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A Study on Sylow Theorems for Finding out Possible Subgroups of a Group in Different Types of Order

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Abstract This paper aims at treating a study on Sylow theorem of different algebraic structures as groups, order of a group, subgroups, along with the associated notions of automorphisms group of the dihedral groups, split extensions of groups and vector spaces arises from the varying properties of real and complex numbers. We must have used the Sylow theorems of this work when it's generalized. Here we discuss possible subgroups of a group in different types of order which will give us a practical knowledge to see the applications of the Sylow theorems. In algebraic structures, we deal with operations of addition and multiplication and in order structures, those of greater than, less than and so on. It is through the study of Sylow theorems that we realize the importance of some definitions as like as the exact sequences and split extensions of groups, Sylow p-subgroup and semi-direct product. Thus it has been found necessary and convenient to study these structures in detail. In situations, where it was found that a given situation satisfies the basic axioms of structure and having already known the properties of that structure. Finally, we find out possible subgroups of a group in different types of order for abelian and non-abelian cases.

Keywords Dihedral Group, Exact Sequences, Split Extensions of Groups, Lagrange's Theorems, Sylow p-Subgroup, Sylow Theorems

1. Introduction

It's not true for any number dividing the order of a group; there exists a subgroup of that order. For example, the group S_4 of even permutations on the set {1, 2, 3, 4} has order 12, yet there does not exist a subgroup of order 6. As usual, we can use Lagrange's Theorem to evaluate subgroups of group of different orders such as order $2, 4, 6, 8, 9, 10, 12, ...$ etc., i.e. whose order not so high (Not higher order groups). But it is not possible to evaluate subgroups of higher order group as like as $30, 35, 40, 45, 50, ...$ etc. by using Lagrange's Theorems. For this case, applying P-Sylow theorems we can easily evaluate all possible subgroups of any higher order groups.

The Sylow theorems are very important part of finite group theory and the finite simple groups [1, 3]. Here at present, we discussed subgroups of a group in different types of order by using Sylow theorems where subgroups of a groups of order 30 and 42 by using Sylow theorems [4]. The order of sylow p-subgroup of a finite group G is P^n, where n the multiplicity of p in the order of is G and any subgroup of order p^n is a Sylow p-subgroup of G.

2. Preliminaries

Dihedral Group

A dihedral group is the group of symmetries of a regular polygon, which includes rotations and reflections. Dihedral groups are among the simplest examples of finite groups, and they play an important role in group theory, geometry, and chemistry. It's denoted by D_n

Definition of p-group

Any finite group G with $o(G)=p^m$ where p is a prime number and m is a positive integer is called a p-group.

Example: The group G with $o(G)=9=3^2$ is a 3-group.

Definition of Sylow p-subgroup

Suppose G is a finite group of order $p^m n$, where p is not a divisor of n. A subgroup H of G is said to be a Sylow p-subgroup if

i). $o(H)=p^m$.

ii). p^m is a divisor of o (G), whereas p^{m+1} is not a divisor of o (G).

Index of a Group

Let G be a group. Let H be a subgroup of G.

The number of distinct left (or right) cosets of H in G is called index of H in G and is denoted by $[G:H]$ or by $i_G(H)=o(G)/o(H)$

Sylow's First Theorem [5]

Let G be a finite group and p be a prime number. If m is the largest non-negative integer such that p^m is a divisor of $\circ(G)$, then G has a subgroup of order p^m.

Sylow's Second Theorem:

Let G be a finite group and let p be a prime number such that p is a divisor of $\circ(G)$. Then, all sylow p-subgroups of G are conjugates of one another.

Sylow's Third Theorem

Let G be a finite group and p be a prime number such that $p \mid \circ(G)$. Then the number of sylow p-subgroups is of the form $1+mp$, where m is some non-negative integer.

Sylow's Fourth Theorem

The number of Sylow p-subgroups of a finite group is congruent to $1(\bmod p)$.

Sylow's Fifth Theorem

The number of Sylow p-subgroups of a finite group is a divisor of their common index.

Automorphisms Group of the Dihedral Group D_4

Let $D_4=\{e, x, x^2, x^3, y, yx, yx^2, yx^3\}$ with the defining relation $x^4=y^2=e$, $y^{-1}xy=x^{-1}$, be the dihedral group of order 8.

Now, the conjugate classes of D_4 are:

$$\{e\}, \{x^2\}, \{x, x^3\}, \{y, yx, yx^2, yx^3\}.$$

So, $D_4/\{e, x^2\} \cong D_4$ to a group of order 4. So, D_4 has 4 inner automorphisms, one of which is the identity. Then, let the other 3 inner automorphisms be α, β, γ .Now, if x is fixed by α then $\alpha(e)=e$, $\alpha(x)=x$ and $\alpha(y)=y, yx, yx^2, or\ yx^3$.

But $\alpha(y)\neq y$, for if $\alpha(y)=y$ then $\alpha = Id$, which is not possible. Then, let $\alpha(y)=yx^2$ and hence $\alpha(yx)=\alpha(y)\alpha(x)=yx^3$ and therefore, $\alpha^2 = Id$ .Next, if y is fixed by β then $\beta(e)=e$, $\beta(y)=y$ and $\beta(x)=x^{-1}$ $\beta(yx)=\beta(y)\beta(x)=yx^{-1}$ and $\beta^2=Id$. Then $\gamma(e)=e$ and $\gamma(yx)=yx$ and $\gamma(x)=x^{-1}$, $\gamma(y)=yx^2$ and $\gamma^2=Id$ .Hence, we have $\gamma^2=\beta^2=\alpha^2=Id$ and also we have $\alpha\beta=\beta\alpha=\gamma$ and $\alpha\gamma=\gamma\alpha.$ Therefore inner $Aut(D_4)=\{Id, \alpha, \beta, \beta\alpha\}\cong C_2\times C_2$ with $\alpha^2=\beta^2=Id$ and $\alpha\beta=\beta\alpha$.

Now, we consider the mapping $f: D_4 \to D_4$. With $f(e)=e$ and $f(x)=x\, or\, x^3$ .So, let $f(x)=x$ and assume that $g(x)=\alpha\, f(x)$ then $g(x)=\alpha(x)=x$ and $g(y)\neq x^2$ for x^2 is a central element and hence $g(y)=y, yx, yx^2, or\ yx^3$.

If $g(y)=y,$ then $g=Id$, and hence $g(y)\neq y$

If $g(y)=yx^2$ then $g=\alpha$ and hence $g(y)\neq yx^2$.

If $g(y)=yx$ then $g(yx)=yx^2$ and $g^4=Id$.

Then, we have, $\beta g\beta=g^{-1}$ with $g^4=\beta^2=Id$, and also $\gamma g\gamma=g^{-1}$, with $g^4=\gamma^2=Id$. Therefore,

$Aut(D_4) \cong \{\beta, g\}$ with $g^4 = \beta^2 = Id$ and $\beta^{-1} g \beta = g^{-1}$.

Automorphisms Group of the Dihedral Group D_6

Let $D_6 = \{e, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$

With defining relation $x^6 = y^2 = e$ and $y^{-1}xy = x^{-1}$, be a dihedral group of order 12.
Now, the conjugate classes are:

$$\{e\}, \{x, x^5\}, \{x^2, x^4\}, \{x^3\}, \{y, yx^2, yx^4\}, \{yx, yx^3, yx^5\}.$$

So, $D_6 / \{e, x^3\} \cong D_6$ to a group of order 6. Then D_6 has 6 inner automorphisms one of which is the identity. Let the other inner automorphisms be Y, Z, U, V, T .Now, if x is fixed by Y then $Y(e) = e$, $Z(x) = x$ and $Y(y) = yx^2$ and hence $Y(yx) = yx^5$.Then $Y^3 = Id$. Next, if y is fixed by Z then $Z(e) = e$, $Z(y) = y$, and $Z(x) = x^{-1}$ and $Z(yx) = yx^{-1}$ and then $Z^2 = \mathrm{Id}$. Next, if yx is fixed by U then $U(e) = e$, $U(yx) = yx$, and $U(x) = x^{-1}$ and $U(y) = yx^2$. Lastly, if yx^5 is fixed by T then $T(e) = e$, $T(yx^5) = yx^5$ and $T(x) = x^{-1}$ and $T(yx) = yx^3$ and then $T^2 = Id$, and hence we have, $Y^3 = Z^2 = U^2 = V^3 = T^2 = Id$ and by calculation we have, $Y^2 = V$, $TU = V = ZT$, $UT = Y = TZ$ and hence $Z^{-1}YZ = Y^{-1}$, $U^{-1}VU = V^{-1}$, $T^{-1}YT = Y^{-1}$.

Therefore, inner $Aut(D_6) = \{Z, Y\} \cong D_3 \cong S_3$ with $Z^{-1}YZ = Y^{-1}$ and $Y^3 = Z^2 = Id$.

Now, consider the mapping $S : D_6 \to D_6$

Let $S(e) = e$ then $S(x) = x$ or x^5 and so let $S(x) = x^5$ and put $M = US$ then $M(x) = US(x) = U(x^5) = x$

Now, $M(y) \neq x^3$ for x^3 is a central element.

If $M(y) = y$ then $M = Id$ and hence $M(y) \neq y$

If $M(y) = yx^2$ then $M = U$ and hence $M(y) \neq yx^2$

If $M(y) = yx^4$ then $M = Y$ and hence $M(y) \neq yx^4$

If $M(y) = yx$ then $M(yx) = yx^2$ and $M^6 = Id$.

Now, $MZ = ZM^5$, $MT = TM^5$ and $MU = UM^5$, $Aut(D_6) = \{M, Z\} \cong D_6$ with $M^6 = Z^2 = Id$ and $Z^{-1}MZ = M^{-1}$.

3. Result and Discussion

Here, we discuss a Study of Sylow's theorem with all possible subgroups of a group in different types of order for abelian and non-abelian cases.

All Group of Order 4

Abelian Case

Let G be a group of order 4.

1. If G has an element of order 4, then $G \cong C_4$, order 4 of a cyclic group, i.e. $C_4 = \{e, a, a^2, a^3\}$ with $a^4 = e$, the identity element of C_4.
2. If G has no element of order 4 but G has an element of order 2(2divides4), let $x \in G$ be an element of order 2 and let G_1 be the group generated by $x \in G$ then $G_1 \subset G$ and assume that there is an element $y \in G$ such that $y \notin G_1$ but $y^2 \in G_1$ and let G_2 be the group generated by y then the only possible case is that $y^2 = e$, the identity element of G. Hence, $G = G_1 G_2$ and $G_1 \cap G_2 = e$ and so $G \cong G_1 \times G_2$.

Non-Abelian Case

Let G be a group of order 4 then G cannot be non-abelian.

All Group of Order 6 [6]

Abelian Case

Let G be a group of order 6

1. If G has an element of order 6 then $G \cong C_6$ order 6 of a cyclic group, i. e. $C_6 = \{e, a, a^2, a^3, a^4, a^5\}$ with $a^6 = e$, the identity element of C_6.
2. If G has no element of order 6 but G has an element of order 2 (2 divides 6), Let x be an element of order 2 and $x \in G$ and let G_1 be the group generated by x, then $G_1 \subset G$ and assume that $\exists\ y \in G$ s. t. $y \notin G_1$. But $y^2 \in G$ and let G_2 be a group generated by y. If $y^2 = e$ then $G = G_1 G_2$ and $G_1 \cap G_2 = e \Rightarrow G \cong G_1 \times G_2$

which is not possible. So $y^2 \neq e$ and $y^2 \in G_1$ then $y^2 = x$, so $y^4 = e$. But G has no element of order 4 (4 does not divide 6). Hence there exists no element $y \in G$ s. t. $y \notin G_1$; $y^2 \in G_1$.

Assume that G has an element of order 3 (3 divides 6) and let z be such an element .let G_3 be the group generated by z then if $G = G_1G_3$ and $G_1 \cap G_3 = e$ then $G \cong G_1 \times G_3 \cong C_6$ for 3 and 2 and 3 are relatively prime. So, there is an abelian group of order 6 s. t. $G \cong C_6 \cong C_3 \times C_2$

Non-Abelian Case

Let G be a group of order 6.

1. If G has no element of order 6. Assume that G has an element of order 3 and let $x \in G$ and x is of order 3 then $[G : G_1] = 2$, there G_1 is the group generated by x and G_1 is normal subgroup of G and so $\exists$ $y \in G$ s. t. $y \notin G_1$; $y^2 \in G_1$. Now if $y^2 = x$ then $y^6 = x^3 = e$, the identity of G which is not possible. Similarly $y^2 \neq x^{-1}$, there the only possible cases $y^2 = e$. Hence, we have $y^2 = x^3 = e$ and then $G = \{e, x, x^2, y, yx, yx^2\}$ where $y^{-1}xy = e$ or x, x^2, y, yx, yx^2 . If $y^{-1}xy = e$ then $x = e$ this is not possible. If $y^{-1}xy = x$ then $xy = yx$ which is an abelian case, this is not to be considered under non-abelian case.

If $y^{-1}xy = y$ then $xy = y^2 \Rightarrow x = y$, this is not possible.

If $y^{-1}xy = yx$ then $xy = y^2x$ is not possible.

If $y^{-1}xy = x^2$ then $xy = yx^2 = yx^{-1}$ this is only possible case.

Note that $x^3 = e \Rightarrow x^2 = x^{-1}$.

So, G= $\{e, x, x^2, y, yx, yx^2\}$ with defining relation: $x^3 = y^2 = e$ and $y^{-1}xy = x^{-1}$ and $G \cong S_3 \cong D_3$ where $S_3 =$ Symmetric group on three symbols and D_3 is called the dihedral group of order 6.

All Group of Order 8

Abelian Case

If G is abelian then there are three possibilities.

Let G be a group of order 8.

1. If G has an element of order 8. Then $G \cong C_8$ order 8 of a cyclic group, i. e.

$G = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7\}$ with $a^8 = e$, the identity of G. So $G \cong C_8$.

If G has no element of order 8, but G does have an element of order 4. Every non-identity element of G has order 2, now (ii) Let $a \in G$, a is of order 4. Then, since $(a) \subset G$, then there is an element $b \in G$, $b \notin (a)$ .If $b^2 = e$ then $G = (a)(b)$ and $(a) \cap (b) = \{e\}$. Thus, $G \cong C_4 \times C_2$ .If $b^2 \neq e$ then $b^2 \in (a)$, for otherwise $|(b)| = 4$ and $(a) \cap (b) = \{e\}$ and $|G| \geq 16$. Hence, $b^2 \in (a)$ and since $|b^2| = 2$, $b^2 = a^2$. Now let $c = ba^{-1}$. Then $c \notin (a)$ and $c^2 = b^2a^{-2} = e$. Thus, $G = (a)(c) \cong C_4 \times C_2$.

We know that, if $G \neq \{e\}$ be a finite group all of whose elements different from identity have order p a prime, then for integer ≥ 1, $G \cong A_1 \times A_2 \times \ldots \ldots \ldots \times A_n$, where $A_i \cong C_P$, The cyclic group of order P, for i =1,2, … … … …,n and $|G| = p^n$ .So, If G has order 2 (a prime), then G according by $G \cong C_2 \times C_2 \times C_2$, where 2 is a prime and $2^3 = 8$.

Non-abelian Case

We assume that G has no element of order 8, for if did, this would imply $G \cong C_8$ which is an abelian group. Also, not every element of G, $g \in G, g \neq e$, can have order 2. For if these are the case, G would be abelian group which has already been considered. Thus G has an element of order 4, but none of order 8. Let $a \in G$, a is of order 4. Then $[G : (a)] = 2$ and (a) is normal in G.

Hence $G/(a)$ has order 2, and so there exists an element $b \in G$ and $b \notin (a)$ and $b^2 \in (a)$.

If $b^2 = a$ then since a has order 4, b would have order 8 which is a contradiction.

Similarly $b^2 \neq a^3$. Hence (i) $b^2 = e$ or (ii) $b^2 = a^2$

1. If $b^2 = e$ then, We may express all elements of G in terms of a and b which are as follows: $\{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$. Here, we need to determine the product of ba . Since $b \notin (a)$, $ba \notin (a)$. If $ba = e$ then $a = e$, is a contradiction. If $ba = ab$, then G is abelian, another contradiction. If $ba = a^2b$ then $bab^{-1} = a^2$. But $|bab^{-1}| = 4$

while $|a^2| = 2$, again contradiction. Thus the only possibility that can hold $ba = a^3b$.

Hence G has 2 generators a and b, where $a^4 = e$, $b^2 = e$ and $ba = a^3b \Rightarrow bab^{-1} = a^{-1}$. So $G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ with $a^4 = b^2 = e$ and $ba = bab^{-1}$ which is isomorphic to D_4, a dihedral group of order 8.

2. If $b^2 = a^2$ then again
$G = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$.

Now as in (i) we must determine the product ba.

Exactly as in one, we find that we must have $ba = a^3b$, that is $bab^{-1} = a^{-1}$.

We know that G is generated by a and b, where $b^2 = a^2$, $a^4 = e$ and $bab^{-1} = a^{-1}$.

This group is called the quaternion group of order 8, Which is not isomorphic to D_4.

This group can also be with the group say, $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ with 1,-1 $\in Z$ and the remaining symbols are distinct and satisfy: $ij = k, jk = i, ki = j$ and $ji = -k, kj = -i, ik = -j$ and $i^2 = j^2 = k^2 = -1$, where i, j, k are unit vectors.

Finally we may list all the group of order.

1. C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$

2. Non-abelian

D_4 and Q_8

All Group of Order 10 [7]

Abelian Case

Let G be a group of order 10.

1. If G has an element of order 10 then $G \cong C_{10}$.

If G has no element of order 10 but G has an element of order 5 (5 divides 10) and let $x \in G$ be of order 5 and suppose that G_1 be the group generated by x. Then $G_1 \cong C_5$, order 5 of a cyclic group, i. e. $G_1 = \{e, x, x^2, x^3, x^4\}$ with $x^5 = e$. Assume that $\exists\ y \in G$ s. t. $y \notin G_1$ and $y^2 \in G_1$ and let G_2 be the group generated by y then the possible cases are:

$y^2 = e$, $y^2 = x$, $y^2 = x^2$, $y^2 = x^3$, $y^2 = x^4$.

If $y^2 = x$ then $y^{10} = e$ this is not possible

If $y^2 = x^2$ which is not possible

If $y^2 = x^3$ which is not possible

If $y^2 = x^4$ which is not possible

If $y^2 = e$ is the only possible case, then $G = G_1G_2$ and $G_1 \cap G_2 = e$

Then $G \cong C_5 \times C_2 \cong C_{10}$, for 2 and 5 are relatively prime.

Non-abelian Case

G can never have an element of order 10, for if it has an element of order 10 then $G \cong C_{10}$ which is an abelian group. Then assume that G has an element of order 5 and Let $x \in G$ and let G_1 be group generated by x, then $|G : G_1| = 2 \Rightarrow G_1$ is a normal subgroup of G. So, $\exists\ y \in G$ s. t. $y \notin G_1$; $y^2 \in G_1$. If $y^2 = x$ then, it implies that y is of order 10 which is not possible.

Hence, $y \neq x$ and $y^2 \neq x^{-1}, y^2 \neq x^2, y^2 \neq x^3$ and hence $y^2 = e$ is the only possible case. Then $G = G_1 \times G_2$ and $G_1 \cap G_2 = e$ Where G_2 is the group generated by y and
$G = \{e, x, x^2, x^3, x^4, y, yx, yx^2, yx^3, yx^4\}$.

Now $y^{-1}xy = e$ or $x, x^2, x^3, x^4, y, yx, yx^2, yx^3, yx^4$. $y^{-1}xy = x$ then G is abelian, so it is not be considered. Also the other case namely $y^{-1}xy = x^2$ or x^3 or x^4 or yx or yx^2 or yx^3 or yx^4 are not possible. So, the only possible case is that $y^{-1}xy = x^{-1}$ and hence $G \cong D_5$, the Dihedral group of order 10 and
$G = \{e, x, x^2, x^3, x^4, y, yx^2, yx^3, yx^4\}$ with defining relations $x^4 = y^2 = e$, $y^{-1}xy = x^{-1}$.

Note:

Abelian group-1 and Non-abelian group-1

All Group of Order 12

Abelian Case

Let G be a group of order 12. If G has an element of order 12, then $G \cong C_{12} \cong C_4 \times C_3$, (4 and 3 are relatively prime).

If G has no element of order 12 but G has an element of 6(6 divides 12).

Let $x \in G$ be an element of order 6 and let G_1 be the group generated by x then $G_1 \subset G$ and assume that $\exists\ y \in G$ s. t. $y \notin G_1$ but $y \in G_1$ and let G_2 be the group generated by y, then the only possible case is that

$y^2 = e$ and hence $G = G_1 G_2$ and $G_1 \bigcap G_2 = e$ then $G \cong C_6 \times C_2$ and since 3 and 2 are relatively prime then we have $G \cong C_6 \times C_2 \cong C_3 \times C_2 \times C_2$.

Non-Abelian Case

In this case, the group G can not have an element of order 12 for if it has then $G \cong C_{12}$ which is an abelian group. Assume that G has an element of order 6 and let G_1 be the group generated by $x \in G$ where x is of order 6 then $[G : G_1] = 2$ and G_1 is normal subgroup of G. Then $\exists$ $y \in G$ s. t. $y \notin G_1$ but $y^2 \in G_1$. Now, if $y^2 = x$ then $y^{12} = e$, this is not possible.

Hence $y^2 \neq x$ and also $y^2 \neq x^5$ and if $y^6 = e$ and hence $y \in G_1$ which is not also possible.

Hence $y^2 \neq x^2$ and also $y^2 \neq x^4$.

Then the only possibilities are $y^2 = e$ and $y^2 = x^3$.

1. If $y^2 = e$ then the elements of G in terms of x and y can be put as $G = \{e, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$ Now let us determine the product xy. Since $y \notin G_1$ then $xy \notin G_1$. If $xy = y$ then $x = e$, which is not possible and hence $xy \neq y$. If $xy = yx$ then G is abelian and hence $xy \neq yx$. If $xy = yx^2$ then $y^{-1}xy = x^2$ which is not possible for the order of $y^{-1}xy$ is 6 and the order of x^2 is 3 and hence $xy \neq yx^2$. Similarly we have that $xy \neq yx^3$ and $xy = yx^4$.

Now, the only possibility is that $xy = yx^5$ which gives that $y^{-1}xy = x^5 = x^{-1}$. Hence, G is an non abelian group of order 12 generated by the elements x and y with the following defining relations: $x^6 = y^2 = e$, $y^{-1}xy = x^{-1}$. This is exactly the dihedral group of order 12 which is denoted by D_6.

2. If $y^2 = x^3$ then G is an other non abelian group of order 12 generated by x and y with the following defining relations: $x^6 = y^2 = e$, $y^2 = x^3$ and $y^{-1}xy = x^{-1}$. This group is called the dicyclic group of order 12 and is denoted by Δ_6.

Further investigation for non-abelian group of order 12, we have $12 = 2^2.3$ and both 2 and 3 are primes. Let G has no element of order 4 and it has only elements of order 2. Then the order of a 2-Sylow subgroup is 4 and $S_2 \cong C_2 \times C_2$ and $a^2 = b^2 = e$ where S_2 is the 2-Sylow subgroup and $ab = ba$ and let $c^3 = e$ and let G be generated by a, b, c. Let G_1 be the group generated by the element c. Put $d = cac^{-1}$ which has order 2 and assume that $d \in S_2$ and c is invariant. Then $a^{-1}ca = c^t$, $b^{-1}cb = c^q$, if $t = q = 1$ then G is abelian, so put $t = q = 1$, and hence $t = q = 2$ then $a^{-1}ca = c^2 = c^{-1}$ and $b^{-1}cb = c^{-1}$, so $(ba)^{-1}c(ba) = a^{-1}b^{-1}cba = a^{-1}(b^{-1}cb)a = a^{-1}c^{-1}a = c.$ Put $ba = f$ then $f^{-1}cf = c$ and hence $cf = fc$. So, $G \cong D_3 \times C_2 \cong D_6$, which has already been found. If c is not invariant, then by the third Sylow's theorem, S_3 the 3-Sylow subgroup is generated by c and has 4 conjugates. Now $4 = \lfloor G : N(S\ ) \rfloor$ and hence $[N(S_3) : \{e\}] = 3$ and so $N(S_3) = C(S_3) = C_3$ and hence by Burnside's theorem, there is an invariant normal complement $C_2 \times C_2$. Now $c^{-1}ac = a^t b^q$ such that $t, q = 0,1$, then we have if $t = 1,\quad q = 0$ then $c^{-1}ac = a$ and hence $ca = ac$.

Put $x = ac = ca$ then we have $x^6 = e$ and hence we have the cyclic group C_6 generated by x.Now we have to determine the product xb, since $b \notin C_6$ then so is xb. If $xb = bx$, we have $G \cong C_6 \times C_2$ which is impossible for G is non abelian and so $xb \neq bx$. If $xb = bx^2$ then $bxb = b^2x^2 = x^2$, where $(b^2 = e)$, which is not possible for x^3 is of order 2 and bxb is of order 6, hence $xb \neq bx^2$. If $xb = bx^3$ then $bxb = x^3$ which is not possible for x^3 is of order 2 and bxb is of order 6 and hence $xb \neq bx^3$. If $xb = bx^4$ then $bxb = x^4$, which is also impossible. If $xb = bx^5$ then $bxb = x^{-1}$ is the only possibility.

Hence, we have $bxb = x^{-1}$ with $b^2 = x^6 = e$ then $G \cong D_6$ which has already been found. If $t = 1, q = 1$, then $c^{-1}ac = ab$ and hence $c(ab) = ac$ then the element of G is in terms of a, c, ab, where

$$G = \begin{Bmatrix} e, a, (ab), c, c^2, ca, c^2a, \\ cb, c^2b, b, c(ab), c^2(ab) \end{Bmatrix}$$

Now, let us put

$$a = (12)(34),\ b = (13)(24),\ c = (123)$$

Then we have

$ca = (234)$, $cb = (142)$, $ab = (14)(23)$, $c^2a = (314)$, $c^2b = (234)$, $c(ab) = (134)$, $c^2(ab) = (124)$, which are the elements of S_4.

Then the group generated by a,b,c will be the non-abelian group of order 12 which is denoted by A_4 (alternating group on four symbols) elements are as follows:

$$A_4 = \left\{ \begin{array}{l} (1),(12)(34),(13)(24),(14)(23),(123),(132), \\ (124),(142),(143),(234)(134)(243) \\ \end{array} \right\}$$

The group is called Alternating group on four symbols and is a subgroup of S_4. Hence $G \cong A_4$

Remarks

The groups of order 12may be listed as follows:

Abelian

1. C_{12} 2. $C_6 \times C_2$

Non-Abelian

1. D_6 2. Δ_6 3. A_4

All Group of Order 20 [8]

Abelian Case

Let G be a group of order 20.

1. If G has an element of order 20 then $G \cong C_{20}$ and since 5 and 4 are relatively prime then $G = C_5 \times C_4$.
2. If G has no element of order 20 but it has an element x of order 10. Let $x \in G$ be of order 10 and let the group generated by x be G_1 then $G_1 \subset G$ and let there be an element $y \in G$ such that $y \notin G_1$ but $y^2 \in G_1$ and also let G_2 be the group generated by y. Now, if $y^2 = e$ then $G = G_1G_2$ and $G_1 \bigcap G_2 = e$ and hence $G \cong G_1 \times G_2 \cong C_{10} \times C_2$. Again, if $y^2 \neq e$ then $y^2 \in G_1$ for otherwise the order of y is 10 and $y^2 = x^2$ then let $z = yx^{-1}$ but $z \notin G_1$ and $z^2 = y^2x^{-2} = e$. Thus $G = G_1G_3 \cong C_{10} \times C_2$ where G_3 is the group generated by z. Let us analyses $C_5 \times C_4$ with may be $C_5 \times (C_2 \times C_2)$ with $c^5 = y^2 = z^2 = e$ and $yz = zy$.

Now, the generator of $C_5 \times (C_2 \times C_2)$ has order 10 and the generator of $C_{10} \times C_2$ has order 10 and so, we have $G \cong C_{10} \times C_2 \cong C_5 \times (C_2 \times C_2)$.

Non-Abelian Case

In this case G can not have an element of order 20 for if it has then G is an element of order 10. Let $x \in G$ and x is of order 10 and let G_1 be the group generated by x then $[G : G_1] = 2$ and hence G_1 is an normal subgroup of G. Then there exists an element $g \in G$ such that $y \notin G_1$ and $y^2 \in G_1$ .Now, if $y^2 = x$ then $y^{20} = x^{10} = e$ which is not possible for G does not have an element of order 20 and hence $y^2 \neq x$ and similarly, we have $y^2 \neq x^3$, $y^2 \neq x^7$ and $y^2 \neq x^9$. Hence the possible cases are $y^2 = e$, $y^2 = x^2$, $y^2 = x^4$, $y^2 = x^5$, $y^2 = x^6$ and $y^2 = x^8$.

1. If $y^2 = e$ then all the elements of G can be expressed in term of x and y in the following way:
$$G = \left\{ \begin{array}{l} e,x,x^2,x^3,x^4,x^5,x^6,x^7,x^8,x^9,y,yx, \\ yx^2,yx^3,yx^4,yx^5,yx^6,yx^7,yx^8,yx^9 \end{array} \right\}$$
Now, let us determine the product xy Since $y \notin G_1$ then $xy \notin G_1$ .If $xy = y$ then $x = e$ which is not possible and hence $xy \neq y$ If $xy = yx$ then G is abelian which not possible and hence $xy \neq yx$. If $xy = yx^2$ then $y^{-1}xy = x^2$ which is not, for x^2 is of order 10. Hence, $xy \neq yx^2$ .Similarly, $xy \neq yx^4$, $xy \neq yx^5$, $xy \neq yx^6$, and $xy \neq yx^8$ And hence the only possible case is that $xy = yx^9$.If $xy = yx^9$ then $y^{-1}xy = x^{-1}$ and so G is non-abilian group of order 20 and G is generated by x and y with the defining relations: $x^{10} = y^2 = e$, $y^{-1}xy = x^{-1}$. This is exactly the dihedral group of order 20 and is denoted by D_{10}.
2. If $y^2 = x^2$, then
$$y^2 = y^{-1}y = y^{-1}x^2y = (y^{-1}x)(xy) = (y^{-1}x)(yx^{-1}).$$
$$= y^{-1}(yx^{-1})x^{-1} = x^2$$
$y^2 = x^8$ which is not possible by (b.1).
3. If $y^2 = x^4$ then $y^2 = y^{-1}y^2y = y^{-1}x^4y = x^6$ this is not possible

4. If $y^2 = x^5$ then $y^2 = y^{-1}y^2y = y^{-1}x^5y = x^5$ this is not possible

Hence, G is a non-abelian group generated by x and y with the following defining relations $x^{10} = e$, $y^2 = x^5$ and $y^{-1}xy = x^{-1}$. This group is called dicyclic group of order 20 and is denoted by Δ_{10}.

5. If $y^2 = x^6$ then $y^2 = y^{-1}x^6y = x^4$ this is not possible
6. If $y^2 = x^8$ then $y^2 = y^{-1}x^8y = x^2$ this is not possible

Further Investigation for Non-Abelian Group of Order 20

Let G have no element of order 10 but an element x of order 5 and hence 5-Sylow subgroup is isomorphic to C_5.Assume that there is another element y of order 4 and $y \in G$, Then the elements of G may be expressed as:

$$G = \begin{Bmatrix} e, x, x^2, x^3, x^4, y, y^2, y^3, yx, yx^2, yx^3, yx^4, \\ y^2x, y^2x^2, y^2x^3, y^2x^4, y^3x^3, y^3x^4 \end{Bmatrix}$$

Now, let us determine the product xy. Here $xy \neq yx$ for if $xy = yx$ then is abelian which is not possible and also $xy \neq yx$ for $xy = y$ gives $x = e$ which is not possible again $xy = yx^2$ gives that $y^{-1}xy = x^2$ which is possible for $y^{-1}xy$ is of order 5 and also x^2 is of order 5. Hence, G is a non-abelian group of order 20 with no element of order 10 and G is generated by the elements x and y with the following defining relations: $x^5 = y^4 = e$ and $y^{-1}xy = x^2$ .This group is isomorphic to a subgroup of S_5, the symmetric group on 5 symbols.

If we put $x = (12345)$, $y = (1523)$, then the group $G \cong G_5$ (a subgroup of S_5)

$$= \begin{Bmatrix} (1),(12345),(13524),(14253),(15432),(1523), \\ (12)(35),(1325),(2453), \\ (1254),(1342),(1435),(13)(45), \\ (14)(23),(15)(24),(25)(34), \\ (14)(52),(1534),(2354),(1243) \end{Bmatrix}$$

Note

G_5 has only inner automorphism and identity is the centre, So it is called a complete group.

Remarks

The group of order 20 can be listed as follows:

Abelian

1. C_{20}, 2. $C_{10} \times C_2$

Non-Abelian

1. D_{10}, 2. Δ_{10}, 3. G_5

All Group of Order 28 [9]

(a) Abelian Case

We can list the Abelian groups of order 28 as follows:

$i.\ G \cong C_{28}$ $ii.\ G \cong C_2 \times C_{14}$

(b) Non – Abelian Case

We keep in mind that 28=2^2.7

2- Sylow Subgroups:

The number x of 2-Sylow subgroup of a group G of order 28 is $x \equiv 1(\mathrm{mod}\,ulo 2)$, where x= 1, 7.

1, 2-Sylow subgroup:

It implies that there is a proper normal subgroup in G which may be called N of order 2. Therefore, $N \cong C_2$

If $N \cong C_2$, then the sequence of group extension $\{e\} \to N \to G \to C_{14} \to \{e\}$

But, (1, 14) =1, so the extension splits.

Now, $Y : C_{14} \to Aut(C_2) \cong Id$ and (14, 1) =1, Where Y is a constant homomorphism and the relation is given by $b^{-1}ab = a^{-1}$ which is a commutative case. So, we exclude this case.

7, 2- Sylow subgroup

The normalized of 2-sylow subgroup $N(S_2)$ must have an invariant subgroup of order 2.

Now, the order of $N(S_2) = 4$ and so, $N(S_2) = C_4$ *or* $C_2 \times C_2$ *or* D_2 but $N(S_2) = D_2$ because none of them can have an invariant subgroup of order 2.

The possibilities are

$i.\ N(S_2) = C_4$ $ii.\ N(S_2) = C_2 \times C_{14}$ and so by Burnside's Theorem normal 2-complement exists. This will be abelian, so we exclude this case.

7-Sylow Subgroups

The number x of 7-sylow subgroups of a group G of order 28 is $x \equiv 1\ (\mathrm{mod}\,ulo\, 7)$, where x=1

1, 7-Sylow Subgroups

Any D_7 is a normal subgroup of G. The group extension is $\{e\} \to D_7 \to G \to H \to \{e\}$ and H is of order 4. But (7, 4) =1, so the extension splits. And so $Y : H \to Aut(D_7) \cong D_4$

(a) Let $H = D_4 \cong D_2 \times D_2$ then

KerY contains D_2 and it commutes with $D_7, D_2, \ldots$

This gives that $G = C_2 \times N$ where N is a non-abelian group of order 14, so that $N = D_{14}$. Therefore $G = D_{14}$.

(b) Let $H = C_2$ and C_2 has an element of order 7, so $Y(C_2)$ has order 1.

If $Y(C_2)$ has order 1 then, $G \cong D_7 \times C_2$ which is a non-abelian group of order 28.

Remarks

We can list the different groups of order 28 as follows:
Non-abelian groups:

1) $G = D_{14}$. 2) $G \cong D_7 \times C_2$

All Group of Order 50 [10]

(a) Abelian Case

We can list the Abelian groups of order 50 as follows:

$$i.\ \ G \cong C_5 \quad ii.\ \ G \cong C_5 \times C_{10}$$

(b) Non – Abelian Case

We keep in mind that $50=2.5^2$

2- Sylow Subgroups

The number x of 2-Sylow subgroup of a group G of order 28 is $x \equiv 1(\mathrm{mod}\,ulo2)$, where x= 1, 5, 25.

1, 2-Sylow Subgroup

It implies that there is a proper normal subgroup in G which may be called N of order 2. Therefore, $N \cong C_2$

If $N \cong C_2$, then the sequence of group extension $\{e\} \to N \to G \to C_{25} \to \{e\}$

But, (1, 25) =1, so the extension splits. Now, $Y : C_{25} \to Aut(C_2) \cong Id$ and (25, 1) =1, Where Y is a constant homomorphism and the relation is given by $b^{-1}ab = a^{-1}$ which is a commutative case. So, we exclude this case.

5, 2- Sylow Subgroup

The normalized of 2-sylow subgroup $N(S_2)$ must have an invariant subgroup of order 2.

Now, the order of $N(S_2) = 4$ and so, $N(S_2) = C_4 \ or\ C_2 \times C_2 \ or\ D_2$ but $N(S_2) = D_2$ because none of them can have an invariant subgroup of order 2.

The possibilities are

$i.\ \ N(S_2) = C_4 \quad ii.\ \ N(S_2) = C_2 \times C_{25}$ and so by Burnside's Theorem normal 2-complement exists. This will be abelian, so we exclude this case.

25, 2-Sylow subgroups

(a) The normalized of 2-sylow subgroup $N(S_2)$ must have an invariant subgroup of order 2.

(b) Now, the normal 2 components N of order 25 gives that $N \cong G_{25}$, then the group extension is given by

$\{e\} \to D_{25} \to G \to H \to \{e\}$ Where $H = D_2$. But $(2, 25) = 1$, so the extension splits.

(b.1). If $H = D_2$ then $Y : D_2 \to Aut(D_{25}) \cong D_3$ and $Y(D_2) = \{e\}$

(b.1.1). If $Y(G_2) = \{e\}$, then G=$D_2 \times D_{25}$,which is a non-abelian group of order 50

5-Sylow Subgroups

The number x of 5-Sylow subgroups of a group G of order 50 is $x \equiv 1(\mathrm{mod}\,ulo5)$; where $x = 1$.

1, 5-Sylow Subgroups

Any C_5 is a normal subgroup of G.

The group extension is $\{e\} \to C_5 \to G \to H \to \{e\}$ and H is of order 10.

But $(5,10) \neq 1$, so the extension splits and so $Y : H \to Aut(C_5) \cong C_4$.

(a) Let $H = C_{10} \cong C_5 \times C_2$ then $KerY$ contains C_2 and it does not commute with C_5, C_2. So, it has no non-abelian group of order 50.

(b) Let $H = A_4$ and A_4 has no quotient group of order 2 or 4, so $Y(A_4) = \{e\}$ and $G \cong C_5 \times A_4$ which has no non- abelian group of order 50.

(c) Let $H = D_5$ and D_5 has an element of order 5, so $Y(D_5)$ has order 2 or 1

(c.1) If $Y(D_5)$ has order 1 then $G \cong C_5 \times D_5$ which has no non-abelian group of order 50

(c.2) If $Y(D_5)$ has order 2, then $KerY = C_5$ and

$a^5 = c^7 = d^2$, $d^{-1}ad = a^{-1}$, $c^{-1}ac = a^{-1}$, $d^{-1}cd = c^{-1}$. So, a group generated by a and c is D_{25}

Therefore $G = D_{25}$ is a non- abelian group

Remarks

We can list the different groups of order 28 as follows:
Non-abelian groups:

(1) $G = D_{25}$.

(2) $G \cong C_5 \times D_5$

(3) $G \cong C_2 \times D_{25}$

4. Conclusions

The Lagrange's theorem seems somewhat limited. It seems intuitively clear that any order of subgroups of a group in use of the Sylow theorems. We hope that this work will be useful for group theory related to subgroups with abelian and non-abelian groups. Our results are possible subgroups of a group in different types of order by using Sylow's theorem. This result has found an extensive use in probability, statistics, information theory and geometrics etc. Then all expected results in this paper will help us to understand better solution of complicated order.

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Availability of Data and Materials

Data sharing not applicable to this article as no datasets were analyzed during the current study.

Competing Interests

The authors declare that they have no competing interests.

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