Mathematics and Statistics 10(4): 875-883, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100419

2-Odd Labeling of Graphs Using Certain Number Theoretic Concepts and Graph Operations

Ajaz Ahmad Pir[1], **Tabasum Mushtaq**[2], **A. Parthiban**[1,*]

[1]Department of Mathematics, Lovely Professional University, Phagwara 144 411, Punjab, India

[2]Department of Statistics, Govt. Degree College, Sopore 193 201, Jammu and Kashmir, India

Received May 5, 2022; Revised June 8, 2022; Accepted July 21, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] Ajaz Ahmad Pir, Tabasum Mushtaq, A. Parthiban, "2-Odd Labeling of Graphs Using Certain Number Theoretic Concepts and Graph Operations," Mathematics and Statistics, Vol.10, No.4, pp. 875-883, 2022. DOI: 10.13189/ms.2022.100419*

***(b)**: Ajaz Ahmad Pir, Tabasum Mushtaq, A. Parthiban, (2022). 2-Odd Labeling of Graphs Using Certain Number Theoretic Concepts and Graph Operations. Mathematics and Statistics, 10(4), 875-883. DOI: 10.13189/ms.2022.100419*

Abstract Graph theory plays a significant role in a variety of real-world systems. Graph concepts such as labeling and coloring are used to depict a variety of processes and relationships in material, social, biological, physical, and information systems. Specifically, graph labeling is used in communication network addressing, fault-tolerant system design, automatic channel allocation, etc. 2-odd labeling assigns distinct integers to the nodes of $G(V, E)$ in such a manner, that the positive difference of adjacent nodes is either 2 or an odd integer, $2k \pm 1, k \in N$. So, G is a 2-odd graph if and only if it permits 2-odd labeling. Studying certain important modifications through various graph operations on a given graph is interesting and challenging. These operations mainly modify the underlying graph's structure, so understanding the complex operations that can be done over a graph or a set of graphs is inevitable. The motivation behind the development of this article is to apply the concept of 2-odd labeling on graphs generated by using various graph operations. Further, certain results on 2-odd labeling are also derived using some well-known number theoretic concepts such as the Twin prime conjecture and Goldbach's conjecture, besides recalling a few interesting applications of graph labeling and graph coloring.

Keywords 2-odd Labeling, Mobius Ladder, Union, Intersection, Duplication, Extension, Twin Prime Conjecture, Goldbach's Conjecture

1 Introduction

Let $G(V, E)$ be a graph with $V(G)$ as the set of nodes and $E(G)$ as the set of lines. Labeling is the process of assigning integers to either $V(G)$ or $E(G)$ of G, or both, depending on certain conditions. In this paper, we look at graphs that are simple, finite, undirected, and linked. For graph theoretic concepts, refer [35]. Let D represent a group of positive integers and Z represent the set of all integers. The integer distance graph is defined as $G(Z, D)$ with node set Z and any 2 nodes s and t are adjacent if and only if $|s - t| \in D$ [12].

2- odd labeling of G is an injective function $h : V(G) \to Z$ in such a way that the positive difference between any adjacent nodes u and v, the integer $|h(v) - h(u)|$ is either $2n \pm 1; n \in N$ or exactly 2. Thus G is a 2 - odd if there exists 2 - odd labeling of G. For 2 - odd labeling of various graphs, one can see [1, 16, 22]. One can also relate the concept of 2-odd labeling with finite prime distance labeling. For a detailed study on finite prime distance graphs, refer to [16, 22]. Moreover, clearly all the finite prime distance graphs are 2-odd graphs but the converse need not be true [16]. In this paper, 2-odd labeling of certain graphs are derived using some famous number theoretic concepts and graph operations. By WLG we mean without loss of generality.

2 Main Results

First, the following results are recalled.

Lemma 2.1 *[1, 16] The subgraph of any 2 - odd graph is also 2 - odd graph.*

Theorem 2.1 *[1, 16] All bipartite graphs permit 2 - odd labeling.*

Proposition 2.1 *[16] The complete graph $K_n, n \geq 5$ is not 2-odd.*

2.1 2-Odd Labeling Using Number Theoretic Concepts

Graph labeling acts like a bridge between graph theory and number theory. This section discusses the 2-odd labeling of graphs using some concepts of number theory.

Conjecture 2.1 ***(Twin prime conjecture)****[9, 22] "There are infinitely many pairs of primes whose difference is 2".*

Conjecture 2.2 ***(Goldbach's conjecture)****[9, 22] The sum of two primes is any even number bigger than 2.*

Conjecture 2.3 ***(de Polignac's Conjecture)****[9, 16, 22] "For any positive even integer $2k$, there exist infinitely many pairs of consecutive primes that differ by $2k$".*

Theorem 2.2 ***(Green–Tao Theorem)****[4, 16] "There is a prime arithmetic progression of length k for any positive integer k".*

Theorem 2.3 ***(Vinogradov's Theorem)****[16, 23] The sum of three primes equals any suitably large odd number.*

Theorem 2.4 ***(Ramaré's Theorem)****[9, 16] Every even number has a maximum of six primes..*

Theorem 2.5 *[1, 16]"If and only if de Polignac's Conjecture is correct, every paper mill graph is a prime distance graph".*

Theorem 2.6 *[1, 16] If and only if the Twin Prime Conjecture is correct, "every Dutch windmill graph is a prime distance graph".*

Definition 2.1 *[22] The friendship graph F_n has a central node v_0 and pendant nodes v_1 through v_{2n}, with a line between each consecutive pair of nodes v_{2k-1} and v_{2k}, $1 \leq k \leq n$. So F_n has n copies of K_3 joined at the common node v_0.*

Theorem 2.7 *The graph G constructed from gluing a path of finite length at every node of the friendship graph admits 2 - odd labeling if "Twin prime conjecture" is true.*

Proof. Let F_n be the given friendship graph on $2n + 1$ nodes with $V(F_n) = V_1 \cup V_2$, where $V_1 = \{v_0$; the central node $\}$ and $V_2 = \{v_i; 1 \leq i \leq 2n\}$. Let G be formed by gluing $2n + 1$ copies of a path P_m at every node of F_n with $V(G) = V(F_n) \cup V_3$, where $V_3 = \{w_j^i; 0 \leq i \leq 2n, 2 \leq j \leq m\}$. Define an 1-1 function $g : V(G) \to Z$ as follows: WLG, let $g(v_0) = 0$. Then let $g(v_i) = p_i$ and $f(v_{i+1}) = p_i + 2$, where (p_i, p_j) are twin primes for all odd $1 \leq i \leq 2n$. Next let $2r$ be the sufficiently large even number. Then $g(w_2^1) = 2r$ and $g(w_j^1) = g(w_{j-1}^1) + 2$, for $3 \leq j \leq m$. Similarly, let $2s$ be the sufficiently large even number. Then $g(w_2^2) = 2s$ and $g(w_j^2) = g(w_{j-1}^2) + 2$, for $3 \leq j \leq m$. Proceeding thus, one can check that g is the needed 2-odd labeling of G. This labeling is clearly possible when the "twin prime conjecture" is true. Hence the proof.

Example 2.1 *2-odd labeling of a graph formed by gluing P_3 at every node of F_4 is shown in Figure 1.*

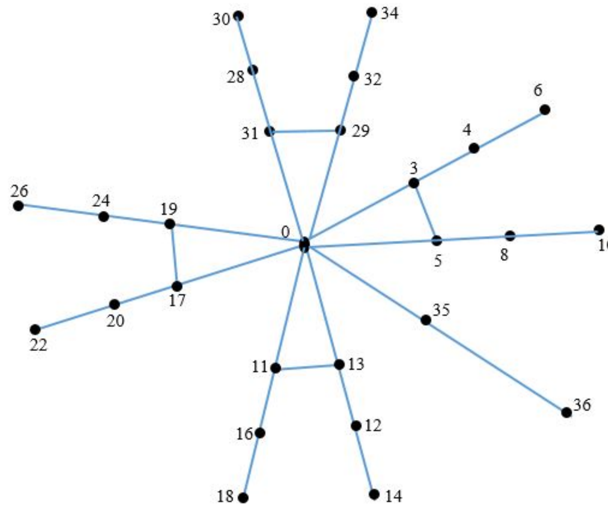

Figure 1. 2-odd labeling of a graph formed by gluing P_3 at every node of F_4

.

Definition 2.2 *[27] The Mongolian tent M_n is formed by joining a node on top of $P_n \times P_n$ grid with the top row nodes of the grid. The number of nodes and lines in M_n is $n^2 + 1$ and $2n^2 - n$, respectively.*

Theorem 2.8 *The Mongolian tent M_n admits 2-odd labeling $\forall n \geq 2$.*

Proof. Let M_n be the given Mongolian tent on $n^2 + 1$ nodes with $V(M_n) = \{v_0 \cup v_i^j\}$, where $\{1 \leq i, j \leq n\}$. Define a function $f : V(M_n) \to Z$ as follows: WLG, let $f(v_0) = 0$. Then $f(v_i^1) = 2i - 1; 1 \leq i \leq n$. Similarly, let $f(v_1^2) = f(v_n^1) + 1$ and $f(v_i^2) = f(v_{i-1}^2) + 2; 2 \leq i \leq n$ and $f(v_1^3) = f(v_n^2) + 1$ and $f(v_i^3) = f(v_{i-1}^3) + 2; 2 \leq i \leq n$. Continuing the same, finally $f(v_1^n) = f(v_n^{n-1}) + 1$ and $f(v_i^n) = f(v_{i-1}^n) + 2; 2 \leq i \leq n$. A simple check proves that f is a 2-odd labeling of M_n.

Example 2.2 *A 2-odd labeling of M_4 is shown in Figure 2.*

2.2 Duplication of Graph Elements

Duplication is an important graph operation in graph theory. Node by node, line by line, node by line, and line by node are the four methods of duplication. The concept of graph element duplication has a fascinating application in computer science. In distributed graph processing systems (DGPS) [36], data duplication is a popular optimization in which node state is repeated across computers to decrease inter-node communication. DGPS [15, 24], many real-world graphs have a "power-law degree distribution". Only for the highly connected hub to offer data duplication nodes. Only duplicating the data from

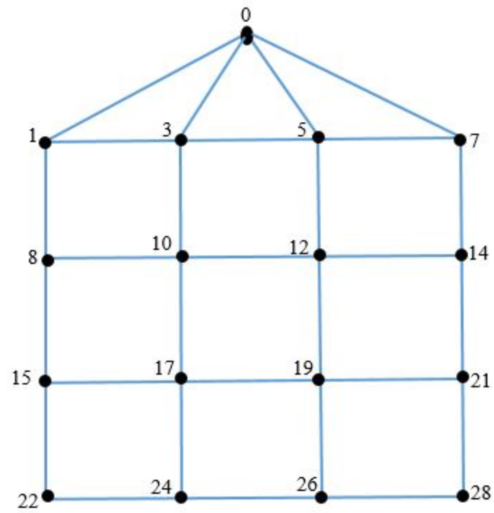

Figure 2. A 2-odd labeling of M_4

.

the hub node saves memory. Duplication overheads are solely incurred for the purposes of efficiency. A tiny percentage of hub nodes (which contribute the most to internode communication). DGPS requires advanced pretreatment methods to appropriately partition a graph's lines throughout the network in order to limit communication over the network. In addition to duplicating data from hub nodes Due to the restricted capacity of the "Last Level Cache" (LLC), replicating all of the hubs in G may result in high memory overheads. Instead of duplicating all hubs, just a subset of hubs (hub nodes with the greatest degrees) should be duplicated, so that the total size of all duplicated threads is less than the LLC's ability to function. For a detailed study, one can see [33].

Definition 2.3 *[22] Duplication of a node $v_i; 1 \leq i \leq n$ of G by a node is formed by inserting a new node $v'_i; 1 \leq i \leq n$ to G and adding new lines so that $N(v'_i) = N(v_i)$.*

Theorem 2.9 *The graph G obtained by performing duplication of a node by a node at all the nodes of C_n admits 2-odd labeling.*

Proof. Let G be formed by taking duplication of a node by a node at all the n nodes of C_n. Clearly $V(G) = V \cup V'$, where $V = V(C_n) = \{v_1, v_2, \dots v_n\}$ and $V' = \{v'_1, v'_2, \dots v'_n\}$. The cardinality $|V(G)| = 2n$. Define a function $f : V(G) \to Z$: There arise two cases.
Case i: When C_n, n is even
One can see that G is bipartite, so it is $2-odd$ by using theorem 2.1.
Case ii: When C_n, n is odd
lines, assign $f(v_i) = 2(i-1) : 1 \leq i \leq n-1\}$. Using Goldbach's conjecture 2.2, $f(v_{n-1}) = p_1 + p_2$. Now assign $f(v_n) = p_1$ and $f(v'_n) = p_2$. Then also let $f(v'_{n-1}) = p_1 + 2$ and $f(v'_1) = p_1 - 2$. Now assign any sufficiently large odd positive integers to the remaining unlabeled nodes v'_i; $2 \leq i \leq n-2$. A simple verification shows that f is the needed 2 - odd labeling of G.

Example 2.3 *A 2-odd labeling of the graph obtained by taking duplication of a node by a node at all the nodes of C_9 is given in Figure 3.*

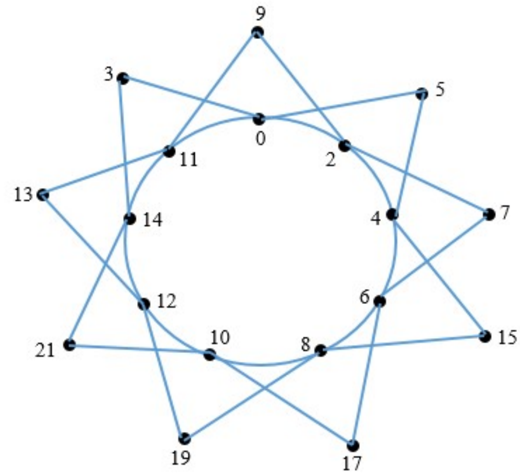

Figure 3. A 2-odd labeling of G formed by taking duplication of a node by a node at all the nodes of C_9

.

Definition 2.4 *[14] Extended jewel graph J_n is a graph with node set $V(J_n) = \{u, v, u', v_i, u_i, v'_i : 1 \leq i \leq n\}$ and line set $E(J_n) = \{uv_i, vv_i, uv', vu', vv', uu', uu_i, vu_i, v'u' : 1 \leq i \leq n\}$.*

Theorem 2.10 *The extended jewel graph J_n admits 2-odd labeling $\forall n \geq 1$.*

Proof. Let $G = J_n$ be the extended Jewel graph with $V(J_n) = V_1 \cup V_2 \cup V_3$, where $V_1 = \{u, v, u', v'\}, V_2 = \{u_i : 1 \leq i \leq n\}$, $V_3 = \{v_i : 1 \leq i \leq n\}$ and $E(J_n) = E_1 \cup E_2 \cup E_3$, where $E_1 = \{uv', uu', vv', vu', u'v'\}, E_2 = \{uu_i, vu_i : 1 \leq i \leq n\}, and E_3 = \{uv_i, vv_i : 1 \leq i \leq n\}$. Clearly $|V(G)| = n+4$ and $|E(G)| = 2n + 5$. Define a function $f : V(G) \to Z$ as follows: WLG, let $f(u) = 1$, $f(v) = -1$ and $f(v') = 0, and f(u') = 2$. Then $f(u_i) = \{2i + 2 : 1 \leq i \leq n\}$ and $f(v_i) = -f(u_i) : 1 \leq i \leq n$. One can see that the extended jewel graph J_n admits a 2-odd labeling.

Example 2.4 *A 2-odd labeling of the extended jewel graph J_3 is given in the Figure 4.*

Definition 2.5 *[10] The triangular book $B_n^{(3)}$ for $n \geq 1$ is a planar graph with $n + 2$ nodes $u, v, v_1, v_2, \dots v_n$ and $2n + 1$ lines formed by n times K_3's sharing a common line v_0, v'_0. Or simply, $B_n^{(3)}$ is the complete tripartite graph $K_{1,1,n}$.*

Theorem 2.11 *The triangular book $B_n^{(3)}$ admits 2-odd labeling $\forall n \geq 1$.*

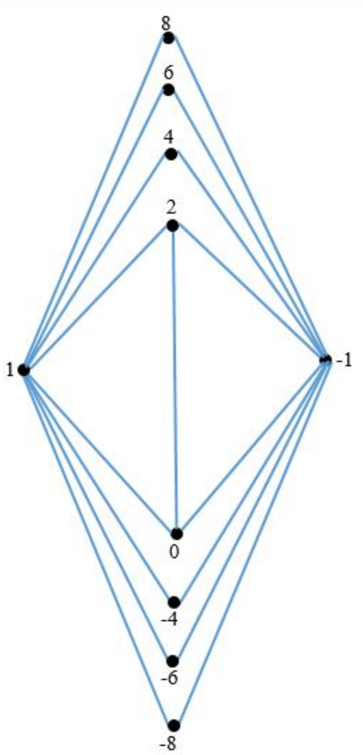

Figure 4. 2-odd labeling of extended jewel graph J_3

.

Proof. Let $B_n^{(3)}, n \geq 1$ be a triangular book graph with $|V(B_n^{(3)})| = n+2$ and $|E(B_n^{(3)})| = 2n+1$. Define a mapping $g : V(B_n^{(3)}) \to Z$ as follows: WLG, let $f(u) = 1$, $f(v) = -1$ and $f(v_i) = 2i : 1 \leq i \leq n$. Evidently, $B_n^{(3)}$ admits 2-odd labeling.

Example 2.5 *2-odd labeling of $B_n^{(3)}$ is given in the Figure 5.*

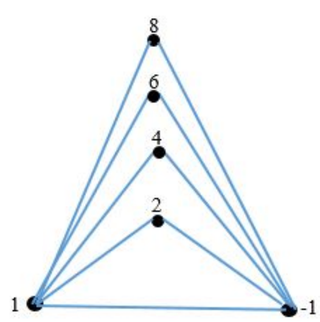

Figure 5. 2-odd labeling of $B_4^{(3)}$

.

2.3 Mobius Ladder

In this section we prove that all Mobius ladder graphs M_n are 2-odd $\forall n \geq 3$. First we recall the definition of M_n and its uses in Physics, Chemistry, and Computer science. With the special case of M_6 ($K_{3,3}$), M_n is a cubic circulant graph with exactly $n/2$ 4-cycles [8] that link together by their common lines to form a Möbius strip. Guy and Harary investigated M_n's for the first time in 1967. Walba et al. [34] created the first molecular structures in the form of M_n in 1982, and this structure has piqued interest in chemistry and chemical stereography ever since, especially given the ladder-like shape of DNA molecules. Flapan [13] discussed the mathematical symmetry of embeddings of M_n's in R^3 in 1989 with this purpose in mind. Every 3-dimensional embedding of M_n with an odd number of rungs is topologically chiral: it cannot be turned into its mirror image without passing one line through another by a continuous deformation of space. M_n's with an even number of rungs, on the other hand, all have embeddings in R^3 that can be distorted into their mirror copies. M_n has also been used as the shape of a superconducting ring in experiments to investigate the impact of conductor topology on electron interactions. M_n's are utilised in integer programming approaches to solve set packing and linear ordering problems in computer science. Möbius ladder constraints, which are facets of the polytope expressing a linear programming relaxation of the issue, can be defined using some configurations inside these problems.

Definition 2.6 *[11] The Möbius ladder $M_n, n \geq 3$, has n pairs of nodes, is constructed from C_n by adding lines (known as "rungs") connecting opposite pairs of nodes in C_n.*

One can see the 2-odd labeling of $M_n, 3 \leq n \leq 5$ in Figure 6. For $M_n, n \geq 6$, we prove the following theorem.

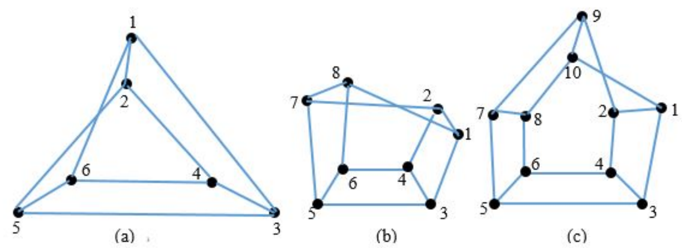

Figure 6. A 2-odd labeling of (a) M_3, (b) M_4, and (c) M_5

.

Theorem 2.12 *$M_n, n \geq 6$ admits 2-odd labeling.*

Proof. Let $M_n, n \geq 6$ be the given Mobius ladder. Let v_i and $u_i : 1 \leq i \leq n$ be the outer and inner nodes of M_n. Define an injective map $h : V(M_n) \to Z$. There arise two cases:
Case (i): When either n is odd or $n \equiv 2 \pmod 4$ in M_n
One can observe that M_n is bipartite and the result follows directly from Theorem 2.1.
Case (i): When $n \equiv 0 \pmod 4$ in M_n
WLG, let $h(v_i) = 2i - 1$ and $h(u_i) = 2n; 1 \leq i \leq n$. Hence $M_n, n \geq 6$ is 2-odd graph.

Definition 2.7 *[3] "The Jahangir graph $J_{n,m}$, where $n > 1$ and $m > 3$, is a graph with $nm + 1$ nodes that is made up of a cycle C_{nm} with an extra central node, say u, that is next* *to cyclically labeled nodes $v_1, v_2, \ldots v_m$ so that $d(v_i, v_{i+1}) = n, 1 \leq i \leq m-1$ in C_{nm}".*

Theorem 2.13 *The Jahangir graph $J_{n,m}$ admits 2-odd labeling $\forall n \geq 1$ and $m \geq 3$.*

Proof. Let $J_{n,m}$, where $n \geq 1, m \leq 3$ as defined in Definition 2.7. Let $v_i : 1 \leq i \leq nm$ be the cyclically labeled nodes in $J_{n,m}$. Define an injective map $h : V(J_{n,m}) \to Z$. There arise two cases:
Case (i): When n is even in $J_{n,m}$
One can observe that $J_{n,m}$ is bipartite and the result follows directly from Theorem 2.1.
Case (ii): When n is odd in $J_{n,m}$
WLG, let $h(u) = 0$ and $h(v_1) = 1$. Then $h(v_i) = h(v_{i-1}) + 2 : 2 \leq i \leq nm-1$ and $h(v_{nm}) = h(v_{nm-1}) + 1$. Hence $J_{n,m}$ a 2-odd graph.

Example 2.6 *A 2-odd labeling of Jahangir graph $J_{3,6}$ is given in the Figure 7 .*

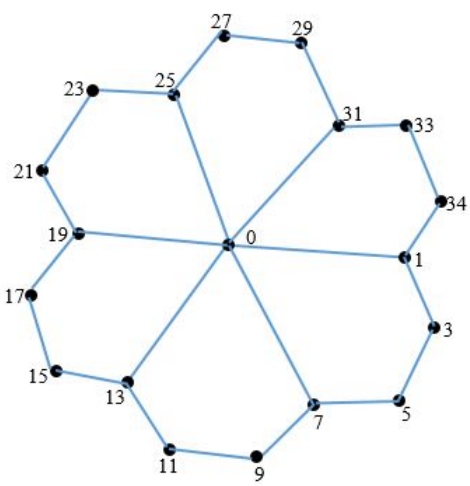

Figure 7. A 2-odd labeling of Jahangir graph $J_{3,6}$

.

2.4 2-Odd Labeling of Graphs Obtained Using Some Graph Operations

The word "operation" is derived from the Latin word called opus ("work"). Operations may involve mathematical objects like graphs other than numbers. Likewise, in graph theory, from the initial graphs, graph operations generate new graphs. They are mostly grouped into two groups: (a) Unary operations: These operations create a new graph from the original. Simple local changes, such as adding or removing a node or a line, merging and dividing nodes, line contraction, and so on, build a new graph from the original one. Advanced operations, such as transpose, complement, line, and dual graphs, construct a new graph from an existing one by performing sophisticated alterations. (b) Binary operations: given two basic graphs H_1 and H_2, generate a new graph. For more information on the same, one can see [21]. By $N(v)$ and $N[v]$, we mean an open and closed neighborhood of $v \in G$, respectively.

Definition 2.8 *[37] Duplication of a line $e = xy$ in H by a node z is formed by adding a new node z to H such that $N(z) = x, y$.*

Definition 2.9 *[37] Duplication of a node v_k in H by a line $e = v_k' v_k''$ is constructed by inserting new nodes v_k', v_k'' such that $N(v_k') = v_k, v_k''$ and $N(v_k'') = v_k, v_k'$.*

These graph operations mainly modify the structure of the underlying graph and so having some understanding of the complex operations which can be done over a graph or a set of graphs are inevitable. This section discusses 2 - odd labeling of graphs generated by using various graph operations such as union, disjoint union, intersection, duplication, and extension.

Theorem 2.14 *[1] The graph obtained by performing duplication of the node by a line at all the nodes of a 2-odd graph permits 2-odd labeling if the "Twin prime conjecture" is true.*

Definition 2.10 *[37] Duplication of a line $e = xy$ in H by a line produces G by adding a line $e' = x'y'$ where $x'y'$ are newly inserted nodes to H such that $N(x') = N(x) \cup y' - y$ and $N(y') = N(y) \cup x' - x$.*

Theorem 2.15 *The graph obtained by duplicating line by a line at all the lines in any 2-odd graph admits 2-odd labeling.*

Proof. Let h be the given labeling of 2-odd graph H. Let $V(H) = \{v_1, v_2, \ldots v_n\}$ and G be obtained by duplicating line by a line at all the lines of H. So, $V(G) = \{V(H) \cup \{v_i'\} \cup \{v_i''\} : 1 \leq i \leq n\}$ and $|V(G)| = 3n$. Defining a function $f : V(G) \to Z$ gives the following three cases.
Case 1 : When $h(v_k)$ and $h(v_{k+1})$ are even
WLG, let $f(v_k') = p_1$ and $f(v_k'') = p_2; 1 \leq k \leq n$, where p_1, p_2 are sufficiently large twin primes.
Case 2: When $h(v_k)$ and $h(v_{k+1})$ are odd
Let $2k_r$ be the sufficiently large even number. Then $f(v_k') = 2k_r$ and $f(v_k'') = 2k_r + 2; k_r \in N$.
Case 3: When $h(v_k)$ is odd (even) and $h(v_{k+1})$ is even (odd)
Let $2k_s$ be the sufficiently large even number. Then $f(v_k') = 2k_s$ and $f(v_k'') = 2k_s + 1 : k_s \in N$. A similar argument holds good when $h(v_k)$ is even and $h(v_{k+1})$ is odd. Thus G admits 2-odd labeling.

Definition 2.11 *[37] Duplication of a node v of H is formed by inserting a new node v' to H and introducing lines in such a way that $N(v') = N(v)$.*

Inspired by Theorem 2.14 and Theorem 2.15, the following conjecture is raised.

Conjecture 2.4 *The graph obtained by performing duplication of the node by a node at all the nodes of any 2-odd graph permits 2-odd labeling.*

Definition 2.12 *[37] An extension of a node x by a new node y in H produces G such that $N(y) = N[x]$.*

Theorem 2.16 *The graph formed by performing an extension of the node by a node at all the nodes of any 2-regular graph admit 2-odd labeling.*

Proof. Let H be the 2-regular graph on n nodes, namely $v_1, v_2, \ldots v_n$. Obtain G by performing extension of the node by a node at all the nodes of H by introducing the following new nodes $u_1, u_2, \ldots u_n$, respectively to H as given in Definition 2.12 (See Fig. 8). Note that $V(G) = V_1 \cup V_2$ where $V_1 = \{v_i; 1 \leq i \leq n\}$ and $V_2 = \{u_i; 1 \leq i \leq n\}$ and also $|V(G)| = 2n$ and $|E(G)| = 4n$. Defining a function $g : V(G) \to Z$ gives rise to the below two cases.
Case 1 : When H is of order $n \equiv 0(mod \quad 2)$
Let $g(v_i) = 3i; 1 \leq i \leq n$ and WLG, $g(u_1) = 1$. Then $g(u_i) = g(u_{i-1}) + 3; 2 \leq i \leq n$.
Case 2 : When H is of order $n \equiv 1(mod \quad 2)$
WLG, let $g(v_1) = 2, g(v_2) = 3, g(v_3) = 6$. Then $g(v_i) = g(v_{i-1}) + 2; 4 \leq i \leq n-1$, and $g(v_n) = g(v_{n-1}) + 1$. Again let, $g(u_1) = 0, g(u_2) = 4, g(u_3) = 5$. Then $g(u_i) = g(u_{i-1}) + 2; 4 \leq i \leq n-1$, and $g(u_n) = g(u_{n-1}) + 4$. Thus one can easily check that G admits 2-odd labeling.

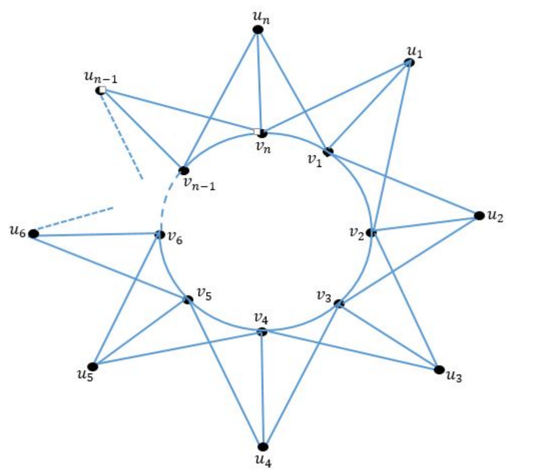

Figure 8. Extension of the node by a node at all the nodes in a 2-regular graph

.

Motivated by Theorem 2.16, the following conjecture is formulated.

Conjecture 2.5 *The graph formed by performing an extension of node by a node at all the nodes of the given 2-odd graph permits 2- odd labeling.*

Definition 2.13 *[25] The disjoint union of graphs combines two or more graphs to form the largest graph. In other words, if G and H are two graphs, then G + H denotes their disjoint union.*

Theorem 2.17 *The disjoint union of any finite copies of 2-odd graph permits 2 - odd labeling.*

Proof. Let the given 2-odd graph be $H(n, m)$ with a 2-odd labeling h. Let $V(H) = \{u_1, u_2, ..., u_n\}$. WLG, let $h(u_k) = \max_{1 \leq i \leq n} \{h(u_i)\} = r_n$. Construct the disconnected graph G as a disjoint union of H_i's, i.e., $G = kH$ as shown in Fig 9 with $V(G) = \{v_j^i : 1 \leq j \leq n; 1 \leq i \leq k\}$ and $E(G) = \{e_j^i : 1 \leq j \leq m; 1 \leq i \leq k\}$. Define a function $g : V(G) \to Z$ as follows: $g(v_j^1)$ for $1 \leq j \leq n; g(v_j^2) = g(v_j^1) + r_{n_1}$, where $r_{n_1} = r_n + 1$. Next $g(v_j^3) = g(v_j^1) + r_{n_2}$, where $r_{n_2} = \max_{1 \leq j \leq n} g(v_j^2) + 1$. Proceeding thus, we have $g(v_j^k) = g(v_j^1) + r_{n_{k-1}}$, where $r_{n_{k-1}} = \max_{1 \leq j \leq n} g(v_j^{k-1}) + 1$. A quick check reveals that g is the desired G 2-odd labelling..

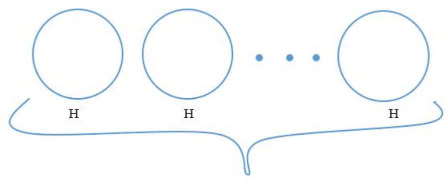

Figure 9. $G = kH$

.

Definition 2.14 *[32] The graph $V(H_1 \cup H_2) = V(H_1) \cup V(H_2)$ and $E(H_1 \cup H_2) = E(H_1) \cup E(H_2)$ is the union of two graphs H_1 and H_2.*

Theorem 2.18 *The graph obtained as a union of any finite copies of a 2-odd graph permits 2 - odd labeling.*

Proof. Let H be the given 2-odd graph on n nodes. Take k copies of H, namely $H_i; 1 \leq i \leq k$ and obtain $G = \bigcup_{i=1}^{k} H_i$ with $|V(G)| = kn$. Let $h_1, h_2, ..., h_k$ be the 2-odd labeling of $H_1, H_2, ..., H_k$, respectively. Defining $g : V(G) \to Z$ gives rise to the following three cases.
Case 1: When h_i's are distinct
Clearly, the labels induced by h_i are unique for each H_i. Since each component, H_i admits 2-odd labeling, the graph is clearly a disjoint union of 2-odd graphs and the proof eventually follows from Theorem 2.17.
Case 2: When $h_i's$ are same
Clearly, $h_1(H_1) = h_2(H_2) = \cdots = h_k(H_k)$. The resultant graph is H itself considering the fact from definition 2.14 of the union of graphs. Further, $|V(G)| = n$ that is any one of the k copies of $H_i : 1 \leq i \leq k$, which is already a 2-odd graph.
Case 3: When some of the $h_i's$ are distinct and the others are not
WLG, assume that the first r copies of $H_i's$ have the 2-odd labeling with same labels and the remaining $k - r$ copies of $H_i's$ have 2-odd labeling with distinct labels. Now the resultant graph G has one of the r copies of $H_i's$ and all the $k - r$ copies of $H_i's$. Here $|V(G)| = n(k - r + 1)$. Establishing 2-odd labeling for the resultant graph G is done on similar lines that of Theorem 2.17.
Case 4: When some labels are repeated within the graphs $H_i's$
Here we get a new resultant graph G' obtained from G. Clearly, $|V(G')| < |V(G)|$ and $|E(G')| < |E(G)|$. Define $g' : V(G') \to Z$ as follows: WLG, let $g'(V(G')) = g(V(G))$. Thus in all four cases, graph G admits 2-odd labeling.

Definition 2.15 *[26, 31] "A graph H is said to be an arbitrary super subdivision of $G(p, q)$, denoted by $ASS(G)$, if H is drawn from G by replacing every line e_i of G by K_{2,m_i} (for some $m_i : 1 \leq i \leq q$) in such a way that the ends of each*

e_i are merged with the two nodes of the 2-nodes part of K_{2,m_i} after removing the line e_i from G. Thus, $|V(H)| = p + \sum_{i=1}^{q} m_i$ and $|E(H)| = \sum_{i=1}^{q} 2m_i$".

Lemma 2.2 *Arbitrary super subdivision of any graph is bipartite.*

Proof. Let H be the given graph and G be the graph obtained from H by taking an arbitrary super subdivision of H. There are two cases:
Case 1: When H is bipartite.
There is nothing to prove.
Case 2: When H is not bipartite.
By considering the fact from Definition 2.15 that the graph G formed from H by taking an arbitrary super subdivision does not contain any odd cycle, thus the proof follows from the Theorem 2.1.

Definition 2.16 *[18, 30] If H is formed by subdividing every line of G exactly once, it is considered to be a subdivision of $G(n, m)$, denoted by $S(G)$. As a result, $|V(H)| = n + m$ and $|E(H)| = 2m$.*

Conjecture 2.6 *The subdivision of any graph is bipartite.*

Definition 2.17 *[17, 30] A graph H is a super subdivision of $G(p, q)$, denoted by $SS(G)$, if it can be made from G by replacing every line e of G with a complete bipartite graph $K_{2,m}$. So, $|V(H)| = p + mq$ and $|E(H)| = 2mq$.*

Conjecture 2.7 *The $SS(G)$ of any graph is bipartite.*

Remark 2.1 *The proofs of Corollary 2.6 and Corollary 2.7 follow in similar lines of Lemma 2.2.*

Theorem 2.19 *The graph obtained by performing an arbitrary super subdivision of any graph admits 2-odd labeling.*

Proof. The proof clearly follows from Lemma 2.2 and Theorem 2.1.

Conjecture 2.8 *The graph obtained by performing subdivision of any graph admits 2-odd labeling.*

Conjecture 2.9 *The graph obtained by performing supersubdivision of any graph admits 2-odd labeling.*

Remark 2.2 *The proofs of Corollary 2.8 and Corollary 2.9 follow in the similar lines of Theorem 2.19.*

Definition 2.18 *[28] A "graph G in which a node is distinguished from other nodes is called a rooted graph and the node is called the root of G. The graph G^k is formed by identifying the roots of k copies of a rooted graph G is called a one-point union of the k copies of G".*

Theorem 2.20 *One-point union of any finite number of 2-odd graphs permits 2-odd labeling if each 2-odd graph has distinct node labels except for one node.*

Proof. Let $G_i; 1 \leq i \leq k$ be the given 2-odd graphs and $n_1, n_2, n_3, \ldots n_k$ be the cardinalities of $G_i's$, respectively. Note that each G_i is a 2-odd graph with the condition that every G_i has distinct node labels except for exactly one node, say v_1. That is., if g_i is the 2-odd labeling of the corresponding $G_i : 1 \leq i \leq k$, respectively, then $g_1(G_1) \neq g_2(G_2) \neq g_3(G_3), ... \neq g_k(G_k)$ except that $g_1(v_1) = g_2(v_1) = g_3(v_1) = ... = g_k(v_1)$. Now WLG, obtain G by fusing all the $G_i's$: $1 \leq i \leq k$ at v_1 as a one-point union of G_i with $|V(G)| = \sum_{i=1}^{k} n_i - k + 1$. Define $\beta : V(G) \to Z$ by letting $\beta(v_1) = g_1(v_1)$ and $\beta(G - v_1) = g_i(G_i - v_1) : 1 \leq i \leq k$. So one can evidently see that β induces 2-odd labeling of G.

Inspired by Theorem 2.20, the following conjecture is proposed.

Conjecture 2.10 *One point union of any finite number of 2-odd graphs permits 2-odd labeling.*

Definition 2.19 *[29] "Let $H_1, H_2, \ldots H_k$ be $k \geq 2$ copies of a graph H. The graph $H(k)$ is formed by introducing a line between H_i and $H_{i+1}; i = 1, 2, ..., k - 1$. The graph $H(k)$ is called the path-union of k copies of H".*

Theorem 2.21 *Path union of finite copies of any 2-odd graph admits 2-odd labeling.*

Proof. Let H be the given 2-odd graph on n nodes, namely $v_1, v_2, ..., v_n$. Since H is a 2-odd graph, it has 2-odd labeling say, $h : V(H) \to Z$ such that the induced line labels are either 2 or $2r \pm 1, r \in N$. Also assume that $h(v_1) = 0$. Take k copies of H, namely $H_i; 1 \leq i \leq k$, and obtain G as a path union of $H_i's$ with $V(G) = v_i^j : 1 \leq i \leq n, 1 \leq j \leq k$. Let the nodes of the newly introduced path be merged with the nodes that are assigned 0 in each copy. Clearly $|V(G)| = \sum_{i=1}^{k} n_i$. Define $f : V(G) \to Z$ as follows: WLG, let $h(v_1^1) = 0$ and $f(H_1) = h(H_1)$. If k_1 is the largest label used in H_1, then r_1 be sufficiently larger than k_1 so that $|f(v_1^1) - f(v_1^2)|$ is either 2 or odd and also $f(v_i^2) = h(v_i^2) + r_1 : 2 \leq i \leq n$. Similarly, if k_2 is the largest label used in H_2, then r_2 be sufficiently larger than k_2 so that $|f(v_1^2) - f(v_1^3)|$ is either 2 or odd and also $f(v_i^3) = h(v_i^3) + r_2 : 2 \leq i \leq n$. By continuing this way, If k_t is the largest label used in H_{k-1}, then r_t be sufficiently larger than k_t so that $|f(v_1^{k-1}) - f(v_1^k)|$ is either 2 or odd and also $f(v_i^k) = h(v_i^k) + r_t : 2 \leq i \leq n$. By the above process, one can see that the path union of finite copies of any 2-odd graph admits 2-odd labeling.

Definition 2.20 *[32] Given two graphs H_1 and H_2, with a minimum one of node in common, then the intersection of H_1 and H_2 is a graph such that $V(H_1 \cap H_2) = V(H_1) \cap V(H_2)$ and $E(H_1 \cap H_2) = E(H_1) \cap E(H_2)$.*

Theorem 2.22 *The graph obtained as an intersection of a finite number of 2-odd graphs permits 2-odd labeling.*

Proof. Let H_i denote the given 2-odd graphs on n_i nodes with 2-odd labeling h_i, respectively for $1 \leq i \leq k$. Obtain $G =$

$\bigcap_{i=1}^{k} H_i$. Defining $g : V(G) \to Z$ gives rise to the following two cases.
Case 1: When each H_i has distinct labeling h_i
Obviously, the resultant graph is null and there is nothing to prove.
Case 2: When H_i's have some common labels
The result followed by Lemma 2.1 as the resultant graph is clearly a subgraph of the given 2-odd graphs.
Thus g is clearly 2-odd labeling of G.

3 Applications of Graph Labeling and Graph Coloring

Graph labeling is one of the most interesting concepts in graph theory which has numerous applications in different fields. A graph labeling is a function that assigns integers to the lines or nodes, or both of G under some conditions. The importance of graph labeling includes its numerous applications in many areas like circuit design, radar, communication network addressing, fault- tolerant system design, and automatic channel allocation, etc. For a detailed study, see [2, 6, 7, 35].

A node $k-$ colouring of G is a mapping $c : V(G) \to 1, 2, \ldots, k$, and c is a proper $k-$ colouring if the nodes next to it are coloured differently. The chromatic number $chi(G)$ of G is the smallest k for which G has a correct colouring at node $k-$. Graph colouring is used a lot in computer science research, especially in fields like data mining, image segmentation, clustering, taking pictures, networking, and so on. As an example, The shape of a data structure can be made to look like a tree, turn nodes and lines were used. The most important thing to know about graph colouring is used for allocating resources and making plans. In addition, paths, walks, and circuits are used in a lot of different ways in graph theory, applications like the Traveling Salesman Problem and database design ideas, and connecting resources.
The graph colouring is also used to protect an art gallery is [5]. Exam scheduling needs to be based on a good allocation of schedule times for a set of tests. Every test is taken based on a set of rules, by a number of students. Ozcan et. al. [20] came up with an algorithm called the mimetic algorithm (MA) to solve final Yeditepe University sets up exam times. Taking into account the job reducing the number of exams and getting rid of the scheduling conflicts, final exam scheduling comes down to colouring a graph problem [19].

4 Conclusions

Further results on 2 - odd labeling of graphs are explored using certain well-known number theoretic concepts, besides establishing the same using various graph operations such as union, intersection, disjoint union, one point union, path union, duplication, extension, and arbitrary subdivision. Moreover, a few interesting conjectures are also formulated. The study undertaken may serve as a path in finding the complete characterization of 2-odd labeling which is still an open problem. The concept of 2-odd labeling of graphs may find its applications in graph-based cryptography and network security.

REFERENCES

[1] P. Abirami, A. Parthiban and N. Srinivasan, On 2 - Odd Labeling Of Graphs, European Journal of Molecular and Clinical Medicine, vol. 07, pp. 3914 – 3918, 2020.

[2] B. D. Acharya, S. Arumugam, and A. Rosa, Labeling of discrete structures and Applications, Narosa publishing house, New Delhi, 2008.

[3] A. Ali, I. Hashim, W. Nazeer and S.M. Kang, On More Topological Indices Of Jahangir Graphs, International Journal of Pure and Applied Mathematics, vol. 119, pp. 1-8, 2018.

[4] Ben Green, Terence Tao,The primes contain arbitrarily long arithmetic progressions, Ann. of Math. (2), vol. 167, pp. 481-547, 2008.

[5] Besjana Tosuni, Graph coloring problems in modern computer science, European Journal of Interdisciplinary Studies, vol.1(2), pp. 87-95, 2015.

[6] G.S. Bloom, and S.W. Glomb, Application of numbered undirected graphs, Proc. IEEE, vol. 65, pp. 562-570, 1977.

[7] Gross, J., and Yellen, J., (1999), Graph theory and its applications, CRC Press, London.

[8] N.L. Biggs, R.M. Damerell and D.A. Sands, Recursive Families of Graphs, Journal of Combinatorial Theory Series B, Vol. 12, pp. 123-131, 1972.

[9] D.M. Burton, Elementary Number Theory, seventh ed., McGraw-Hill, New York, 2011.

[10] S.N. Daoud and Ahmed N. Elsawy, line even graceful labeling of some book graphs, Journal of Taibah University For Science, vol.12, pp. 315-30, 2018.

[11] P.K. Dewi et al., On Super (a,d)-line-antimagic Total Labeling of Mobius Ladder, J. Phy. Conf. Ser., vol. 1040, pp. 1-6, 2018.

[12] R.B. Eggleton, P. Erdos and D.K. Skilton, Colouring Prime Distance Graphs, Graphs and Combinatorics, vol. 6, pp. 17-32, 1990.

[13] Flapan and Erica, Symmetries of Mobius Ladders, Mathematische Annalen, vol. 283, pp. 271-283, 1989.

[14] J. Jeba Jesinath, N.K. Vinodhini and S. Munavar Hussain, Odd Graceful Labeling for the Jewel Graph and The Extended Jewel Graph without the Prime line, Bulletin of Pure and Applied Sciences Section-E- Mathematics and Statistics, vol. 39E2, pp. 212-217, 2020.

[15] E. Joseph Gonzalez, Yucheng Low, Haijie Gu, Danny Bickson, and Carlos Guestrin. Power Graph: Distributed Graph-Parallel Computation on Natural Graphs, In OSDI, vol. 12, 2012.

[16] J.D. Laison, C. Starr and A. Walker, Finite Prime Distance Graphs and 2-odd Graphs, Discrete Mathematics, vol. 313, pp. 2281-2291, 2013.

[17] K. M. Kathiresan, and S. Amutha, Arbitrary supersubdivision of stars are graceful, Indian J. Pure and Appl. Math., 35, pp. 81 – 84, 2004.

[18] K. M. Kathiresan, Subdivision of ladders are graceful, Indian Journal of Pure Appl Math., vol. 23(1), pp. 21 – 23, 1992.

[19] F.T. Leighton, A graph coloring algorithm for large scheduling problems, Journal of Research of the National Bureau Standard, vol. 84, pp. 79-100, 1979.

[20] E. Ozcan, and E. Ersoy, Final exam scheduler - FES, Proc. of IEEE Congress on Evolutionary Computation, vol. 2, pp. 1356-1363, 2005.

[21] Meenakshi Paramasivan, Operations on graphs, arrays and automata, Ph.D. Thesis submitted to University of Trier, pp. 1-201, 2017.

[22] A. Parthiban and N. Gnanamalar David, On Prime Distance Labeling of Graphs.in: Arumugam S., Bagga J., Beineke L., Panda B.(eds) Theoretical Computer Science and Discrete Mathematics.ICTCSDM 2016 Lecture Notes in Computer Science,Vol 10398.Springer,Cham., vol. 31, pp. 238-241, 2017.

[23] K. Ramachandra and A. Sankaranarayanan, Vinogradov's three primes theorem, Math. Student, vol. 66,pp. 27-72, 1997.

[24] Rong Chen, Jiaxin Shi, Yanzhe Chen, and Haibo Chen, Powerlyra: Differentiated graph computation and partitioning on skewed graphs, In Proceedings of the Tenth European Conference on Computer Systems ACM, 2015.

[25] H. Rosen Kenneth, Handbook of discrete and combinatorial mathematics, Discrete Mathematics and Its Applications, CRC Press, pp. 515, 1999.

[26] G. Sethuraman, and P. Selvaraju, Gracefulness of super subdivision of graphs, Indian Journal of Pure and Appl. Math., vol. 32(7), pp. 1059 – 1064, 2001.

[27] R. Sreenivasan and M.S. Paulraj, node Antimagic line Labeling of the Mongolian Tent Graph, Int. J. of Scientific Research in Mathematical and Statistical Sciences, Vol. 6, pp. 229-233, 2019.

[28] Sze-Chin Shee, and Yong-Song Ho., The cordiality of the one-point union of n copies of a graph, Discrete Mathematics, vol. 117, pp. 225-243, 1993.

[29] Sze-Chin Shee, Yong-Song Ho, The cordiality of the path-union of n copies of a graph, Discrete Mathematics, vol. 151, pp. 221-229, 1996.

[30] Ujwala Deshmukh, and S. A. Bhatavadekar, Gracefulness of some arbitrary super subdivision of cycle, Journal of Informatics and Mathematical Sciences, vol. 10(4), pp. 599-611, 2018.

[31] S. K. Vaidya, P. L. Vihol, Dani, and D. D. Bantva, L(2,1)-Labeling in the context of some graph operations, Journal of Mathematics Research, vol. 2(3), pp. 109-119, 2010.

[32] Vasudev, C., Graph theory with applications, New Age International Publishers.

[33] Vignesh Balaji and Brandon Lucia, Combining Data Duplication and Graph Reordering to Accelerate Parallel Graph Processing, International Symposium on High-Performance Parallel and Distributed Computing (HPDC'19), June 2019, Phoenix, AZ, USA. ACM, New York, NY, USA, pp. 133-144, 2019.

[34] D. Walba, R. Richards and R.C. Haltiwanger, Total Synthesis of the first Molecular Mobius Strip, Journal of the American Chemical Society, vol. 104, pp. 3219-21, 1982.

[35] D.B. West, Introduction to Graph Theory, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 2000.

[36] Yucheng Low, Danny Bickson, Joseph Gonzalez, Carlos Guestrin, Aapo Kyrola and Joseph M Hellerstein, Distributed GraphLab: a framework for machine learning and data mining in the cloud. Proceedings of the VLDB Endowment, pp. 716–727, 2012.

[37] V. Yegnanarayanan, and A. Parthiban, On graph theory application-I, Applied Research and Development Institute Journal, vol. 3(4), pp. 28-42, 2012.