Gamma_{\mathbf{T}}$.

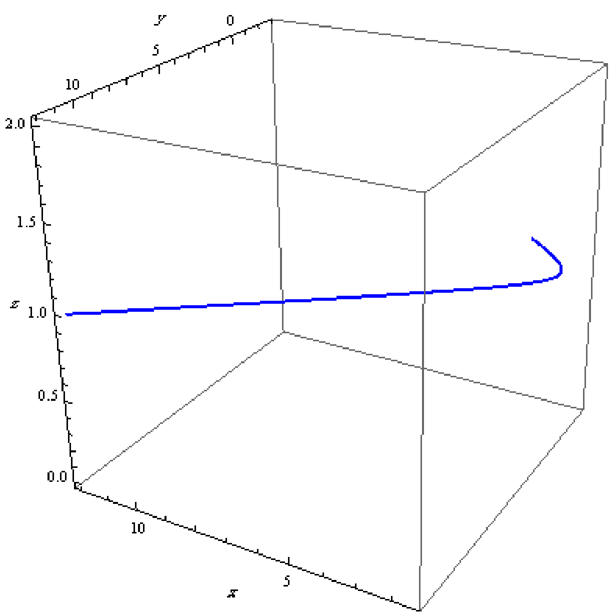

Figure 2. The pseudo spherical image of $\Gamma_{\mathbf{T}}$.

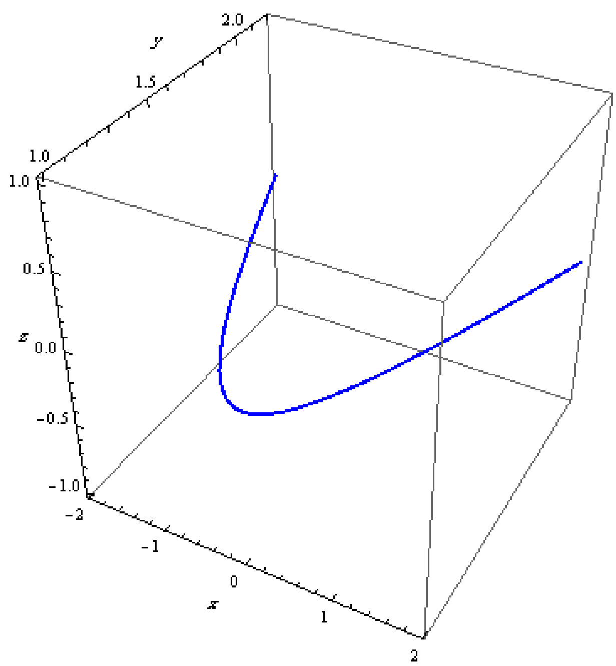

Figure 4. The pseudo spherical image of $\Gamma_{\mathbf{P}}$.

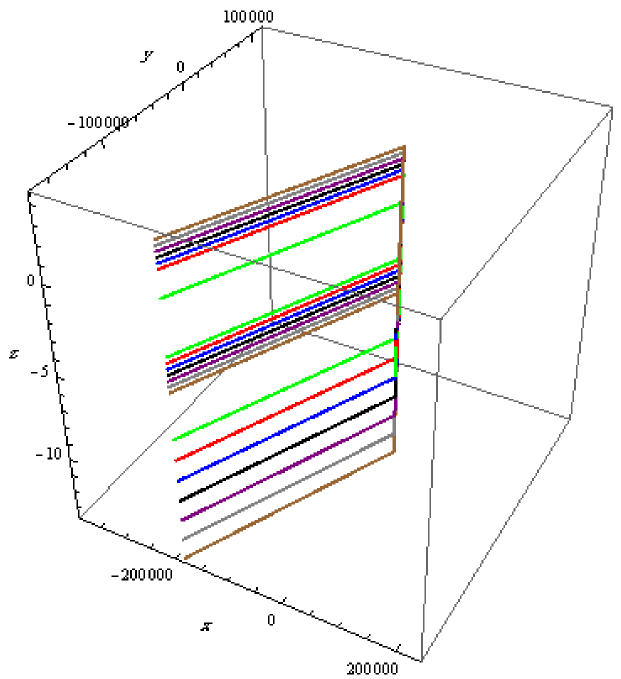

Figure 5. The time evolution of $\Gamma_{\mathbf{p}}$.

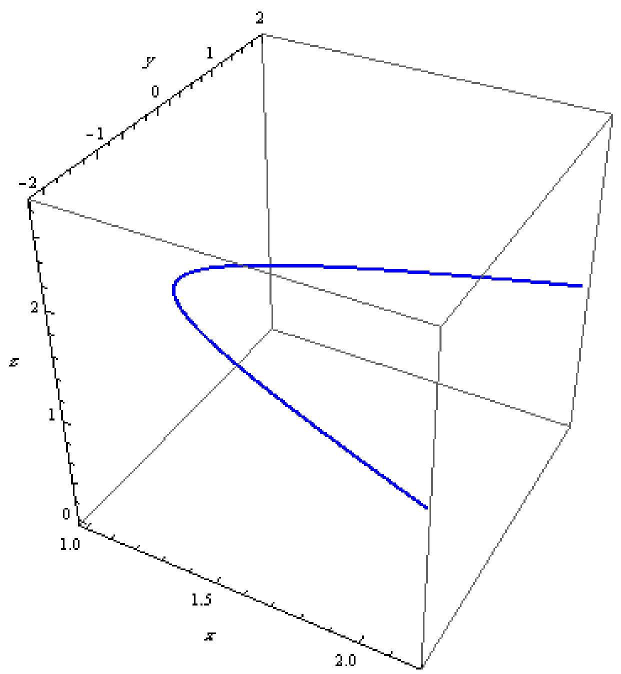

Figure 6. The pseudo spherical image of $\Gamma_{\mathbf{n}}$.

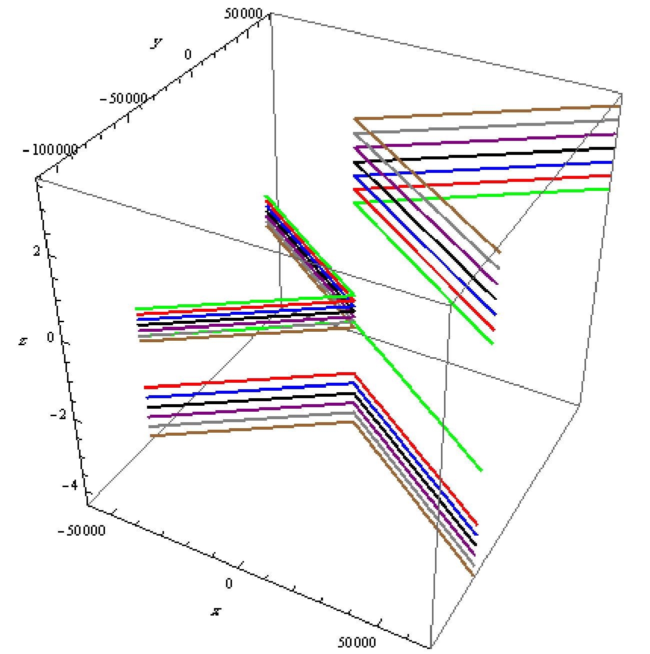

Figure 7. The time evolution of $\Gamma_{\mathbf{n}}$.

however, its Frenet vectors and curvatures respectively, are

$$\begin{cases} \mathbf{T_T} = \mathbf{n} = (0, -\cos s, -\sin s), \\ \mathbf{n_T} = \sqrt{2}\mathbf{T} + \sqrt{3}\mathbf{p} = (0, \sin s, -\cos s), \\ \mathbf{p_T} = -\sqrt{3}\mathbf{T} - \sqrt{2}\mathbf{p} = (-1, 0, 0), \\ \kappa_{\mathbf{T}} = \frac{1}{\sqrt{2}}, \\ \tau_{\mathbf{T}} = 0. \end{cases}$$

The evolution equations of $\Upsilon_{\mathbf{T}}(\phi_{\mathbf{T}}(s)$ are given as follows

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{T_T} \\ \mathbf{n_T} \\ \mathbf{p_T} \end{pmatrix} = \begin{pmatrix} \zeta_3 & 0 & -\zeta_1 \\ \sqrt{3}\zeta_2 + \sqrt{2} & \sqrt{2}\,\zeta_3 + \sqrt{3}\zeta_1 & \sqrt{2}\,\zeta_2 + \sqrt{3} \\ -\sqrt{2}\,\zeta_2 - \sqrt{3} & -\sqrt{3}\,\zeta_3 - \sqrt{2}\zeta_1 & -\sqrt{3}\zeta_2 - \sqrt{2} \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}. \tag{40}$$

Similarly as in the case of the pseudo spherical image of the tangent, the pseudo spherical images of the binormal and normal to the curve Υ can be served respectively, as follows

$$\Upsilon_{\mathbf{p}}(\phi_{\mathbf{p}}(s) = \left(-\sqrt{2}, \sqrt{3}\sin s, -\sqrt{3}\cos s\right),$$

$$\begin{cases} \mathbf{T_p} = -\mathbf{n} = (0, \cos s, \sin s), \\ \mathbf{n_p} = -\sqrt{2}\mathbf{T} - \sqrt{3}\mathbf{p} = (0, -\sin s, \cos s), \\ \mathbf{p_p} = -\sqrt{3}\mathbf{T} - \sqrt{2}\mathbf{p} = (-1, 0, 0), \\ \kappa_{\mathbf{p}} = \frac{1}{\sqrt{3}}, \\ \tau_{\mathbf{p}} = 0, \end{cases}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{T_p} \\ \mathbf{n_p} \\ \mathbf{p_p} \end{pmatrix} = \begin{pmatrix} -\zeta_3 & 0 & \zeta_1 \\ \sqrt{3}\zeta_2 - \sqrt{2} & -\sqrt{2}\,\zeta_3 - \sqrt{3}\zeta_1 & -\sqrt{2}\,\zeta_2 - \sqrt{3} \\ -\sqrt{2}\,\zeta_2 - \sqrt{3} & -\sqrt{2}\,\zeta_1 - \sqrt{3}\zeta_3 & -\sqrt{3}\zeta_2 - \sqrt{2} \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}, \tag{41}$$

and

$$\Upsilon_{\mathbf{n}}(\phi_{\mathbf{n}}(s) = (0, -\cos s, -\sin s),$$

$$\begin{cases} \mathbf{T_n} = \sqrt{2}\mathbf{T} + \sqrt{3}\mathbf{p} = (0, \sin s, -\cos s), \\ \mathbf{n_n} = -\mathbf{n} = (0, -\cos s, -\sin s), \\ \mathbf{p_n} = -\sqrt{3}\mathbf{T} - \sqrt{2}\mathbf{p} = (1, 0, 0), \\ \kappa_{\mathbf{n}} = 1, \\ \tau_{\mathbf{n}} = 0, \end{cases}$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \mathbf{T_n} \\ \mathbf{n_n} \\ \mathbf{p_n} \end{pmatrix} = \begin{pmatrix} \sqrt{3}\zeta_2+\sqrt{2} & \sqrt{2}\,\zeta_3+\sqrt{3}\zeta_1 & \sqrt{2}\,\zeta_2+\sqrt{3} \\ -\zeta_2 & -\zeta_1 & -1 \\ -\sqrt{3}-\sqrt{2}\zeta_2 & -\sqrt{3}\,\zeta_3-\sqrt{2}\zeta_1 & \sqrt{3}\zeta_2-\sqrt{2} \end{pmatrix}\begin{pmatrix} \mathbf{T} \\ \mathbf{n} \\ \mathbf{p} \end{pmatrix}. \tag{42}$$

The pseudo spherical images and their evolutions of Υ are clearly shown through Figs. (9, 10), Figs. (11, 12) and Figs. (13, 14).

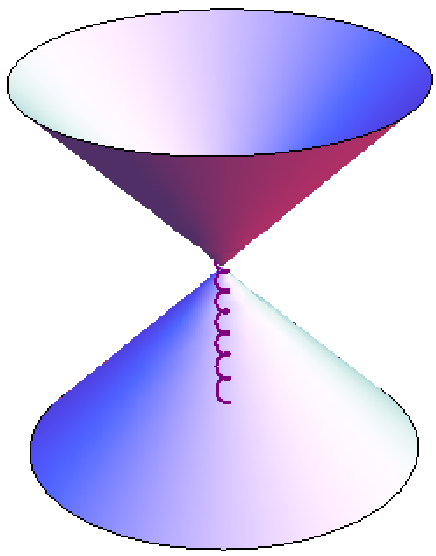

Figure 8. The timelike curve $\Upsilon = \Upsilon(s)$ on the unit pseudo sphere S_1^2.

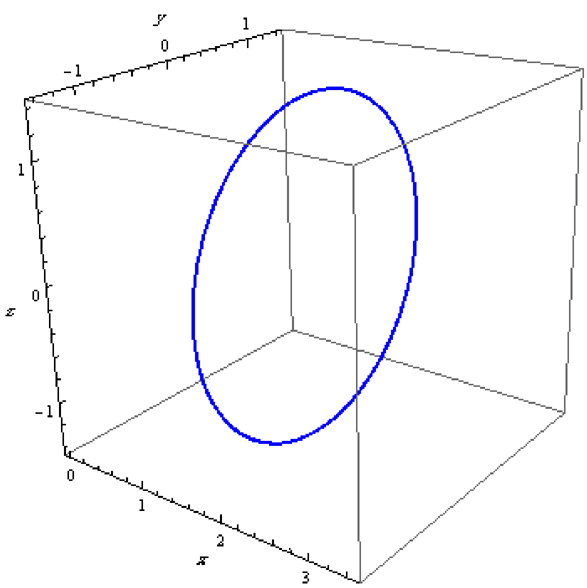

Figure 9. The pseudo spherical image of $\Upsilon_{\mathbf{T}}$.

Remark 4.1 *The time evolutions of pseudo-spherical images of the curves Γ and Υ that are shown in the above figures depend on the choice of the velocities $(\zeta_1, \zeta_2, \zeta_3)$.*

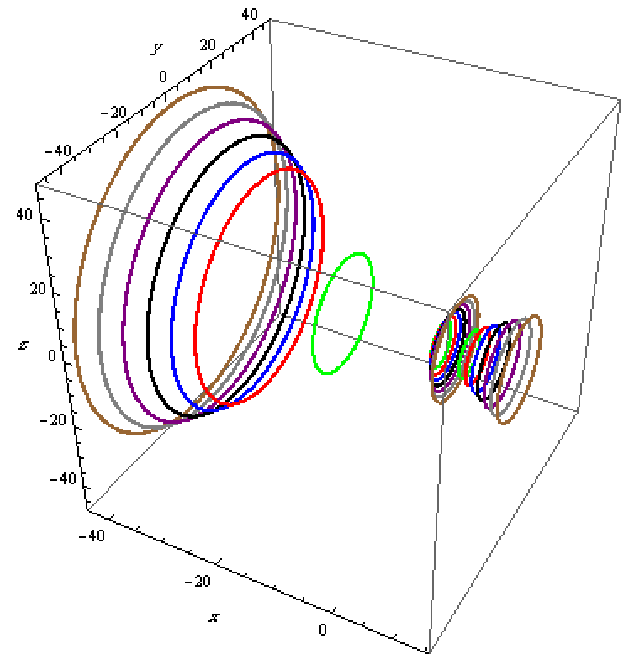

Figure 10. The time evolution of $\Upsilon_{\mathbf{T}}$.

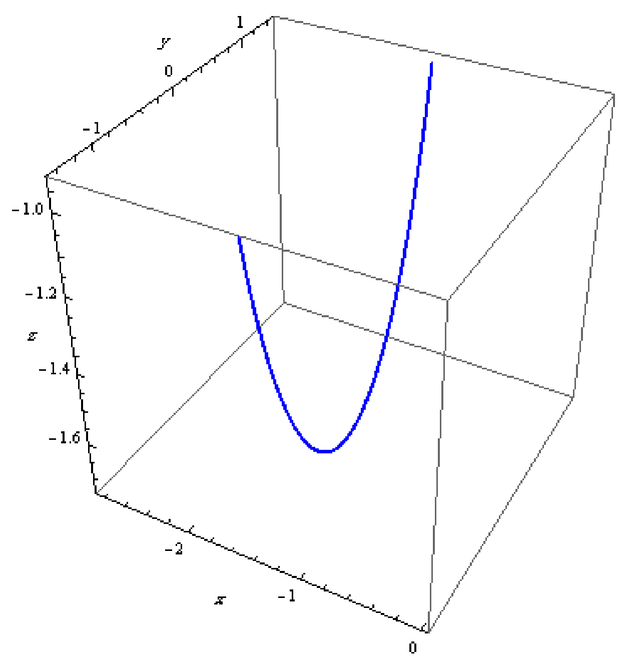

Figure 11. The pseudo spherical image of $\Upsilon_{\mathbf{p}}$.

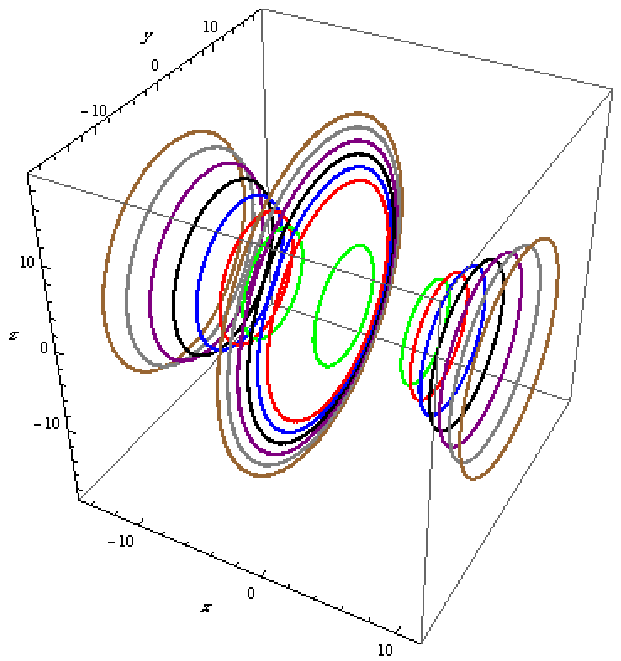

Figure 12. The time evolution of $\Upsilon_{\mathbf{p}}$.

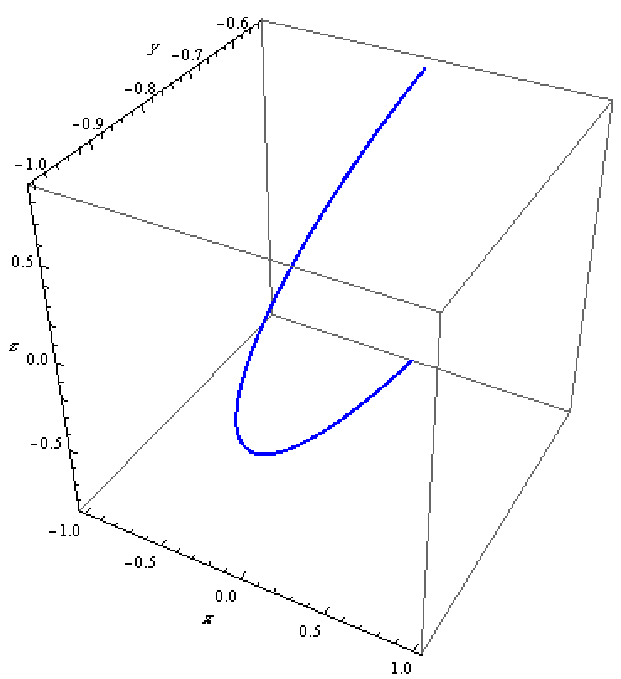

Figure 13. The pseudo spherical image of $\Upsilon_{\mathbf{n}}$.

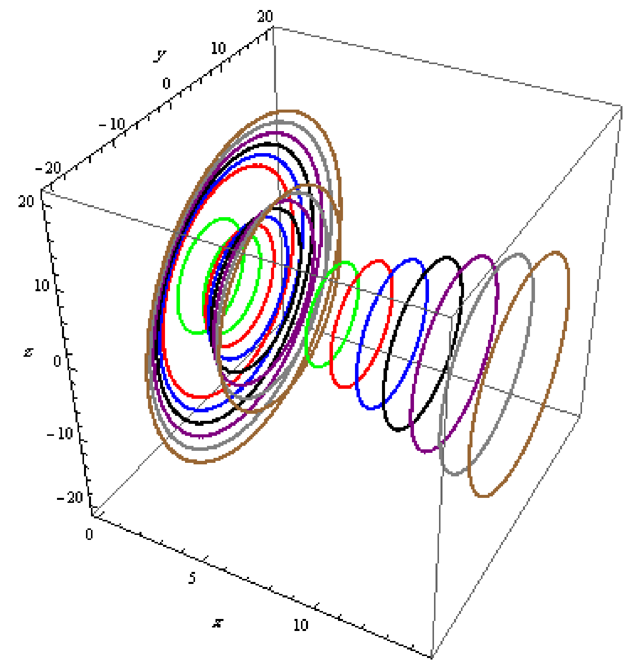

Figure 14. The time evolution of $\Upsilon_{\mathbf{n}}$.

5 Conclusion

In the three dimensional Minkowski space, we have introduced the pseudo spherical images for tangent, binormal and normal vectors of a timelike curve. Furthermore, the evolution equations for them via their frame fields using a new approach have been derived. The technique that has been used for the derivation of these equations is based on the integration of Frenet frame numerically by operating the Mathematica package. Additionally, some computational examples to illustrate our main results are given and plotted.

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