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Analysis of Limiting Ratios of Special Sequences

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Abstract In this paper, we have determined the limit of ratio of (n+1)th term to the nth term of famous sequences in mathematics like Fibonacci Sequence, Fibonacci – Like Sequence, Pell's Sequence, Generalized Fibonacci Sequence, Padovan Sequence, Generalized Padovan Sequence, Narayana Sequence, Generalized Narayana Sequence, Generalized Recurrence Relations of Fibonacci – Type sequence, Polygonal Numbers, Catalan Sequence, Cayley numbers, Harmonic Numbers and Partition Numbers. We define this ratio as limiting ratio of the corresponding sequence. Sixteen different classes of special sequences are considered in this paper and we have determined the limiting ratios for each one of them. In particular, we have shown that the limiting ratios of Fibonacci sequence and Fibonacci – Like sequence is the fascinating real number called Golden Ratio which is 1.618 approximately. We have shown that the limiting ratio of Pell's sequence is a real number called Silver Ratio and the limiting ratios for generalized Fibonacci sequence are metallic ratios. We have also obtained the limiting ratios of Padovan and generalized Padovan sequence. The limiting ratio of Narayana sequence happens to be a number called super Golden Ratio which is 1.4655 approximately. We have shown that the limiting ratios of Generalized Narayana sequence are the numbers known as super Metallic Ratios. We have also shown that the limiting ratio of generalized recurrence relation of Fibonacci type is 2 and that of Polygonal numbers and Harmonic numbers are 1. We have proved that the limiting ratio of the famous Catalan sequence and Cayley numbers are 4. Finally, assuming Rademacher's Formula, we have shown that the limiting ratio of Partition numbers is the natural logarithmic base e. We have proved fourteen theorems to derive limiting ratios of various well known sequences in this paper. From these limiting ratio values, we can understand the asymptotic behavior of the terms of all these amusing sequences of numbers in mathematics. The limiting ratio values also provide an opportunity to apply in lots of counting and practical problems.

Keywords Sequence, Recurrence Relation, Limiting Case, Limiting Ratio, Asymptotic Behaviour

1. Introduction

There are hundreds of amusing sequences that exist in mathematics each occurring in plenty of interesting counting problems. Each such sequence exhibits remarkable mathematical properties and these properties in turn are applied in several disciplines of science and technology, art and designs and in almost all human endeavors. In this paper, we shall mainly focus on the aspect of determining the limiting ratio of several important sequences in mathematics. The answers will provide an insight of how the terms of those sequences behave asymptotically.

2. Definition

The limiting ratio of any sequence is defined as the ratio of (n+1)th term to its nth term as $n \to \infty$. Thus the limiting ratio of the sequence

$$\left\{ s_n \right\}_{n=1}^{\infty} \text{ is } \frac{s_{n+1}}{s_n} \text{ as } n \to \infty \qquad (2.1)$$

We note an important fact that if λ is the limiting ratio of $\left\{S_n\right\}_{n=1}^{\infty}$ and if $r \geq 1$ is any integer, then as $n \to \infty$ we have

$$\frac{S_{n+r}}{S_n} = \frac{S_{n+r}}{S_{n+r-1}} \times \frac{S_{n+r-1}}{S_{n+r-2}} \times \frac{S_{n+r-2}}{S_{n+r-3}} \times \cdots \times \frac{S_{n+1}}{S_n} = \lambda \times \lambda \times \lambda \times \cdots \times \lambda = \lambda^r \quad (2.2)$$

3. Fibonacci Sequence

The Fibonacci sequence named after Italian mathematician and merchant Leonardo Fibonacci is defined recursively through

$$F_{n+2} = F_{n+1} + F_n, n \geq 0 \quad (3.1)$$

where $F_0 = 0, F_1 = 1$.

Using the initial two terms, and (3.1), the terms of the Fibonacci Sequence are given by 0,1,1,2,3,5,8,13,21,34,55,89,144,,. . .

Theorem 1:

The limiting ratio of the Fibonacci sequence is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$

Proof: From (3.1), we have $\frac{F_{n+2}}{F_n} = \frac{F_{n+1}}{F_n} + 1$. Thus if, λ is the limiting ratio then from (2.2), as $n \to \infty$ we have $\lambda^2 = \lambda + 1$, leading to the quadratic equation $\lambda^2 - \lambda - 1 = 0$ (3.2). Taking the positive root (as all the terms of the Fibonacci sequence are positive), we get $\lambda = \frac{1+\sqrt{5}}{2}$. This number is called Golden Ratio which is approximately 1.618. Thus the limiting ratio of Fibonacci sequence is the Golden Ratio given by $\varphi = \frac{1+\sqrt{5}}{2}$ (3.3). This completes the proof.

4. Fibonacci – Like Sequence

The Fibonacci – Like sequence is defined recursively by

$$FL_{n+2} = FL_{n+1} + FL_n, n \geq 0 \quad (4.1)$$

where $FL_0 = a, FL_1 = b$, and *a*, *b* are positive real numbers.

The terms of Fibonacci – Like sequence are $a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, \ldots$

We notice that the coefficients of *a*, *b* in the above terms are numbers in the Fibonacci sequence, fetching this name to them.

Theorem 2:

The limiting ratio of the Fibonacci – Like sequence is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$

Proof: From (4.1), we have $\frac{FL_{n+2}}{FL_n} = \frac{FL_{n+1}}{FL_n} + 1$. If λ is the limiting ratio, then from (2.2), as $n \to \infty$, we have $\lambda^2 = \lambda + 1$, leading to $\lambda^2 - \lambda - 1 = 0$ (4.2). Just as in section 3 with Fibonacci sequence, we note that the positive root of (4.2) will be the Golden Ratio $\frac{1+\sqrt{5}}{2}$. Thus the limiting ratio of Fibonacci – Like sequence is the Golden Ratio given by $\varphi = \frac{1+\sqrt{5}}{2}$ (4.3). This completes the proof.

We also note that the famous Lucas sequence is a special case of Fibonacci – Like sequence by taking *a* = 2, *b* = 1.

5. Pell's Sequence

The Pell's sequence is defined recursively by $P_{n+2} = 2P_{n+1} + P_n, n \geq 0$ (5.1) where $P_0 = 0, P_1 = 1$. The terms of the Pell's sequence are 0, 1, 2, 5, 12, 29, 70, 169, 408, 985, . . .

Theorem 3:

The limiting ratio of the Pell's sequence is the silver ratio $1+\sqrt{2}$

Proof: From (5.1), we have $\frac{P_{n+2}}{P_n} = \frac{P_{n+1}}{P_n} + 1$. If λ is the limiting ratio of Pell's sequence, then from (2.2) as $n \to \infty$, we have $\lambda^2 = 2\lambda + 1$ giving the quadratic equation $\lambda^2 - 2\lambda - 1 = 0$ (5.2). The positive root of this quadratic equation is $1+\sqrt{2}$. The real number $1+\sqrt{2}$ is called as Silver Ratio. Thus the limiting ratio of Pell's sequence is the Silver Ratio given by $1+\sqrt{2}$ (5.3). This completes the proof.

6. Generalized Fibonacci Sequence

The Generalized Fibonacci sequence is defined recursively by

$$GF_{n+2} = k\,GF_{n+1} + GF_n, n \geq 0, k > 0 \quad (6.1)$$

where $GF_0 = 0, GF_1 = 1$. Note that if *k* = 1, 2 then we get Fibonacci and Pell's sequence defined in sections 3 and 5 respectively.

The terms of the Generalized Fibonacci sequence are $0,1,k,k^2+1,k^3+2k,k^4+3k^2+1,\ldots$

Theorem 4:

The limiting ratio of the Generalized Fibonacci sequence is the metallic ratio $\frac{k+\sqrt{k^2+4}}{2}$.

Proof: From (6.1), we get $\frac{GP_{n+2}}{GP_n}=k\frac{GP_{n+1}}{GP_n}+1$. If λ is the limiting ratio, then using (2.2) and considering the limit as $n\to\infty$, we have

$$\lambda^2-k\lambda-1=0\ . \quad (6.2)$$

The positive real root of this quadratic equation is $\frac{k+\sqrt{k^2+4}}{2}$. We call the real number $\frac{k+\sqrt{k^2+4}}{2}$ as Metallic Ratio. Thus the limiting ratio of Generalized Pell's sequence is the Metallic Ratio given by $\frac{k+\sqrt{k^2+4}}{2}$ (6.3). This completes the proof.

We notice that for k = 1, 2 from (6.3) we get Golden Ratio and Silver Ratio which we have obtained as limiting ratios of Fibonacci and Pell's sequences respectively. In (6.3), if k = 3, we get a number $\frac{3+\sqrt{13}}{2}$ (6.4) called Bronze Ratio. For knowing more about Metallic ratios see **[1]**.

7. Padovan Sequence

The Padovan's sequence is defined recursively by

$$PD_{n+3}=PD_{n+1}+PD_n\ ,n\geq 0 \quad (7.1)$$

where $PD_0=1,PD_1=1,PD_2=1$. The sequence was named after British mathematician Richard Padovan which became famous after Ian Stewart mentioned about them in his two books.

We notice that in Fibonacci sequence, except the first two terms, each term is the sum of two previous terms, where as in Padovan sequence, except the first three terms, each term is the sum of one but previous two terms. Thus the fourth term of the sequence is the sum of second and first terms. Similarly the fifth term is the sum of fourth and third terms. The terms of Padovan sequence as defined through (7.1) are given by 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200,

Theorem 5:

The limiting ratio of the Padovan sequence is the plastic number 1.324718 approximately.

Proof: From (7.1), we get $\frac{PD_{n+3}}{PD_n}=\frac{PD_{n+1}}{PD_n}+1$. Thus, if λ is the limiting ratio, then as $n\to\infty$ using (2.2), we have $\lambda^3=\lambda+1$ leading to the cubic equation $\lambda^3-\lambda-1=0$ (7.2). Using Descarte's Rule of Signs, we see that equation (7.2) has only one positive real root the other two being complex conjugate roots. Using Newton – Raphson method, we see that the lonely positive root of (7.2) is given by 1.324718 approximately. The number 1.324718 which is root of (7.2) is known as Plastic Number. Thus the limiting ratio of the Padovan sequence is the plastic number 1.324718 approximately. This completes the proof.

8. Generalized Padovan Sequence

The Generalized Padovan sequence is defined recursively by

$$GPD_{n+3}=k\ GPD_{n+1}+GPD_n\ ,n\geq 0 \quad (8.1)$$

where $GPD_0=1,GPD_1=1,GPD_2=1$ and k is a positive real number. Using (8.1), the first few terms of Generalized Padovan sequence are given by

$$1,1,1,k+1,k+1,k^2+k+1,k^2+2k+1,$$
$$k^3+k^2+2k+1,k^3+3k^2+2k+1,\cdots$$

Theorem 6:

The limiting ratio of the Generalized Padovan sequence is $\sqrt{k}$, whenever k is very large.

Proof: From (8.1), we get $\frac{GPD_{n+3}}{GPD_n}=k\frac{GPD_{n+1}}{GPD_n}+1$. If λ is the limiting ratio, then as $n\to\infty$ using (2.2), we get the cubic equation

$$\lambda^3-k\lambda-1=0\ . \quad (8.2)$$

Since $k > 0$, we notice that equation (8.2) possesses only one positive real root, the other two being complex conjugate roots. The roots of (8.2) for the values of k = 2, 3, 4, 5, 6, … are 1.618 (Golden Ratio which is the limiting ratio of Fibonacci Sequence), 1.879, 2.115, 2.33, 2.529, . . . respectively.

We may call these numbers as Polymer Numbers. We notice that if k = 1, then the limiting ratio is the plastic number discussed in section 7. Also if k is very large, then the limiting ratio of Generalized Padovan sequence is approximately $\sqrt{k}$ (8.3). We can call the limiting ratios for each value of k as Polymer Numbers.

This completes the proof.

9. Narayana Sequence

The Narayana Sequence named 14th century Indian mathematician Narayana Panditha is defined recursively by

$$N_{n+3} = N_{n+2} + N_n,\ n \geq 0 \tag{9.1}$$

where $N_0 = 0, N_1 = 1, N_2 = 1$. Similar to how Fibonacci defined Fibonacci sequence through immortal rabbits, Narayana Panditha defined Narayana sequence through immortal cows.

We notice that the number of cows present each year is equal to the number of cows in previous year plus the number of cows present three years ago giving the recurrence relation (9.1). The terms of Narayana sequence defined tbrough (9.1) are given by 0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, 1278, 1873, 2745, 4023, ...

Theorem 7:

The limiting ratio of the Narayana sequence is the Super Golden Ratio 1.46557 approximately.

Proof: From (9.1), we get $\frac{N_{n+3}}{N_n} = \frac{N_{n+2}}{N_n} + 1$. Thus, if λ is the limiting ratio, then as $n \to \infty$ using (2.2), we have $\lambda^3 = \lambda^2 + 1$ leading to the cubic equation

$$\lambda^3 - \lambda^2 - 1 = 0. \tag{9.2}$$

Using Descarte's Rule of Signs, we see that equation (9.2) has only one positive real root the other two being complex conjugate roots. Using Newton – Raphson method, we see that the lonely positive root of (9.2) is given by 1.46557 approximately. The number 1.46557 which is root of (9.2) is known as Super Golden Ratio. Thus the limiting ratio of the Narayana sequence is the Super Golden Ratio 1.46557 approximately. This completes the proof.

10. Generalized Narayana Sequence

The Generalized Narayana sequence is defined recursively by

$$GN_{n+3} = k\ GN_{n+2} + GN_n,\ n \geq 0 \tag{10.1}$$

where $GN_0 = 0, GN_1 = 1, GN_2 = 1$ and k is a positive real number. Using (10.1), the first few terms of Generalized Narayana sequence are given by

$$0, 1, 1, k, k^2+1, k^3+k+1, k^4+k^2+2k,$$
$$k^5+k^3+3k^2+1, k^6+k^4+4k^3+2k+1, \cdots$$

Theorem 8:

The limiting ratio of the Generalized Narayana sequence is the positive real root of $\lambda^3 - k\lambda^2 - 1 = 0$.

Proof: From (8.1), we get $\frac{GN_{n+3}}{GN_n} = k\frac{GN_{n+1}}{GN_n} + 1$. If λ is the limiting ratio, then as $n \to \infty$ using (2.2), we get the cubic equation $\lambda^3 - k\lambda^2 - 1 = 0$ (10.2). Since $k > 0$, we notice that equation (10.2) possesses only one positive real root, the other two being complex conjugate roots. The roots of (10.2) for the values of k = 2, 3, 4, 5, 6, … are 2.206, 3.104, 4.061, 5.039, 6.028, . . . respectively. This completes the proof.

We may call these numbers as Super Metallic Ratios. We notice that if k = 1, then the limiting ratio is the Super Golden Ratio 1.46557 discussed in section 9. Also if k is very large, then the limiting ratio of Generalized Narayana sequence is approximately k (10.3). We can call the limiting ratios for each value of k as Super Metallic Ratios. Refer **[2–3]** for more details about Generalized Narayana Sequence and its limiting ratios in different scenario.

11. Generalized Recurrence Relations of Fibonacci Type

The generalized recurrence relation of Fibonacci type sequence is defined by

$$\begin{aligned} P(n+m) &= P(n+m-1) + P(n+m-2) + \\ &+ \cdots + P(n+1) + P(n),\ n \geq 0, m \geq 2 \end{aligned} \tag{11.1}$$

Where

$P(0) = P(1) = P(2) = \cdots = P(m-2) = P(m-1) = 1$.

In (11.1), we observe that except for the first m terms, each term is sum of previous m terms of the sequence. We now prove that the limiting ratio of the generalized recurrence relation as defined in (11.1) converges to 2.

The generalized recurrence relation (as defined in (11.1) is given by

$$\begin{aligned} P(n+m) &= P(n+m-1) + P(n+m-2) + \cdots + \\ &+ P(n+1) + P(n),\ n \geq 0, m \geq 2 \end{aligned}$$

where

$$P(0) = P(1) = P(2) = \cdots = P(m-2) = P(m-1) = 1$$

Theorem 9:

The limiting ratio of the Generalized Recurrence Relations of Fibonacci type is 2.

Proof: From (11.1), we have

$$\begin{aligned} \frac{P(n+m)}{P(n)} &= \frac{P(n+m-1)}{P(n)} + \frac{P(n+m-2)}{P(n)} + \\ &+ \cdots + \frac{P(n+1)}{P(n)} + 1 \end{aligned} \tag{11.2}$$

Now using (2.2), as $n \to \infty$ (11.2) can be written as

$$\lambda^m - \lambda^{m-1} - \lambda^{m-2} - \cdots - \lambda^3 - \lambda^2 - \lambda - 1 = 0 \quad (11.3)$$

The positive real root of (11.3) will be limiting ratio of the generalized recurrence relation (11.1).

If $f(\lambda) = \lambda^m - \lambda^{m-1} - \lambda^{m-2} - \cdots - \lambda^3 - \lambda^2 - \lambda - 1$ then we find that $f(1) = -(m-1) < 0$ since $m \geq 2$.

Similarly, $f(2) = 2^m - 2^{m-1} - 2^{m-2} - \cdots - 2^3 - 2^2 - 2 - 1 = 2^m - \left(2^m - 1\right) = 1 > 0$

Hence the positive real root λ of (11.3) must lie between 1 and 2 for all $m \geq 2$.

Since,

$$(\lambda - 1)(\lambda^{m-1} + \lambda^{m-2} + \cdots + \lambda^3 + \lambda^2 + \lambda + 1) = \lambda^m - 1, \quad (11.4)$$

using (11.3) in (11.4) we get $(\lambda - 1)\lambda^m = \lambda^m - 1$. Simplifying this equation we get $1 + \lambda^{m+1} = 2\lambda^m$ giving

$$\lambda + \frac{1}{\lambda^m} = 2 \quad . \quad (11.5)$$

Since $\lambda > 1$, $\frac{1}{\lambda^m} \to 0$ as $m \to \infty$. Hence from (11.5) we see that $\lambda \to 2$ as $m \to \infty$.

Thus the limiting ratio λ of the generalized recurrence relation converges to 2.

This completes the proof.

We now provide few examples to justify this result by considering some values of *m*.

11.1. When *m* = 3

If *m* = 3, then from (11.1), we get

$$P(n+3) = P(n+2) + P(n+1) + P(n), \; n \geq 0, P(0) = 1, P(1) = 1, P(2) = 1 \quad . \quad (11.6)$$

From (11.6), we get

$$\frac{P(n+3)}{P(n)} = \frac{P(n+2)}{P(n)} + \frac{P(n+1)}{P(n)} + 1 \; .$$

If λ is the limiting ratio of (11.6) then as $n \to \infty$ from (2.2), we get $\lambda^3 - \lambda^2 - \lambda - 1 = 0$. By Newton – Raphson method, we see that the positive real root of this cubic polynomial is 1.83928 approximately. Hence the limiting ratio of (11.6) is the positive real root of $\lambda^3 - \lambda^2 - \lambda - 1 = 0$ which is 1.83928 approximately.

11.2. When *m* = 4

If *m* = 4, then from (11.1), we get

$$P(n+4) = P(n+3) + P(n+2) + P(n+1) + P(n), \; n \geq 0, P(0) = 1, P(1) = 1, P(2) = 1, P(3) = 1 \; . \quad (11.7)$$

From (11.7), we get

$$\frac{P(n+4)}{P(n)} = \frac{P(n+3)}{P(n)} + \frac{P(n+2)}{P(n)} + \frac{P(n+1)}{P(n)} + 1 \; .$$

If λ is the limiting ratio of (11.7) then as $n \to \infty$ from (2.2), we get $\lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0$. By Newton – Raphson method, we see that the positive real root of this fourth degree polynomial is 1.92756 approximately. Hence the limiting ratio of (11.7) is the positive real root of $\lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0$ which is 1.92756 approximately.

11.3. When *m* = 5

If *m* = 5, then from (11.1), we get

$$P(n+5) = P(n+4) + P(n+3) + P(n+2) + P(n+1) + P(n), \; n \geq 0$$
$$P(0) = 1, P(1) = 1, P(2) = 1, P(3) = 1, P(4) = 1 \; . \quad (11.8)$$

From (11.8), we get

$$\frac{P(n+5)}{P(n)} = \frac{P(n+4)}{P(n)} + \frac{P(n+3)}{P(n)} + \frac{P(n+2)}{P(n)} + \frac{P(n+1)}{P(n)} + 1 \; .$$

If λ is the limiting ratio of (11.8) then as $n \to \infty$ from (2.2), we get $\lambda^5 - \lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0$. By Newton – Raphson method, we see that the positive real root of this fifth degree polynomial is 1.96594 approximately. Hence the limiting ratio of (11.8) is the positive real root of (3.11) which is 1.96594 approximately.

11.4. When *m* = 10

If *m* = 10, then from (11.1), we get

$$P(n+10) = P(n+9) + P(n+8) + P(n+7) + \cdots + P(n+1) + P(n), \; n \geq 0, \quad (11.9)$$
$$P(0) = P(1) = \ldots = P(9) = 1$$

From (11.9), we get

$$\frac{P(n+10)}{P(n)} = \frac{P(n+9)}{P(n)} + \frac{P(n+8)}{P(n)} + \cdots + \frac{P(n+2)}{P(n)} + \frac{P(n+1)}{P(n)} + 1 \; .$$

If λ is the limiting ratio of (11.9) then as $n \to \infty$ from (2.2), we get

$$\lambda^{10} - \lambda^9 - \lambda^8 - \lambda^7 - \lambda^6 - \lambda^5 - \lambda^4 - \lambda^3 - \lambda^2 - \lambda - 1 = 0$$

By Newton – Raphson method, we see that the positive real root of the tenth degree polynomial described above is 1.99901 approximately. Hence the limiting ratio of (11.9) is the positive real root of

$$\lambda^{10}-\lambda^{9}-\lambda^{8}-\lambda^{7}-\lambda^{6}-\lambda^{5}-$$
$$-\lambda^{4}-\lambda^{3}-\lambda^{2}-\lambda-1=0$$

which is 1.99901 approximately.

Thus, through the four cases for m = 3, 4, 5 and 10 in sections 11.1 to 11.4 respectively, we notice that as m increases, the limiting ratios of the generalized recurrence relation defined in (11.1) approach 2. To know more about Generalized recurrence relations and other classes of special numbers see **[4–14]**.

12. Polygonal Numbers

The Polygonal number sequence of order $k\geq 3$ is defined by

$$P_k(n)=\frac{(k-2)n^2-(k-4)n}{2} \quad . \qquad (12.1)$$

Polygonal numbers are also called Figurate numbers.

Theorem 10:

The limiting ratio of the Polygonal numbers sequence is 1.

Proof: From (12.1), we notice that for k = 3, 4, 5 we get $P_3(n)=\frac{n^2+n}{2}$, $P_4(n)=n^2$, $P_5(n)=\frac{3n^2-n}{2}$, which are called Triangular, Square and Pentagonal numbers respectively, since, they can be arranged to form such shapes in a plane.

Similarly, for other values of k = 6, 7, 8, . . . we get the respective polygonal shapes in the plane, giving the name Polygonal numbers or Figurate numbers.

If λ is the limiting ratio of the polygonal numbers of order k, then as $n\to\infty$ we get

$$\lambda=\frac{P_k(n+1)}{P_k(n)}=\frac{\dfrac{(k-2)(n+1)^2-(k-4)(n+1)}{2}}{\dfrac{(k-2)n^2-(k-4)n}{2}}$$

$$=\frac{(k-2)\left(1+\frac{1}{n}\right)^2-(k-4)\left(\frac{1}{n}+\frac{1}{n^2}\right)}{(k-2)-(k-4)\left(\frac{1}{n}\right)}\to 1$$

Thus the limiting ratio of Polygonal number sequence is 1 (12.2). This completes the proof.

13. Catalan Sequence

The sequence of numbers whose nth term defined by $C_n=\frac{1}{n+1}\binom{2n}{n}$ (13.1) is called Catalan sequence named after Belgian mathematician Eugene Charles Catalan. The terms of the sequence are called Catalan numbers. Catalan numbers play a significant role, since it occurs in plenty of counting problems. Using (13.1), we will now determine the limiting ratio λ of Catalan sequence.

Theorem 11:

The limiting ratio of the Catalan sequence is 4.

Proof: From (13.1), considering the limit as $n\to\infty$, we have

$$\lambda=\frac{C_{n+1}}{C_n}=\frac{\frac{1}{n+2}\binom{2n+2}{n+1}}{\frac{1}{n+1}\binom{2n}{n}}=\frac{n+1}{n+2}\times\frac{(2n+2)!}{(n+1)!\times(n+1)!}\times\frac{n!\times n!}{(2n)!}$$

$$=\frac{2(2n+1)}{n+2}=\frac{2\left(2+\frac{1}{n}\right)}{1+\frac{1}{n}}\to\frac{2(2+0)}{1+0}=4$$

Thus the limiting ratio of terms in Catalan sequence is 4 (13.2). This completes the proof.

14. Cayley Numbers

Cayley numbers named after English mathematician Arthur Cayley are terms of a sequence whose nth term is defined by $CY_n=\frac{2}{n}\binom{2n}{n-2}$ (14.1) . Catalan numbers appear along the second column of the Catalan Triangle.

Theorem 12:

The limiting ratio of the Cayley sequence is 4.

Proof: If λ is the limiting ratio of the Cayley sequence of terms then using (14.1) and considering the limit as $n\to\infty$ we have

$$\lambda = \frac{CY_{n+1}}{CY_n} = \frac{\frac{2}{n+1}\binom{2n+2}{n-1}}{\frac{2}{n}\binom{2n}{n-2}} = \frac{n}{n+1} \times$$

$$\times \frac{(2n+2)!}{(n-1)! \times (n+3)!} \times \frac{(n-2)! \times (n+2)!}{(2n)!}$$

$$= \frac{2n(2n+1)}{(n-1)\times(n-3)} \rightarrow \frac{2\left(2+\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)\times\left(1-\frac{3}{n}\right)} = 4 .$$

Thus the limiting ratio of sequence whose terms are Cayley's numbers is 4 (14.2).

This completes the proof.

(See **[4]** for knowing more about Catalan and Cayley sequences)

15. Harmonic Numbers

The Harmonic numbers are reciprocals of natural numbers. The first few Harmonic numbers are given by $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ We notice that these numbers form Harmonic Progression. The nth Harmonic number is given by $H_n = \frac{1}{n}$ (15.1).

Theorem 13:

The limiting ratio of the Harmonic numbers sequence is 1.

Proof: If λ is the ratio of Harmonic numbers then from (15.1) and considering the limit as $n \rightarrow \infty$ we have

$$\lambda = \frac{H_{n+1}}{H_n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \rightarrow \frac{1}{1+0} = 1 .$$

Thus, the limiting ratio of sequence whose terms are Harmonic numbers is 1 (15.2).

This completes the proof.

16. Partition Numbers

Let n be a positive integer. The number of ways of expressing n as unordered sum using smaller numbers is called Partition of n. The study of partitions was carried out by many great mathematicians including Euler, Jacobi, Ramanujan, Hardy and several modern number theorists.

Theorem 14:

The limiting ratio of the Partition numbers is the logarithmic base e.

Proof: Due to the works of Ramanujan, Hardy and Rademacher using circle method, they derived the asymptotic behavior of partition of a positive integer n given by $p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2n}{3}}}$ (16.1). Hence if λ is the limiting ratio of the partition of integer function then by (16.1), then as $n \rightarrow \infty$. we get

$$\lambda = \frac{p(n+1)}{p(n)} \sim \frac{4n\sqrt{3}}{4(n+1)\sqrt{3}} \frac{e^{\pi\sqrt{\frac{2(n+1)}{3}}}}{e^{\pi\sqrt{\frac{2n}{3}}}} =$$

$$= \frac{n}{n+1} e^{\pi\sqrt{\frac{2}{3}}\left(\sqrt{n+1}-\sqrt{n}\right)} \rightarrow (1) \times e^{\pi\sqrt{\frac{2}{3}}(0)} = e$$

Thus the limiting ratio of partition numbers is the natural logarithmic base e, the Euler number when n is very large.

This completes the proof.

For knowing more about partitions see [**15**–**16**].

17. Conclusions

Considering several sequences of numbers we have determined their limiting ratio (the ratio of the (n + 1)th term to its nth term). The limiting ratio provides us with an idea of asymptotic behavior of the terms of that sequence. For example, in Fibonacci sequence, the (n + 1)th term is roughly 1.618 times the nth term and in Catalan as well as Cayley sequences, the (n + 1)th term is exactly four times the nth term. In section 11, we have considered generalized recurrence relations of Fibonacci type and justified that the limiting ratio is 2 by considering four different cases. Interestingly, in section 16, we found that the partition of a natural number n + 1 is approximately 2.718 times the partition of n, when n tends to infinity. Thus the idea of limiting ratio will help us to make the decision of how quick the terms are growing and also help us to predict the next term of the sequence knowing the current term. In this paper, we have considered 16 different sequences of numbers and determined their limiting ratios by establishing fourteen theorems.

We have not considered the most straightforward sequences like Arithmetic Progression or Geometric Progression for which the limiting ratios can be determined in a direct way or for the sequences for which the limiting ratio doesn't exist. For example, for the sequence of factorials, the limiting ratio doesn't exist. In general we can possibly conclude that if the limiting ratio is finite, then the terms of the sequence grows sub-exponentially. Thus all the sequences considered in this paper have terms which grow sub-exponentially. The rate of their growth is determined by the respective limiting ratios. Fourteen theorems were proved in this paper in obtaining the required results.

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