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Numerical Solution of Nonlinear Fredholm Integral Equations Using Half-Sweep Newton-PKSOR Iteration

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Abstract This paper is concerned with producing an efficient numerical method to solve nonlinear Fredholm integral equations using Half-Sweep Newton-PKSOR (HSNPKSOR) iteration. The computation of numerical methods in solving nonlinear equations usually requires immense amounts of computational complexity. By implementing a Half-Sweep approach, the complexity of the calculation is tried to be reduced to produce a more efficient method. For this purpose, the steps of the solution process are discussed beginning with the derivation of nonlinear Fredholm integral equations using a quadrature scheme to get the half-sweep approximation equation. Then, the generated approximation equation is used to develop a nonlinear system. Following that, the formulation of the HSNPKSOR iterative method is constructed to solve nonlinear Fredholm integral equations. To verify the performance of the proposed method, the experimental results were compared with the Full-Sweep Newton-KSOR (FSNKSOR), Half-Sweep Newton-KSOR (HSNKSOR), and Full-Sweep Newton-PKSOR (FSNPKSOR) using three parameters: number of iteration, iteration time, and maximum absolute error. Several examples are used in this study to illustrate the efficiency of the tested methods. Based on the numerical experiment, the results appear that the HSNPKSOR method is effective in solving nonlinear Fredholm integral equations mainly in terms of iteration time compared to rest tested methods.

Keywords Nonlinear Fredholm Integral Equations, Newton-PKSOR, Half-sweep Newton-PKSOR, Iterative Method, Quadrature Scheme

1 Introduction

Integral equations arise in certain problems in mechanics, mathematical physics, and technology [1]. Besides that, its applications include real-world problems such as fluid mechanics, biological models, solid state physics, mathematical physics, kinetics, potential theory, electrostatics and radiative heat transfer problems [2,3]. One of the important classes of integral equations is known as Fredholm integral equations. The problems of nonlinear Fredholm integral equations are usually more difficult to solve numerically than the linear problem. However, the intention to solve these equations is continuously developing due to its important applications in various research fields. Thus, the discussion on solving nonlinear Fredholm can be found in many literature. In the past several years, many analytical and numerical methods have been introduced to solve the nonlinear Fredholm integral equations, among these are the extrapolation method [4], parameter continuation method [5], successive approximations method [6], Newton's method [7], q-homotopy analysis method [8], Legendre spectral collocation method [9], least squares support vector regression [10], and Hermite wavelets collocation method [11]. In [12], the authors present a new technique based on a combination of a Newton-Kantorovich and Haar wavelet for solving nonlinear Fredholm integral equations. Several other combinations of methods to solve these equations can also be found in many literature. For some examples, see [13-15].

In this study, a combination of the Newton-PKSOR iterative method with half-sweep approach is developed to obtain the approximate solution of following nonlinear Fredholm integral equations

$$u(x) = y(x) + \int_a^b k(x,t,u(t))dt, x \in [a,b]. \quad (1)$$

where $k(x,t,u(t))$ is continuous on $[a,b]$, $y(x)$ is the given function and $u(x)$ is the unknown function to be determined [6]. The recent study in [16] has shown the application of Newton's method with PKSOR iterative method also called as NPKSOR method to be effective in solving nonlinear Fredholm integral equations of the second kind when compared to the Newton-KSOR and Newton-GS methods. However, NPKSOR method encounters weakness when it comes to the computational complexity of the calculation because it is categorized in a full-sweep case. The computation of these methods requires immense amounts of computational complexity to solve nonlinear Fredholm integral equations. Thus, this study proposes the combination of the NPKSOR method with a half-sweep approach to fully utilize the method in solving the proposed problem.

Half-sweep approach was firstly introduced in 1991 [17] which is known to be useful in reducing the computational complexity of the iteration in solving many problems. It has been used widely in numerical computing as a combination with iterative methods to minimize the iteration time in solving many linear systems. Several applications of the half-sweep iteration have been carried out in solving integral equations in [18-20]. Besides that, it is also found in solving fractional diffusion [21], reaction-diffusion equations [22], porous medium equations [23], Burger's equations [24], and Fredholm integro-differential equations [25]. Based on these studies, it has demonstrated that the key concept of the half-sweep iteration is to take just half of the total number of node points of the problem from the solution domain. Consequently, the principle of half-sweep approach will minimize the difficulty of estimation in the iterative process, which inevitably leads to a reduction in the number of iterations and the time of execution. By taking these advantages into account, this study expands the implementation of the half-sweep with NPKSOR methods to solve the proposed problem.

The rest of this paper is organized as follows. In Section 2, we discuss the formulation of the proposed methods for solving nonlinear Fredholm integral equations which consist of two main parts. In the first part, we discuss the discretization of nonlinear Fredholm integral equations using a quadrature scheme along with a half-sweep approach to generate a nonlinear system. Following that, we discuss the formulation of HSNPKSOR iteration to solve the nonlinear system in the second part. Then, we present the numerical examples and discuss the experimental work of this study in Section 3. Finally, we give the conclusions in Section 4.

2 Methodology

This section discusses the formulation of HSNPKSOR iteration to obtain the approximate solution of nonlinear Fredholm integral equations. The formulation of the numerical solution of this study begins with the derivation on integral terms of (1) using the quadrature scheme to get the half-sweep approximation equation. From this stage, a system of nonlinear equations can be generated. Following that, this study discusses the formulation of the HSNPKSOR iterative method to solve the nonlinear system.

2.1 Discretization of nonlinear Fredholm integral equations

Consider the quadrature scheme as follows [18]

$$\int_a^b u(t) = \sum_{j=0}^{n} W_j(t_j) + \epsilon_n(y). \quad (2)$$

where (t_j), $(j = 0, 1, ..., n)$ are abscissas of the partition points of the integration on interval [a,b], W_j are the numerical coefficients and $\epsilon_n(y)$ is the truncation error. The numerical coefficient W_j based on Trapezoidal rule satisfies the following relation [18,20]

$$W_j = \begin{cases} h, & j = 0, n. \\ 2h, & \text{otherwise}. \end{cases} \quad (3)$$

where h is the constant step size defined as

$$h = \frac{b-a}{n} \quad (4)$$

and n is the number of subintervals on $[a,b]$[16]. Following that, to construct the derivation of half-sweep approximation equation, let the interval $[a,b]$ to be distributed into equidistance node as $\{a = u_0, u_1, u_2, \dots u_{n-1}, u_n = b\}$. In half-sweep approach, only a set of nodes, $\{u_0, u_2, u_4, \dots u_{n-2}, u_n\}$ is considered for iteration while the remaining nodes are computed directly after the convergence criterion is satisfied. This intention is to reduce the computational node point into $(\frac{n}{2}+1)$ from its standard nodes to be solved iteratively. As consequence, the application of half-sweep approach can reduce the iteration time by approximately half of its standard form (full-sweep approach).

For $j = 0, 2, 4, \dots, n$, the application of first-order quadrature scheme into (1) can be defined as [16]

$$u(x) - \sum_{j=0,2,4,\dots}^{n} k(x,t,u(t)) = y(x). \quad (5)$$

The summation term in (5) can be expended using Trapezium rule to form the general nonlinear approximation equation as follows

$$\begin{aligned} u_i - \frac{1}{2}hk(x,t_0,u_0) - hk(x,t_2,u_2) - \\ hk(x,t_4,u_4) - \dots - \frac{1}{2}hk(x,t_n,u_n) = y_i, \end{aligned} \quad (6)$$

where $i = 0, 2, 4, \dots, n$. Consider the nonlinear function of Eq. (6) be written as

$$\begin{aligned} f_i(u_i) = u_i - \frac{1}{2}hk(x,t_0,u_0) - \\ hk(x,t_2,u_2) - \dots - \frac{1}{2}hk(x,t_n,u_n) - y_i. \end{aligned} \quad (7)$$

for $u_i = (u_0, u_2, u_4, \ldots, u_n)$, this leads to the following system of nonlinear equation

$$f_i(u_i) = 0. \quad (8)$$

2.2 Formulation of Half-Sweep NPKSOR Iteration

Following the discretization part, the formulation of Half-Sweep Newton-PKSOR iteration is discussed to solve nonlinear Fredholm integral equations. Newton-PKSOR method is based on the combination of Newton's method with PKSOR iterative method. This study uses Newton's method in linearization process to convert nonlinear terms of (8) into a linear system. Following that, PKSOR iterative method will be used to solve the linear system iteratively. To get this linear system, imposing Newton's method over the generated system of nonlinear (8) can be represented in the following form [26]

$$J(\underline{u}^{(k)})\Delta u^{(k)} = -f(\underline{u}^{(k)}), \quad (9)$$

where

$$J(\underline{u}^{(k)}) = \begin{bmatrix} \frac{df_0}{du_0} & \frac{df_0}{du_2} & \frac{df_0}{du_4} & \cdots & \frac{df_0}{du_n} \\ \frac{df_2}{du_0} & \frac{df_2}{du_2} & \frac{df_2}{du_4} & \cdots & \frac{df_2}{du_n} \\ \frac{df_4}{du_0} & \frac{df_4}{du_2} & \frac{df_4}{du_4} & \cdots & \frac{df_4}{du_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{df_n}{du_0} & \frac{df_n}{du_2} & \frac{df_n}{du_4} & \cdots & \frac{df_n}{du_n} \end{bmatrix}, \quad (10)$$

$$\Delta u^{(k)} = \begin{bmatrix} \Delta u_0 & \Delta u_2 & \Delta u_4 & \cdots & \Delta u_n \end{bmatrix}^T, \quad (11)$$

Then, the solution of (9) can be computed using iterative scheme

$$u_i^{(k+1)} = u_i^{(k)} + \Delta u_i, i = 0, 2, 4, ..., n. \quad (12)$$

Let (9) be written in a matrix from as follows

$$A\underline{u} = \underline{f}, \quad (13)$$

$$A = \begin{bmatrix} A_{0,0} & A_{0,2} & A_{0,4} & \cdots & A_{0,n} \\ A_{2,0} & A_{2,2} & A_{2,4} & \cdots & A_{2,n} \\ A_{4,0} & A_{4,2} & A_{4,4} & \cdots & A_{4,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,2} & A_{n,4} & \cdots & A_{n,n} \end{bmatrix}_{(\frac{n}{2}+1)\times(\frac{n}{2}+1)}, \quad (14)$$

$$\underline{u} = \begin{bmatrix} u_0 \\ u_2 \\ u_4 \\ \vdots \\ u_n \end{bmatrix}, \underline{f} = \begin{bmatrix} f_0 \\ f_2 \\ f_4 \\ \vdots \\ f_n \end{bmatrix}. \quad (15)$$

Before disccussing the formulation of NPKSOR iteration, let us consider the fomulation of NKSOR which defined by

$$u^{[n+1]} = T_{KSOR}u^{[n]} + ((1+\omega^*)D - \omega^* L)^{-1}(\omega^* b), \quad (16)$$

$$T_{KSOR} = ((1+\omega^*)D - \omega^* L)^{-1}(D + \omega^* U) \quad (17)$$

where $\omega^* \in \mathbb{R}-[-2, 0]$[27]. NPKSOR is an update of NKSOR method by with two weighted paramaters, ω_1^* and ω_2^* which defined by

$$[(1+\omega_2^*)D - \omega_1^* L]u^{[n+1]} = (D + \omega_1^* U)u^{[n]} + \omega_1^* b, \quad (18)$$

where ω_1^* and $\omega_2^* \in \mathbb{R} - [-2, 0]$, and the coefficient matrix is $A = D - L - U$, in which $A \leftarrow A - dD$ where $\frac{\omega_2^* - \omega_1^*}{\omega_2^*}$. D represents diagonal matrix, L and U are strickly lower and upper triangular matrix of A, respectively [28]. The process of Half-Sweep NPKSOR in solving system of linear (13) is described in following algorithm.

Algorithm 1: Formulation of Half-Sweep NPKSOR
i. Set the initial value $u^{(k)} = 0$, $k = 0$, and $\epsilon = 10^{-10}$.
ii. Assign the optimal value of ω_1^* and ω_2^*.
ii. For $i = 0, 2, 4, \ldots, n$,
  a. Compute matrix A and vector f.
  b. Solve linear system iteratively using (18).
iii. Run the convergence test, $\left|u_i^{(k+1)} - u_i^{(k)}\right| \leq \epsilon$. If yes, go to step iv, otherwise repeat step iii(a).
iv. Find the current value of $\underline{u}^{(k+1)} = \underline{u}^{(k)} + u^{(k+1)}$.
v. Run the convergence test, $\left|f(\underline{u}^{(k+1)} - f(\underline{u}^{(k)})\right| \leq \xi$. If yes, go to step iv, otherwise repeat step iii.
vi. Compute the remaining odd node points.
vii. Display the results.

3 Numerical Results

In this section, we present the numerical results and disccussion based on the implementation of Algorithm 1 to get the approximate solution of the nonlinear Fredholm integral equation. To test the efficiency and reliability of HSNPKSOR iteration, this study considers Full-Sweep Newton-KSOR (FSNKSOR), Half-Sweep Newton-KSOR (HSNKSOR), and Full-Sweep Newton-PKSOR (FSNPKSOR) methods for the comparative study. FSNKSOR iteration will be set as a control method to illustrate the efficiency of the rest of the iterations. The experimental work of this study is run using Borland C++ on several grid sizes which are 256, 512, 1024, 2048, and 4096. Besides that, this study uses several parameters which are the number of iteration, iteration time (in seconds), and maximum absolute error to compare the results. The tested methods of this study is applied to the following numerical examples.
Example 1. In this example, we consider [29]

$$u(x) = 1 - \frac{5}{12}x + \int_0^1 xt[u(t)]^2 dt,$$

where the given exact solution is $u(x) = 1 + \frac{1}{3}t$ [26].
Example 2. In this example, we consider [30]

$$u(x) = cos(x) + \frac{5}{2} - x + \frac{1}{2}cos(1)sin(1) + 2sin(1) + \int_0^1 (x - u(t)^2)dt,$$

where the given exact solution is $u(x) = cos(x) + 1$.
Example 3. In this example, we consider [31]

$$u(x) = 1 - \frac{1}{3}x + \int_0^1 xt^2 u^3(t)dt,$$

where the given exact solution is $u(x) = 1$.

Usually, the method to get the optimum values for NKSOR and NPKSOR methods is not easy and has to be done semi-manually. Thus, this study uses a different way to get the optimum values for HSNKSOR and HSNPKSOR methods. To get the numerical data, first, the program is run to find the optimum ω^* for FSNKSOR for all grid sizes. Then for FSNPKSOR, ω^* from FSNKSOR is set as ω_2 and the program is continued to find the optimum ω_1. Using the same way, this study finds the optimum parameters for HSNKSOR and HSNPKSOR only for grid size 256. Following that, the optimum parameters for the remaining grid sizes are used from the recorded FSNKSOR and FSNPKSOR. After all optimum parameters are obtained, the program is run for all methods to get the number of iteration, iteration time (in seconds), and maximum absolute error for each grid size.

Table 1 to 3 shows the performance of HSNKSOR method compared with FSNKSOR, FSNPKSOR, and HSNKSOR iterations for Example 1, 2, and 3 respectively. While Table 4 shows the reduction percentages of FSNPKSOR, HSNKSOR, and HSNPKSOR compared with FSNKSOR method in terms of number of iteration and iteration time. Based on the results, still study shows that the HSNKSOR and HSNPKSOR methods can produce fewer iterations compared to FSNKSOR and FSNPKSOR. Also, it is clearly shown that the computational approximate solution for all three numerical examples using HSNKSOR and HSNPKSOR are faster than FSNKSOR and FSNPKSOR methods. In table 4, it shows that the HSNPKSOR method recorded the highest reduction percentages in terms of number of iteration and iteration time compared to rest tested methods. In terms of maximum absolute error, HSNKSOR and HSNPKSOR are slightly less accurate than FSNKSOR and FSNPKSOR due to the implementation of direct method. However, the accuracy of HSNKSOR and HSNPKSOR is more accurate as the grid sizes increase. When comparing both HSNKSOR and HSNPKSOR, it is clear that HSNPKSOR is a more efficient solver.

4 Conclusions

This paper proposed the combination of the Newton-PKSOR iterative method with half-sweep approach, called HSNKSOR iteration to obtain the approximate solution of nonlinear Fredholm integral equations. The proposed method of this study is tested on three numerical examples and compared to three other methods: FSNKSOR, HSNKSOR, and FSNPKSOR to test its efficiency. Based on the numerical findings, this study has shown that the implementation of HSNPKSOR can produce a faster computational method with fewer iterations in solving nonlinear Fredholm integral equations. The HSNPKSOR can lower the number of iterations and minimize the iteration time even in large grid sizes compared to the FSNKSOR, HSNKSOR, and FSNPKSOR. This is due to the implementation of the half-sweep technique into Newton-PKSOR iteration which helps in reducing the amount of computational complexity of the iteration. Besides that, the optimum parameters in HSNPKSOR play an important role in reducing the iteration number. Moreover, the technique of finding the optimum weighted parameters in this study is found to be effective and easy to implement. Overall, this study concludes that HSNPKSOR is a more efficient method for solving nonlinear Fredholm integral equations compared to the rest tested methods in this study.

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Table 1. Comparison of FSNKSOR, HSNKSOR, FSNPKSOR, and HSNPKSOR for Example 1.

	Methods	Grid Size				
		256	512	1024	2048	4096
k	FSNKSOR	119 $(\omega^* = -4.0731)$	119 $(\omega^* = -4.0685)$	119 $(\omega^* = -4.0662)$	120 $(\omega^* = -4.0525)$	120 $(\omega^* = -4.0525)$
	HSNKSOR	118 $(\omega^* = -4.2103)$	119 $(\omega^* = -4.0731)$	119 $(\omega^* = -4.0685)$	119 $(\omega^* = -4.0662)$	120 $(\omega^* = -4.0525)$
	FSNPKSOR	115 $(\omega_1^* = -4.0769)$ $(\omega_2^* = -4.0731)$	115 $(\omega_1^* = -4.0723)$ $(\omega_2^* = -4.0685)$	114 $(\omega_1^* = -4.0700)$ $(\omega_2^* = -4.0662)$	109 $(\omega_1^* = -4.0563)$ $(\omega_2^* = -4.0525)$	107 $(\omega_1^* = -4.0563)$ $(\omega_2^* = -4.0525)$
	HSNPKSOR	108 $(\omega_1^* = -4.2143)$ $(\omega_2^* = -4.2103)$	115 $(\omega_1^* = -4.0769)$ $(\omega_2^* = -4.0731)$	115 $(\omega_1^* = -4.0723)$ $(\omega_2^* = -4.0685)$	114 $(\omega_1^* = -4.0700)$ $(\omega_2^* = -4.662)$	109 $(\omega_1^* = -4.0563)$ $(\omega_2^* = -4.0525)$
$Time$	FSNKSOR	0.19	0.66	2.48	9.90	35.61
	HSNKSOR	0.05	0.20	0.64	2.50	9.93
	FSNPKSOR	0.17	0.61	2.37	9.00	35.34
	HSNPKSOR	0.04	0.17	0.62	2.40	9.02
$Error$	FSNKSOR	$1.27165E-05$	$3.17896E-06$	$7.94723E-07$	$1.98674E-07$	$4.96624E-08$
	HSNKSOR	$5.08758E-05$	$1.27165E-05$	$3.17896E-06$	$7.94723E-07$	$1.98674E-07$
	FSNPKSOR	$1.27165E-05$	$3.17897E-06$	$7.94747E-07$	$1.98698E-07$	$4.96621E-08$
	HSNPKSOR	$5.08758E-05$	$1.27165E-05$	$3.17897E-06$	$7.94747E-07$	$1.98698E-07$

Table 2. Comparison of FSNKSOR, HSNKSOR, FSNPKSOR, and HSNPKSOR for Example 2.

	Methods	Grid Size				
		256	512	1024	2048	4096
k	FSNKSOR	635 $(\omega^* = 1.1518)$	641 $(\omega^* = 0.9557)$	644 $(\omega^* = 0.9209)$	645 $(\omega^* = 0.9175)$	646 $(\omega^* = 0.9182)$
	HSNKSOR	623 $(\omega^* = 1.1736)$	635 $(\omega^* = 1.1518)$	641 $(\omega^* = 0.9557)$	644 $(\omega^* = 0.9209)$	645 $(\omega^* = 0.9175)$
	FSNPKSOR	459 $(\omega_1^* = 0.8812)$ $(\omega_2^* = 1.1518)$	492 $(\omega_1^* = 0.7433)$ $(\omega_2^* = 0.9557)$	497 $(\omega_1^* = 0.7511)$ $(\omega_2^* = 0.9209)$	497 $(\omega_1^* = 0.7481)$ $(\omega_2^* = 0.9175)$	498 $(\omega_1^* = 0.7491)$ $(\omega_2^* = 0.9182)$
	HSNPKSOR	458 $(\omega_1^* = 0.9276)$ $(\omega_2^* = 1.1736)$	459 $(\omega_1^* = 0.8812)$ $(\omega_2^* = 1.1518)$	492 $(\omega_1^* = 0.7433)$ $(\omega_2^* = 0.9557)$	497 $(\omega_1^* = 0.7511)$ $(\omega_2^* = 0.9209)$	497 $(\omega_1^* = 0.7481)$ $(\omega_2^* = 0.9175)$
$Time$	FSNKSOR	0.93	3.44	13.64	54.61	218.65
	HSNKSOR	0.26	0.92	3.45	13.77	54.68
	FSNPKSOR	0.74	2.70	10.68	42.49	172.08
	HSNPKSOR	0.22	0.65	2.74	10.67	42.55
$Error$	FSNKSOR	$7.03908E-07$	$1.76078E-07$	$4.40987E-08$	$1.11044E-08$	$2.85597E-09$
	HSNKSOR	$2.84048E-06$	$7.07011E-07$	$1.76469E-07$	$4.41475E-08$	$1.11104E-08$
	FSNPKSOR	$7.03927E-07$	$1.76056E-07$	$4.40509E-08$	$1.10567E-08$	$2.80573E-09$
	HSNPKSOR	$2.84035E-06$	$7.06879E-07$	$1.76231E-07$	$4.40657E-08$	$1.10583E-08$

Table 3. Comparison of FSNKSOR, HSNKSOR, FSNPKSOR, and HSNPKSOR for Example 3.

	Methods	Grid Size				
		256	512	1024	2048	4096
k	FSNKSOR	87 $(\omega^* = -4.9355)$	87 $(\omega^* = -4.9268)$	88 $(\omega^* = -4.9104)$	88 $(\omega^* = -4.9077)$	88 $(\omega^* = -4.9055)$
	HSNKSOR	87 $(\omega^* = -5.1889)$	87 $(\omega^* = -4.9355)$	87 $(\omega^* = -4.9268)$	88 $(\omega^* = -4.9104)$	88 $(\omega^* = -4.9077)$
	FSNPKSOR	85 $(\omega_1^* = -4.9366)$ $(\omega_2^* = -4.9355)$	85 $(\omega_1^* = -4.9279)$ $(\omega_2^* = -4.9268)$	85 $(\omega_1^* = -4.9115)$ $(\omega_2^* = -4.9104)$	85 $(\omega_1^* = -4.9088)$ $(\omega_2^* = -4.9077)$	85 $(\omega_1^* = -4.9066)$ $(\omega_2^* = -4.9055)$
	HSNPKSOR	84 $(\omega_1^* = -5.1901)$ $(\omega_2^* = -5.1889)$	85 $(\omega_1^* = -4.9366)$ $(\omega_2^* = -4.9355)$	85 $(\omega_1^* = -4.9279)$ $(\omega_2^* = -4.9268)$	85 $(\omega_1^* = -4.9115)$ $(\omega_2^* = -4.9104)$	85 $(\omega_1^* = -4.9088)$ $(\omega_2^* = -4.9077)$
$Time$	FSNKSOR	0.17	0.54	2.09	8.22	33.17
	HSNKSOR	0.05	0.18	0.56	2.09	8.25
	FSNPKSOR	0.14	0.52	2.04	8.03	31.83
	HSNPKSOR	0.04	0.15	0.52	2.03	7.95
$Error$	FSNKSOR	$1.01732E-05$	$2.54317E-06$	$6.35784E-07$	$1.58944E-07$	$3.97342E-08$
	HSNKSOR	$4.07015E-05$	$1.01732E-05$	$2.54317E-06$	$6.35784E-07$	$1.58944E-07$
	FSNPKSOR	$1.01732E-05$	$2.54317E-06$	$6.35783E-07$	$1.58943E-07$	$3.97335E-08$
	HSNPKSOR	$4.07015E-05$	$1.01732E-05$	$2.54317E-06$	$6.35783E-07$	$1.58943E-07$

Table 4. Reduction Percentages of FSNKSOR, HSNKSOR and HSNPKSOR compared with FSNKSOR method.

Methods	Number of Iteration		
	Example 1	Example 2	Example 3
HSNKSOR	$3.36 - 10.00\%$	$22.83 - 27.72\%$	$2.30 - 3.41\%$
FSNPKSOR	$0.00 - 0.84\%$	$0.15 - 1.89\%$	$0.00 - 1.14\%$
HSNPKSOR	$3.36 - 9.24\%$	$22.95 - 28.38\%$	$2.30 - 3.45\%$
Methods	Iteration Time		
	Example 1	Example 2	Example 3
HSNKSOR	$4.42 - 10.64\%$	$20.43 - 22.19\%$	$2.31 - 17.65\%$
FSNPKSOR	$67.69 - 75.49\%$	$72.04 - 74.99\%$	$66.67 - 75.13\%$
HSNPKSOR	$73.85 - 77.78\%$	$76.34 - 81.10\%$	$72.22 - 76.47\%$

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