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Dynamics of Nonlinear Operator Generated by Lebesgue Quadratic Stochastic Operator with Exponential Measure

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Abstract Quadratic stochastic operator (QSO) is a branch of nonlinear operator studies initiated by Bernstein in 1924 through his presentation on population genetics. The study of QSO is still ongoing due to the incomplete understanding of the trajectory behavior of such operators given certain conditions and measures. In this paper, we intend to introduce and investigate a class of QSO named Lebesgue QSO which gets its name from the Lebesgue measure as the measure is used to define the probability measure of such QSO. The broad definition of Lebesgue QSO allows the construction of a new measure as its family of probability measure. We construct a class of Lebesgue QSO with exponential measure generated by 3-partition with three different parameters defined on continual state space $X=[0,1]$. Also, we present the dynamics of such QSO by describing the fixed points and periodic points of the system of equations generated by the defined QSO using a functional analysis approach. The investigation is concluded by the regularity of the operator, where such Lebesgue QSO is either regular or nonregular depending on the parameters and defined measurable partitions. The result of this research allows us to define a new family of functions of the probability measure of Lebesgue QSO and compare their dynamics with the existing Lebesgue QSO.

Keywords Quadratic Stochastic Operator, Lebesgue Measure, Regularity, Partition, Periodic Points

1. Introduction

A quadratic stochastic operator (QSO) is the simplest nonlinear operator acknowledged as a crucial tool in understanding various phenomena in diverse fields. Since it was initiated through Bernstein's advanced work [1] on population genetics in the early 20th century, the study of QSO maintains its position as an ongoing investigation interest among scientists in different fields of study. Due to its importance and applications, the study of QSO has been well developed through countless research to introduce different classes of these operators, to describe their asymptotical behavior and other dynamical properties. Consequently, this paper focuses on QSO defined on continual state space.

One may refer to [2,3] for a clear idea of the theory of QSO, where it can be simplified as an operator that describes the probability distribution of a population given the current state of the population. Based on this brief description, the readers can infer the importance of QSO in predicting the future of a given system. The problem of the QSO study is mainly centered on the behavior of such operators defined on a state space.

A systematic exposition of the recent achievements and open problems in QSO theory is given in [3]. The theory of QSO was initially studied within the small dimensional simplex as introduced and explained in numerous works (see, for example [4,5]). Over a century, the QSO study in a finite-dimensional simplex is still not depleted. Hence, many different classes of QSO are investigated (see for example [6-13]).

In [7], the author described a class of QSO called Volterra QSO defined on the infinite-dimensional space. This study eventually opened up a path to studying different classes of QSO on infinite state space. One may refer to [8,9] for the QSO study on countable state space. For the study on continual state space, readers may refer to [10-13]. Recently, in [14,15], the authors investigated a class of Geometric and Poisson QSO, respectively, where such operators were generated by two measurable partitions of the set of infinite points. These studies showed that the behavior of such infinite-dimensional operators generated by some finite partitions could be explored by reducing the simplex into a small dimensional setting corresponding to the number of defined partitions.

In this paper, we intend to continue investigating the infinite-dimensional QSO, where a new class of Lebesgue QSO generated by 3-partition on continual state space will be considered. In section 2, we provide necessary definitions, notions, and results to explain the properties of QSO briefly. The following section will present the construction of such Lebesgue QSO generated by 3-partition (see section 3). The main result of this paper is to describe the regularity of such Lebesgue QSO through the existence of fixed points and periodic points on low dimensional simplex (see Section 4). In the final section, the discussion of the regularity of such QSO is presented (see Section 5).

2. Preliminaries

Throughout this paper, we consider a measurable space (X,F), and denote $S(X,F)$ as the set of all probability measures on such space, where X is a state space, and F is a σ-algebra of subsets of X. Define a family of functions $\{P(i,j,A): i,j\in X,\, A\in F\}$ on $X\times X\times F$ that satisfy the following conditions:

(i). For any fixed $i,j\in X$, $P(i,j,\cdot)\in S(X,F)$ is a probability measure, where $P(i,j,\cdot): F\to[0,1]$,

(ii). $P(i,j,A)$ is a function of two variables, x and y, with a fixed $A\in F$, which is measurable on $(X\times X, F\otimes F)$,

(iii). $P(i,j,A)=P(j,i,A)$.

A quadratic stochastic operator, $V: S(X,F)\to S(X,F)$ is defined as follows

$$V\lambda(A)=\int_X\int_X P(i,j,A)\,d\lambda(i)\,d\lambda(j), \qquad (1)$$

where $\lambda\in S(X,F)$ is an arbitrary initial measure and $A\in F$ is a measurable set. Note that if a state space X is finite, where $X=\{1,\dots,m\}$ and $P(X)$ is the corresponding σ-algebra on X, then the set of all probability measures $S(X,F)$ is called as simplex with the following form,

$$S^{m-1}=\left\{\mathbf{x}=(x_1,\dots,x_m)\in\mathbb{R}^m: x_i\ge 0,\ \sum_{i=1}^m x_i=1\right\}.$$

This follows that, a probability measure $P(i,j,\cdot)$ is a discrete measure with $\sum_{k=1}^m P_{ij,k}=1$. Hence, a quadratic stochastic operator V with the finite state space X is given as follows.

Definition 1 A mapping $V: S^{m-1}\to S^{m-1}$ is called quadratic stochastic operator, if for any $\mathbf{x}\in S^{m-1}$,

$$(V\mathbf{x})_k=\sum_{i,j=1}^m P_{ij,k}x_ix_j, \qquad (2)$$

where $P_{ij,k}\ge 0$, $P_{ij,k}=P_{ji,k}$, and $\sum_{k=1}^m P_{ij,k}=1$ for $i,j,k\in\{1,\dots,m\}$.

Now, let ξ be an m-measurable partition of the state space X, where $\xi=\{A_1,\dots,A_m\}$ such that $A_i\subset X$ and $A_i\cap A_j=\varnothing$ for any $i,j\in\{1,\dots,m\}$. Then, consider a corresponding partition ζ on $X\times X$, where $\zeta=\left\{B_{ij}=\bigcup_{i,j=1}^m (A_i\times A_j): B_{ij}=B_{ji}\right\}$. As one considers such partition on the state space X, then one may define the probability measure $P(x,y,A)$ such that

$$P(x,y,A)=\mu_{ij}(A), \qquad (3)$$

for $(x,y)\in B_{ij}$. Then, in this case, for any $\lambda\in S(X,F)$ and $A\in F$, one may have

$$V\lambda(A)=\int_X\int_X P(x,y,A)\,d\lambda(x)\,d\lambda(y)$$
$$=\sum_{i,j=1}^m \mu_{ij}(A)\,\lambda(A_i)\,\lambda(A_j).$$

Given that λ is an initial point with $V^0\lambda=\lambda$. Assume $\{V^n\lambda: n=0,1,\dots\}$ is the trajectory behavior of the initial measure λ, where $V^{n+1}\lambda = V(V^n\lambda)$ for all $n = 0,1,\ldots$. Then, one will obtain

$$V^{n+1}\lambda(A) = \sum_{i,j=1}^{m} \mu_{ij}(A)\, V^n\lambda(A_i)\, V^n\lambda(A_j),$$

with

$$V^{n+1}\lambda(A_k) = \sum_{i,j=1}^{m} \mu_{ij}(A_k)\, V^n\lambda(A_i)\, V^n\lambda(A_j), \quad (4)$$

where $A_k \subset X$ and $k = 1,\ldots,m$.

Assume $x_k^{(n)} = V^n\lambda(A_k)$ and $P_{ij,k} = \mu_{ij}(A_k)$. Then, for $\mathbf{x}^{(n)} \in S^{m-1}$, one may rewrite the system of equations in Equation (4) as follows

$$(W\mathbf{x})_k = \sum_{i,j=1}^{m} P_{ij,k} x_i x_j, \quad (5)$$

for all $k = 1,\ldots,m$.

This shows that for a fixed measurable finite partition defined on a state space and family of probability measures in Equation (3), one can approximate the QSO V in Equation (4) by a finite-dimensional QSO W in Equation (5).

In [15], the authors described the trajectory behavior of Poisson QSO generated by 2-partition on countable state space by analyzing the following operator,

$$\begin{aligned} (Wx)_1 &= a x_1^2 + 2b x_1 x_2 + c x_2^2, \\ (Wx)_2 &= (1-a)x_1^2 + 2(1-b)x_1x_2 + (1-c)x_2^2, \end{aligned} \quad (6)$$

where $a = P_{11,1}$, $b = P_{12,1}$, and $c = P_{22,1}$. It is easy to observe that such an operator can be reduced to a single variable function by eliminating the second coordinate and obtaining the following function

$$f(x_1) = (a - 2b + c)x_1^2 + 2(b - c)x_1 + c. \quad (7)$$

Solving $x_1 = f(x_1)$, one may investigate the local characteristics of the fixed point by exploring the discriminant, denoted as Δ, where

$$\Delta = 4(1-a)c + (1-2b)^2.$$

In [16], the following results are proved.

Remark 2.1 *Let x^* be a hyperbolic fixed point, where $|f'(x^*)| \neq 1$, and $f'(x)$ be the first derivative of a function $f(x)$. Then, the following statements hold true:*

i). if $|f'(x^*)| < 1$, then x^* is an attracting hyperbolic fixed point,

ii). if $|f'(x^*)| > 1$, then x^* is a repelling hyperbolic fixed point.

Theorem 2.1 *[16] A fixed point of the transformation in Equation* (7) *is attracting when $0 < \Delta < 4$, and repelling when $4 < \Delta < 5$.*

Corollary 2.1 *[16] All trajectories converge to a fixed point when $0 < \Delta < 4$.*

Theorem 2.2 *[16] If $4 < \Delta < 5$, then there exists a cycle of second order and all trajectories tend to this cycle except the stationary point starting with fixed point.*

3. Lebesgue QSO with Exponential Measure Generated by 3-Partition

In this section, we construct a Lebesgue QSO with exponential measure generated by 3-partition.

Definition 3.1. *A transformation V given by Equation (1) is called a Lebesgue QSO, if $X = [0,1]$, and F is a Borel σ-algebra on $[0,1]$.*

Let $X = [0,1]$ be a continuous set and F be a Borel σ-algebra of X. In this research, we consider a probability measure μ_q, such that

$$\mu_q(A) = \frac{\ln(q+1)}{q} \int_A (q+1)^x\, dx, \quad (8)$$

for any arbitrary $q \in \mathbb{N}$.

Let $\xi = \{A_1, A_2, A_3\}$ be a measurable 3-partition of the set X and $\zeta = \{B_{11}, B_{22}, B_{33}, B_{12}, B_{13}, B_{23}\}$ be the corresponding partition on $X \times X$, where $B_{ij} = \bigcup_{i,j=1}^{3} (A_i \times A_j)$. Select a family of measure in Equation (8), where $\{\mu_{q_{ij}} : i,j = 1,2,3 \text{ and } q_{ij} \in \mathbb{N}\}$ with parameters $q_{11} = q_1$, $q_{22} = q_2$, $q_{33} = q_3$, $q_{12} = q_4$, $q_{13} = q_5$, $q_{23} = q_6$, and define the probability measure $P(x,y,A)$ such that

$$P(x,y,A) = \mu_{q_{ij}}(A), \quad (9)$$

for $A \in F$, $(x,y) \in B_{ij}$, and $i,j = 1,2,3$.

Recall that $P_{ij,k} = \mu_{q_{ij}}(A_k)$. Suppose that any continuous initial measure $\lambda \in S(X,F)$ be a continuous probability measure, then one may have $A = [\alpha,\beta] \subset [0,1]$ for any $A \in F$. Then, define $\int_{A_1} d\lambda = \lambda(A_1)$, and $\int_{A_2} d\lambda = \lambda(A_2)$, where $A_1 = [0,\alpha_1)$, $A_2 = [\alpha_1,\alpha_2]$, and $A_3 = (\alpha_2, 1]$ for $\alpha_1,\alpha_2 \in (0,1)$. One may construct the defined Lebesgue QSO such that

$$
\begin{aligned}
V\lambda(A) &= \int_0^1\int_0^1 P(x,y,A)\,d\lambda(x)\,d\lambda(y) \\
&= \int_{A_1}\int_{A_1}\mu_{q_1}(A)\,d\lambda(x)\,d\lambda(y) + \int_{A_2}\int_{A_2}\mu_{q_2}(A)\,d\lambda(x)\,d\lambda(y) \\
&\quad + \int_{A_3}\int_{A_3}\mu_{q_3}(A)\,d\lambda(x)\,d\lambda(y) + \int_{A_1}\int_{A_2}\mu_{q_4}(A)\,d\lambda(x)\,d\lambda(y) \\
&\quad + \int_{A_2}\int_{A_1}\mu_{q_4}(A)\,d\lambda(x)\,d\lambda(y) + \int_{A_1}\int_{A_3}\mu_{q_5}(A)\,d\lambda(x)\,d\lambda(y) \\
&\quad + \int_{A_3}\int_{A_1}\mu_{q_5}(A)\,d\lambda(x)\,d\lambda(y) + \int_{A_2}\int_{A_3}\mu_{q_6}(A)\,d\lambda(x)\,d\lambda(y) \\
&\quad + \int_{A_3}\int_{A_2}\mu_{q_6}(A)\,d\lambda(x)\,d\lambda(y) \\
&= \mu_{q_1}(A)\lambda^2(A_1) + \mu_{q_2}(A)\lambda^2(A_2) + \mu_{q_3}(A)\lambda^2(A_3) \\
&\quad + 2\mu_{q_4}(A)\lambda(A_1)\lambda(A_2) + 2\mu_{q_5}(A)\lambda(A_1)\lambda(A_3) \\
&\quad + 2\mu_{q_6}(A)\lambda(A_2)\lambda(A_3).
\end{aligned}
$$

This follows that

$$
\begin{aligned}
V^{n+1}\lambda(A) &= \mu_{q_1}(A)\left(V^n\lambda(A_1)\right)^2 + \mu_{q_2}(A)\left(V^n\lambda(A_2)\right)^2 \\
&\quad + \mu_{q_3}(A)\left(V^n\lambda(A_3)\right)^2 \\
&\quad + 2\mu_{q_4}(A)V^n\lambda(A_1)V^n\lambda(A_2) \\
&\quad + 2\mu_{q_5}(A)V^n\lambda(A_1)V^n\lambda(A_3) \\
&\quad + 2\mu_{q_6}(A)V^n\lambda(A_2)V^n\lambda(A_3),
\end{aligned}
$$

with

$$
\begin{aligned}
V^{n+1}\lambda(A_k) &= \mu_{q_1}(A_k)\left(V^n\lambda(A_1)\right)^2 + \mu_{q_2}(A_k)\left(V^n\lambda(A_2)\right)^2 \\
&\quad + \mu_{q_3}(A_k)\left(V^n\lambda(A_3)\right)^2 \\
&\quad + 2\mu_{q_4}(A_k)V^n\lambda(A_1)V^n\lambda(A_2) \\
&\quad + 2\mu_{q_5}(A_k)V^n\lambda(A_1)V^n\lambda(A_3) \\
&\quad + 2\mu_{q_6}(A_k)V^n\lambda(A_2)V^n\lambda(A_3),
\end{aligned} \tag{10}
$$

for $k = 1,2,3$, and $n = 0,1,\ldots$.

Evidently, the recurrent equations in Equation (10) can be rewritten as a QSO W on S^2 with the following form,

$$
\begin{aligned}
(W\mathbf{x})_1 &= a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 \\
&\quad + 2a_{23}x_2x_3, \\
(W\mathbf{x})_2 &= b_{11}x_1^2 + b_{22}x_2^2 + b_{33}x_3^2 + 2b_{12}x_1x_2 + 2b_{13}x_1x_3 \\
&\quad + 2b_{23}x_2x_3, \\
(W\mathbf{x})_3 &= c_{11}x_1^2 + c_{22}x_2^2 + c_{33}x_3^2 + 2c_{12}x_1x_2 + 2c_{13}x_1x_3 \\
&\quad + 2c_{23}x_2x_3,
\end{aligned} \tag{11}
$$

where $a_{ij} = P_{ij,1}$, $b_{ij} = P_{ij,2}$, and $c_{ij} = P_{ij,3}$ for $i,j = 1,2,3$ such that $a_{ij} + b_{ij} + c_{ij} = 1$.

Throughout this paper, we consider a case, where $\mu_{q_{11}} = \mu_{q_{13}} = \mu_{q_{33}} \neq \mu_{q_{22}} \neq \mu_{q_{12}} = \mu_{q_{23}}$. Hence, the following system of equations is produced, where

$$
\begin{aligned}
(W\mathbf{x})_1 &= a_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + a_{22}x_2^2 \\
&\quad + 2a_{12}\left(x_1x_2 + x_2x_3\right), \\
(W\mathbf{x})_2 &= b_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + b_{22}x_2^2 \\
&\quad + 2b_{12}\left(x_1x_2 + x_2x_3\right), \\
(W\mathbf{x})_3 &= c_{11}\left(x_1^2 + 2x_1x_3 + x_3^2\right) + c_{22}x_2^2 \\
&\quad + 2c_{12}\left(x_1x_2 + x_2x_3\right).
\end{aligned} \tag{12}
$$

The trajectory behavior of such Lebesgue QSO will be described by solving the above system of equations in the next section.

4. Dynamics of 3-Partition of Lebesgue QSO with Exponential Measure

The dynamics of 3-partition of Lebesgue QSO with exponential measure is described by fixed points and periodic points of the system of equations generated by the defined QSO in Equation (12). A corollary on the dynamics is supported by some examples provided in this section.

4.1. Fixed Point of 3-Partition of Lebesgue QSO with Exponential Measure

Let $x_1 + x_3 = u$ and $x_2 = v$. In what follows, one will have

$$W:\begin{cases} u' = (a_{11}+c_{11})u^2 + (a_{22}+c_{22})v^2 + 2(a_{12}+c_{12})uv, \\ v' = b_{11}u^2 + b_{22}v^2 + 2b_{12}uv. \end{cases} \quad (13)$$

Since $u+v=1$, then such an operator W has the similar form as the operator in Equation (6). Therefore, the system of equations in Equation (13) has either attracting fixed points or repelling fixed points with the existence of a cycle of second-order.

One may solve for v, where

$$\begin{aligned} f(v) &= b_{11}(1-v)^2 + 2b_{12}(1-v)v + b_{22}v^2 \\ &= (b_{11}-2b_{12}+b_{22})v^2 + 2(b_{12}-b_{11})v + b_{11}. \end{aligned}$$

Then, for $v = f(v)$, one may obtain the following solutions

$$v = \frac{-2(b_{12}-b_{11})+1\pm\sqrt{\Delta_v}}{2(b_{11}-2b_{12}+b_{22})}, \quad (14)$$

such that $\Delta_v = 4(1-b_{22})b_{11} + (1-2b_{12})^2$. Recall that $x_i \geq 0$ and $\sum_{i=1}^{m} x_i = 1$. Therefore, we choose the solution of $v = f(v)$, where $v = \frac{-2(b_{12}-b_{11})+1-\sqrt{\Delta_v}}{2(b_{11}-2b_{12}+b_{22})} \in (0,1)$.

Here, one can denote the unique solution of v as v^*. Consequently, one finds that

$$\begin{aligned} x_1^* &= (a_{11}-2a_{12}+a_{22})(v^*)^2 + 2(a_{12}-a_{11})v^* + a_{11}, \\ x_2^* &= v^*, \\ x_3^* &= (c_{11}-2c_{12}+c_{22})(v^*)^2 + 2(c_{12}-c_{11})v^* + c_{11}. \end{aligned} \quad (15)$$

Thus, it is shown that such an operator in Equation (12) has a unique fixed point denoted by $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*)$.

4.2. Periodic Points of 3-Partition of Lebesgue QSO with Exponential Measure

From Theorem 2.1 and Theorem 2.2, one knows that the system of equations in Equation (12) may have periodic points of period-2. To find the periodic points, one shall consider $f^2(v) = v$.

Let

$$f(v) = Av^2 + Bv + C, \quad (16)$$

where $A = b_{11}-2b_{12}+b_{22}$, $B = 2(b_{12}-b_{11})$, and $C = b_{11}$. Then,

$$\begin{aligned} f^2(v) &= A(Av^2+Bv+C)^2 + B(Av^2+Bv+C) + C \\ &= A^3v^4 + 2A^2Bv^3 + (2A^2C + AB^2 + AB)v^2 \\ &\quad + (2ABC + B^2)v + AC^2 + BC + C. \end{aligned}$$

Solving $f^2(v) = v$ will give us

$$\left(A^2v^2 + (AB+A)v + AC + B + 1\right)\left(Av^2 + (B-1)v + C\right) = 0. \quad (17)$$

From this, one can find the roots for

$$A^2v^2 + (AB+A)v + AC + B + 1 = 0 \quad (18)$$

and

$$Av^2 + (B-1)v + C = 0 \quad (19)$$

respectively.

For the quadratic equation in Equation (18), the roots are as follows

$$v_{\pm}^* = \frac{-(B+1)\pm\sqrt{\Delta_v - 4}}{2A}.$$

Meanwhile, one can see that the roots of the quadratic equation in Equation (19) are given in Equation (14). From this, one learns that v_+^*, v_-^*, and v^* are the fixed points of the function f^2 that belong to $(0,1)$. Thus, v_+^* and v_-^* are the periodic points of period 2 of the function f.

Denote the periodic points of the system of equations in Equation (12) as $\mathbf{x}_a^* = (x_{1,a}^*, x_{2,a}^*, x_{3,a}^*)$, and $\mathbf{x}_b^* = (x_{1,b}^*, x_{2,b}^*, x_{3,b}^*)$. By using simple algebra, one can easily obtain the following coordinates,

$$\begin{aligned} x_{1,a}^* &= (a_{11}-2a_{12}+a_{22})(v_-^*)^2 + 2(a_{12}-a_{11})v_-^* + a_{11}, \\ x_{2,a}^* &= v_+^*, \\ x_{3,a}^* &= (c_{11}-2c_{12}+c_{22})(v_-^*)^2 + 2(c_{12}-c_{11})v_-^* + c_{11}, \end{aligned}$$

and

$$\begin{aligned} x_{1,b}^* &= (a_{11}-2a_{12}+a_{22})(v_+^*)^2 + 2(a_{12}-a_{11})v_+^* + a_{11}, \\ x_{2,b}^* &= v_-^*, \\ x_{3,b}^* &= (c_{11}-2c_{12}+c_{22})(v_+^*)^2 + 2(c_{12}-c_{11})v_+^* + c_{11}. \end{aligned}$$

4.3. Applications

Corollary 4.1 The system of equations in Equation (12) has an attracting unique fixed point when $0 < \Delta_v < 4$, and a repelling fixed point and periodic points of period-2 when $4 < \Delta_v < 5$.

Here, we present some applications of Theorem 2.2 on the Lebesgue QSO with exponential measure to show the existence of fixed point and periodic points of the system of equations in Equation (12).

Example 4.1 Let $A_1 = [0, 0.9)$, $A_2 = [0.9, 0.925]$, and $A_3 = (0.925, 1]$. Choose the parameters, $q_1 = q_3 = q_5 = 2\times10^{23}$, $q_2 = 3$, and $q_4 = q_6 = 1$. Using the measure in Equation (8), one may calculate the heredity coefficients, where

$$a_{11} = 0.004676242238,\ a_{22} = 0.8274007515,$$
$$a_{12} = 0.8660659832,$$
$$b_{11} = 0.9821177258,\ b_{22} = 0.1316660499,$$
$$b_{12} = 0.1013157581,$$
$$c_{11} = 0.0132060320,\ c_{22} = 0.0409331986,$$
$$c_{12} = 0.0326182587.$$

Then, to solve for the second coordinate, one may have $f(v) = 0.9111522595v^2 - 1.761603936v + 0.9821177258$.

Eventually, one shall obtain the solution of v that belongs to $(0,1)$, where $v^* = 0.4115025691$. The other coordinates can be found as shown in Equation (15). Consequently,

$$x_1^* = 0.5611941883,\ x_2^* = 0.4115025691,$$
$$x_3^* = 0.02730324231.$$

Since $\Delta_v = 4.047021159$, one knows that such a fixed point is repelling.

To obtain the periodic points, one needs to find the solutions for $f^2(v) = v$, where

$$f^2(v) = 0.7564371844v^4 - 2.92496168v^3 + 2.853147617v^2 - 0.049525088v + 0.1308719496.$$

From this function, one will have the following roots,

$$v_+^* = 0.5369286806,\ v_-^* = 0.2989403276.$$

Therefore, the periodic points are

$$x_{1,a}^* = 0.4392508048,\ x_{2,a}^* = 0.5369286806,$$
$$x_{3,a}^* = 0.02382051424,$$

and

$$x_{1,b}^* = 0.6702069410,\ x_{2,b}^* = 0.2989403276,$$
$$x_{3,b}^* = 0.03085273100.$$

Example 4.2 Let $A_1 = [0, 0.9)$, $A_2 = [0.9, 0.95]$, and $A_3 = (0.95, 1]$. Choose the parameters, $q_1 = q_3 = q_5 = 20$, $q_2 = 3$, and $q_4 = q_6 = 5$. Simple calculations give us the heredity coefficients, where

$$a_{11} = 0.7244036116,\ a_{22} = 0.8274007515,$$
$$a_{12} = 0.8031505623,$$
$$b_{11} = 0.1482662300,\ b_{22} = 0.08928934463,$$
$$b_{12} = 0.1028306079,$$
$$c_{11} = 0.1273301584,\ c_{22} = 0.08330990387,$$
$$c_{12} = 0.0940188298.$$

One gets the following function to solve for the second coordinate, where

$$f(v) = 0.03189435883v^2 - 0.0908712442v + 0.1482662300.$$

Again, one can easily obtain the solution of v that belongs to $(0,1)$, where $v^* = 0.1364598664$. Consequently,

$$x_1^* = 0.7448804080,\ x_2^* = 0.1364598664,\ x_3^* = 0.1186597255.$$

Since $\Delta_v = 1.171084646$, then the fixed point is attracting.

Thus, these examples support Corollary 4.1.

5. Regularity of 3-Partition of Lebesgue QSO with Exponential Measure

Definition 5.1 A quadratic stochastic operator V is called regular if the limit $\lim_{n\to\infty} V^n \lambda$, exists for any initial point $\lambda \in S(X, F)$.

According to Definition 5.1, the existence of the sequence limit of such an operator determines its regularity. In the previous section, the existence of attracting fixed point and repelling fixed point of the QSO W in Equation (12) is shown.

Hence, it is possible to conclude that the operator W in Equation (12) is either regular or nonregular transformation, depending on the parameters and defined partitions. The following statement is then established.

Corollary 5.1 *If* $0 < \Delta_v < 4$, *then a two-dimensional QSO in Equation (12) is a regular, and if* $4 < \Delta_v < 5$, *then there exists a second-order cycle, and all trajectories tend to this cycle except the stationary trajectory starting with unique fixed point*.

6. Conclusions

In this paper, we construct a class of Lebesgue QSO with exponential measure generated by 3-partition defined on the continual state space, $X = [0,1]$. We consider a specific case, where $\mu_{q_{11}} = \mu_{q_{13}} = \mu_{q_{33}} \neq \mu_{q_{22}} \neq \mu_{q_{12}} = \mu_{q_{23}}$, which results on the system of equations in Equation (12). Also, the form of attracting and repelling fixed points, as well as the form of the period-2 periodic points on S^2 are shown. At the end of the investigation, we discuss the regularity of such Lebesgue QSO through the existence of fixed points. It is proven that if there exists an attracting unique fixed point, then the operator W is a regular. It follows that if the fixed point is repelling, then such an operator is nonregular as there exists a cycle of second-order.

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