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Markowitz Random Set and Its Application to the Paris Stock Market Prices

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Abstract In this paper, we will combine random set theory and portfolio theory, through the estimation of the lower bound of the Markowitz random set based on the Mean-Variance Analysis of Asset Portfolios Approach, which represents the efficient frontier of a portfolio. There are several Markowitz optimization approaches, of which we denote the most known and used in the modern theory of portfolio, namely, the Markowitz's approach, the Markowitz Sharpe's approach and the Markowitz and Perold's approach, generally these methods are based on the minimization of the variance of the return of a portfolio. On the other hand, the method used in this paper is completely different from those denoted above, because it is based on the theory of random sets, which allowed us to have the mathematical structure and the graphic of the Markowitz set. The graphical representation of the Markowitz set gives us an idea of the investment region. This region, called the investment zone, contains the stocks in which the rational investor can choose to invest. Mathematical and statistical estimation techniques are used in this paper to find the explicit form of the Markowitz random set, and to study its elements in function of the signs of the estimated parameters. Finally, we will apply the results found to the case of the returns of a portfolio composed of 200 assets from the Paris Stock Market Prices. The results obtained by this simulation allow us to have an idea on the stocks to recommend to the investors. In order to optimize their choices, these stocks are those which will be located above the curve of the hyperbola which represents the Markowitz set.

Keywords Markowitz Set, Paris Stock Market Prices, Mean-Variance Analysis, Efficient Frontier, "R" Software for Statistical Calculation

Mathematics Subject Classification (MSC):62P05

1. Introduction

Random set theory [1,2,3] is interested on random objects whose realizations are sets. Such objects are known since a long time in statistics and econometrics in the form of confidence regions, which can be naturally characterized as random sets.

The first concept of a general random set [4,5] in the form of a region that is affected by chance comes from Kolmogorov, originally published in 1933. A consistent advancement of the theory of random sets took place recently, boosted by the study in general equilibrium theory and decision theory of correspondences and non-additive functions [6,7], as well as, by the demands in image analysis, microscopy and materials science for statistical techniques to build models for random sets, to estimate their parameters, to screen noisy images and to rank biological images [8].

More recently, the advancement of financial systems with transaction costs has offered a natural new area for the application of random set theory.

On the other side, Portfolio optimization or the optimal choice of financial asset portfolio [9, 10] is a topic that has been of particular importance in the research in financial mathematics. In this context, Markowitz was the first to present a model known as the mean-variance approach in 1952, based on the variances portfolio returns observed about their means as a measure of risk for the optimal choice of the portfolio.

In effect, the Markowitz's model involves minimizing the standard deviation or variance for a given return or to maximize the expectation of return on the portfolio for a chosen risk.

The aim of this paper is to combine the theory of random sets and the theory of portfolio optimization [5,10,13,16], by determining the Markowitz random set which consists in searching for the pairs composed by the mean and the variance of the returns of a portfolio using "Mean–Variance Analysis of Asset Portfolios" approach.

Then, we will estimate the unknown parameters of the Markowitz set, and will determine its boundary which represents the efficient frontier of the portfolio, by using various mathematical and statistical techniques.

Finally, we will apply the different results found on the portfolio returns composed by 200 assets from the Paris Stock Market Prices [17], with a graphical representation of the efficient frontier of this Markowitz set.

2. Markowitz Optimization Approaches

2.1. Markowitz's Approach (1952)

Let p_t be the price of a stock "a" at the end of period t, the price variation $(p_t - p_{t-1})$ is the benefit, to which is possibly added the income d_t, known as the dividend paid during period t.

The return on this stock in period t is defined as follows:

$${}_t = \frac{(p_t - p_{t-1}) + d_t}{p_{t-1}} \quad (1)$$

Let P be a portfolio of assets $(A_1 ... A_n)$ represented by a vector $\pi = (\pi_1, \dots, \pi_n)$ where π_i refers to the proportion of the capital C invested in the stock A_i distinguished by its uncertain return R_i $(i = 1, \dots, n)$.

The return on this portfolio is defined as follows:

$${}_p(\pi) = \sum_{j=1}^{n} \pi_j \; {}_j \quad (2)$$

The value and variation of this portfolio are defined as follows respectively:

$$VaL(\pi) = \sum_{j=1}^{n} p_j \pi_j \quad (3)$$

$$\Delta V(\pi) = \sum_{j=1}^{n} \Delta p_j \pi_j \quad (4)$$

Harry Markowitz ([11], [12]) was the first to develop the idea of measuring the performance of a portfolio by its expected return and the risk by its variance.

The Markowitz approach, also known as mean-variance, consists of minimizing the risk of this portfolio by fixing the minimum return expected by the investor or vice versa, i.e. maximizing the expected return by fixing the minimum risk wished by the investor.

The return on the portfolio is a random variable whose expectation is given by:

$$E[R_p(\pi)] = E[\textstyle\sum_{j=1}^{n} \pi_j \; {}_j] = \sum_{j=1}^{n} \pi_j E[R_j] \quad (5)$$

The variance of the portfolio P return is given by:

$$Var\left(R_p(\pi)\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i \pi_j cov(R_i, R_j) \quad (6)$$

The Markowitz optimization algorithm is written as follows:

$$\begin{cases} \text{Min Var}(\; {}_p(\pi)) \\ \text{Under the constraints} \\ E[R_p(\pi)] = \mu \\ \sum_{j=1}^{n} \pi_j = 1 \end{cases}$$

It is a quadratic programming problem that produces a feasible mean-variance combination.

The set of possible combinations of portfolio mean-variances is called efficient, if there are no strict lower risks among all the portfolios with the same expected return as it. And the efficient frontier is the set of efficient portfolios.

2.2. Markowitz Sharpe's Approach (1963-1964)

Sharpe ([12], [13]) was the first to try simplifying the Markowitz model by using index models based on the simplification of the variance-covariance matrix.

Sharpe suggested a diagonalization of the matrix based on the single-index model, supposing that the stock return fluctuations can be represented by a simple regression.

In other words: $R_i = a_i + b_i R_I + \varepsilon_i$, for $i = 1, \dots, n$, where:

${}_I$ is the return on the index I.

ε_i is a random variable known as white noise that verifies the following conditions:

$E(\varepsilon_i) = 0$, and $\sigma_{\varepsilon_i}^2 \neq 0$ for $i = 1, \dots, n$

$\sigma_{\varepsilon_i, \varepsilon_j} = cov(\varepsilon_i, \varepsilon_j) = 0$ for each $i \neq j$

$\sigma_{\varepsilon_i, R_I} = cov(\varepsilon_i, \varepsilon_j) = 0, i = 1, \dots, n.$

The portfolio return becomes:

$${}_p(\pi) = \sum_{i=1}^{n} \pi_i \; {}_i$$

$$= \sum_{i=1}^{n} \pi_i a_i + (\textstyle\sum_{i=1}^{n} \pi_i b_i) R_I + \sum_{i=1}^{n} \pi_i \varepsilon_i \quad (7)$$

Then, the expected return and the variance are written as:

$$\begin{cases} E(\ _p(\pi)) = \sum_{i=1}^{n} \pi_i a_i + (\sum_{i=1}^{n} \pi_i b_i) E(R_I) \\ V\left(R_p(\pi)\right) = \sum_{i=1}^{n} \pi_i^2 \sigma_{\varepsilon_i}^2 + (\sum_{i=1}^{n} \pi_i b_i)^2 \sigma_I^2 \end{cases} \quad (8)$$

2.3. Markowitz and Perold's Approach (1981)

Markowitz and Perold ([14],[15]) developed a multi-index model which supposes that there is a relationship between the stocks in the following form:

$_i = \alpha_i + \beta_{i1} F_1 + \cdots + \beta_{ik} F_k + \varepsilon_i,\ i = 1, \ldots, n$ where:

F_k is the k random factor;

α_i and β_{ik} are constants;

ε_i : is random noise of mean 0 and is uncorrelated with F_k (for all $k = 1, \ldots, K$)

if $\sigma_i^2 = E(\varepsilon_i{}^2)$ and $\tau_{rs} = cov(F_r, F_s)$, then we get the following formula:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \pi_i \pi_j = \sum_{i=1}^{n} \sigma_i{}^2 \pi_i{}^2 + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{r=1}^{K} \sum_{s=1}^{K} \tau_{rs} \beta_{ir} \beta_{js} \pi_i \pi_j \quad (9)$$

The optimization problem is:

$$Min\{\sum_{i=1}^{n} \sigma_i{}^2 \pi_i{}^2 + \sum_{r=1}^{K} \sum_{s=1}^{K} \tau_{rs} \beta_{ir} y_r y_s\}$$

Under the constraints:

$$\begin{cases} \sum_{j=1}^{n} \pi_j E(R_j) \geq \rho_0 \\ \sum_{j=1}^{n} \beta_{jk} \pi_j - y_k = 0, k = 1, \ldots, K \\ \sum_{j=1}^{n} \pi_j = 1 \\ \pi_j \geq 0,\ j = 1, \ldots, n \end{cases}$$

3. Random Set Theory Approach

3.1. Definition of Random Closed Set

In the theory of random set [1,3] X is called a random closed set in Euclidean space IR^d, if X is a map from a probability space (Ω, T, P) to the family of closed set F in IR^d and $X^-(K) = \{\omega : X(\omega) \cap K \neq \emptyset\}$ belongs to the $\sigma-$ algebra T on Ω for each compact set K in IR^d.

3.2. Single Smooth Inequality

We consider a random element $\varepsilon \in \mathcal{E}$, where $\mathcal{E} \subset IR^d$ a compact parameter, and let $f: \mathcal{E} \rightarrow IR$ be a real valued function on $\mathcal{E}$.

The collection of admissible models as solutions to a single smooth inequality given by $H(\varepsilon) = \{\varepsilon \in \mathcal{E}:\ f(\varepsilon) \leq 0\}$ is a random closed set on IR, and the inequality-generating smooth function f which is unknown can be estimated from the data [4,6].

3.3. Markowitz Random Set

3.3.1. Structure of the Markowitz Set

The Markowitz set [2] of admissible standard deviations and means is given by the following formula:

$$H(\mu, \sigma^2) = \{ (\mu, \sigma^2) \in (R \times R+) \cap K : \sigma_M^2(\mu) - \sigma^2 \leq 0 \} = \{\varepsilon \in \mathcal{E}:\ f_M(\varepsilon) \leq 0\} \quad (10)$$

Where $f_M(\varepsilon) = \sigma_M^2(\mu) - \sigma^2$, and $\mathcal{E} = (\ \times R+) \cap K$, with $K \subset IR^2$ is a compact convex subset.

And:

$$\begin{cases} \sigma^2{}_{Mz}(\mu) = \min_\pi \pi^T \sum \pi \\ \pi^T \vartheta = \mu \\ \pi^T 1_N = 1 \end{cases}$$

With: π, ϑ and $\sum$ are respectively the vector of weights, the vector of mean returns, and the covariance of returns of a Portfolio "P".

In what follows, we will look for the explicit form of $\sigma^2{}_{Mz}(\mu)$, in order to be able to estimate it.

3.3.2. Mean–Variance Analysis of Asset Portfolios

Let $_1, \ldots \ldots, R_N$ the returns on available assets $(1, \ldots, N)$ of Portfolio P, the classical Markowitz problem is to minimize the variance of a portfolio P given some attainable level of return:

$$\begin{cases} \min_\pi E\big(R_P - E(\ _P)\big)^2 \text{such that} \\ E(\ _P) = \mu \end{cases}$$

Where R_P is the return of the portfolio P, determined as $_P = \pi^T R$ and $R = (R_1, \ldots \ldots, R_N)$.

Canonical version of this problem is:

$$\begin{cases} \sigma^2{}_{Mz}(\mu) = \min_\pi \pi^T \sum \pi \text{ such that} \\ \pi^T \vartheta = \mu \\ \pi^T 1_N = 1 \end{cases}$$

With: $\vartheta = (E(\ _1), \ldots E(\ _N))^T$,

$$\sum = \begin{pmatrix} V(R_1) & \cdots & COV(R_1, R_N) \\ \vdots & \ddots & \vdots \\ COV(R_N, R_1) & \cdots & V(R_N) \end{pmatrix} \text{ and } 1_N = (1, \ldots, 1)^T.$$

3.3.2.1. Resolution of the Minimization Problem

Using the Lagrangien function, we have:

$$L(\pi, \lambda, \varphi) = \pi^T \sum \pi - \lambda(\pi^T \vartheta - \mu) - \varphi(\pi^T 1_N - 1) \quad (11)$$

With λ and φ are the Lagrange multipliers in IR.

The first order condition gives:

$$\frac{\partial L}{\partial \pi} = 2\sum \pi - \lambda \vartheta - \varphi 1_N = 0$$

Then:

$$\pi^* = 1/2\lambda \sum^{-1} \vartheta + 1/2\varphi \sum^{-1} 1_N$$

Now, we have to search the expression of λ and φ, by replacing π^* in the constraints of the problem.

We have:

$$\begin{cases} \pi^{*T} \vartheta = \mu \\ \pi^{*T} 1_N = 1 \end{cases}$$

Then:

$$\begin{cases} (1/2\lambda \sum^{-1} \vartheta + 1/2\varphi \sum^{-1} 1_N)^T \vartheta = \mu \\ (1/2\lambda \sum^{-1} \vartheta + 1/2\varphi \sum^{-1} 1_N)^T 1_N = 1 \end{cases}$$

Then:

$$\begin{cases} \frac{1}{2}\lambda\vartheta^T\Sigma^{-1}\vartheta + \frac{1}{2}\varphi {1_N}^T\Sigma^{-1}\vartheta = \mu \\ (\frac{1}{2}\lambda\vartheta^T\Sigma^{-1}1_N + \frac{1}{2}\varphi 1_N\Sigma^{-1}1_N)^T = 1 \end{cases}$$

We set $A = \vartheta^T\Sigma^{-1}\vartheta$, $B = {1_N}^T\Sigma^{-1}\vartheta = \vartheta^T\Sigma^{-1}1_N$ and $C = {1_N}^T\Sigma^{-1}1_N$, then our mathematical system becomes:

$$\begin{cases} \frac{1}{2}\lambda A + \frac{1}{2}\varphi B = \mu \\ \frac{1}{2}\lambda B + \frac{1}{2}\varphi C = 1 \end{cases}$$

Suppose that $\begin{vmatrix} A & B \\ B & C \end{vmatrix} \neq 0$, then the system solution by Cramer method is:

$$(\lambda, \varphi) = \left(\frac{\begin{vmatrix} 2\mu & B \\ 2 & C \end{vmatrix}}{\begin{vmatrix} A & B \\ B & C \end{vmatrix}}, \frac{\begin{vmatrix} A & 2\mu \\ B & 2 \end{vmatrix}}{\begin{vmatrix} A & B \\ B & C \end{vmatrix}} \right) = \left(\frac{2\mu C - 2B}{AC - B^2}, \frac{2A - 2B\mu}{AC - B^2} \right)$$

Then, we replace (λ, φ) by $\left(\frac{2\mu C-2B}{AC-B^2}, \frac{2A-2B\mu}{AC-B^2}\right)$ in following formula:

$$\pi^* = 1/2\lambda\Sigma^{-1}\vartheta + 1/2\varphi\Sigma^{-1}1_N$$

We get:

$$\pi^* = 1/2\frac{2\mu C - 2B}{AC - B^2}\Sigma^{-1}\vartheta + 1/2\frac{2A - 2B\mu}{AC - B^2}\Sigma^{-1}1_N$$

3.3.2.2. The Explicit form of $\boldsymbol{\sigma^2}_{Mz}(\boldsymbol{\mu})$

The solution of our Markowitz problem is

$$\sigma^2_{Mz}(\mu) = \min_\pi \pi^T\Sigma\pi = {\pi^*}^T\Sigma\pi^*.$$

We replace π^* by its formula, we get:

$$\sigma^2_{Mz}(\mu) =$$

$$\left(\frac{2\mu C - 2B}{2(AC - B^2)}\Sigma^{-1}\vartheta + \frac{2A - 2B\mu}{2(AC - B^2)}\Sigma^{-1}1_N\right)^T \Sigma\left(\frac{2\mu C - 2B}{2(AC - B^2)}\Sigma^{-1} + \frac{2A - 2B\mu}{2(AC - B^2)}\Sigma^{-1}1_N\right)$$

$$=$$

$$\left(\frac{2\mu C - 2B}{2(AC - B^2)}\vartheta^T\Sigma^{-1} + \frac{2A - 2B\mu}{2(AC - B^2)}{1_N}^T\Sigma^{-1}\right)\Sigma\left(\frac{2\mu C - 2B}{2(AC - B^2)}\Sigma^{-1}\vartheta + \frac{2A - 2B\mu}{2(AC - B^2)}\Sigma^{-1}1_N\right)$$

$$= \left(\frac{\mu C - B}{AC - B^2}\vartheta^T\Sigma^{-1}\Sigma + \frac{A - B\mu}{AC - B^2}{1_N}^T\Sigma^{-1}\Sigma\right)\left(\frac{\mu C - B}{AC - B^2}\Sigma^{-1}\vartheta + \frac{A - B\mu}{AC - B^2}\Sigma^{-1}1_N\right)$$

$$= \left(\frac{\mu C - B}{AC - B^2}\vartheta^T + \frac{A - B\mu}{AC - B^2}{1_N}^T\right)\left(\frac{\mu C - B}{AC - B^2}\Sigma^{-1}\vartheta + \frac{A - B\mu}{AC - B^2}\Sigma^{-1}1_N\right)$$

$$= \left(\frac{\mu C - B}{AC - B^2}\right)^2\vartheta^T\Sigma^{-1}\vartheta + 2\left(\frac{A - B\mu}{AC - B^2}\right)\left(\frac{\mu C - B}{AC - B^2}\right){1_N}^T\Sigma^{-1}\vartheta + \left(\frac{A - B\mu}{AC - B^2}\right)^2{1_N}^T\Sigma^{-1}1_N$$

$$= \left(\frac{\mu C - B}{AC - B^2}\right)^2 A + 2\left(\frac{A - B\mu}{AC - B^2}\right)\left(\frac{\mu C - B}{AC - B^2}\right)B + \left(\frac{A - B\mu}{AC - B^2}\right)^2 C$$

$$= \frac{(\mu C - B)^2 A + 2(A - B\mu)(\mu C - B)B + (A - B\mu)^2 C}{(AC - B^2)^2}$$

$$= \frac{AC\mu^2 - 2ACB\mu + CA^2 - CB^2\mu^2 + 2B^3\mu - B^2A}{(AC - B^2)^2}$$

$$= \frac{(AC - B^2)(C\mu^2 - 2B\mu + A)}{(AC - B^2)^2}$$

Then we conclude that:

$$\sigma^2_{Mz}(\mu) = \frac{C\mu^2 - 2B\mu + A}{AC - B^2} \tag{12}$$

3.3.3. Canonical form of Markowitz Set $\boldsymbol{H(\mu,\sigma^2)}$

The boundary of the set $H(\mu,\sigma^2)$ is known as the efficient frontier, we restart from the following formula:

$$H(\mu,\sigma^2) = \{ (\mu,\sigma^2) \in (R \times R+) \cap K : \sigma^2_M(\mu) - \sigma^2 \leq 0 \}$$

We replace $\sigma^2{}_{Mz}(\mu)$ by its formula, we get:

$$H(\mu,\sigma^2) = \{(\mu,\sigma) \in IR \times IR^+ \cap K / \frac{C\mu^2 - 2B\mu + A}{AC - B^2} - \sigma^2 \leq 0 \}$$

The canonical form of this set is:

$$H(\mu,\sigma^2) = \{(\mu,\sigma^2) \in IR \times IR^+ \cap K / \frac{C}{AC-B^2}\mu^2 - \frac{2B}{AC-B^2}\mu + \frac{A}{AC-B^2} - \sigma^2 \leq 0\}$$

We set:

$$\alpha = \frac{C}{AC-B^2}, \beta = -\frac{2B}{AC-B^2}, \text{and } \gamma = \frac{A}{AC-B^2}$$

With $(\alpha,\beta,\gamma) \in IR^3$.

Then, our canonical form of Markowitz set becomes:

$$H(\mu,\sigma^2) = \{(\mu,\sigma^2) \in IR \times IR^+ \cap K / \alpha\mu^2 + \beta\mu + \gamma - \sigma^2 \leq 0\} \quad (13)$$

3.3.4. Estimation of Markowitz Set $\boldsymbol{\widehat{H}(\mu,\sigma^2)}$

According to the canonical form of the Markowitz set, a natural estimator of $H(\mu,\sigma^2)$ is its empirical analog, and we denote that:

$$\widehat{H}(\mu,\sigma^2) = \{(\mu,\sigma^2) \in IR \times IR^+ \cap K / \ \widehat{\sigma}^2{}_{Mz}(\mu) - \sigma^2 \leq 0 \}$$

$$= \{(\mu,\sigma^2) \in IR \times IR^+ \cap K / \ \widehat{\alpha}\mu^2 + \widehat{\beta}\mu + \widehat{\gamma} - \sigma^2 \leq 0 \} \quad (14)$$

Such that:

$$\widehat{\alpha} = \frac{\widehat{C}}{\widehat{A}\widehat{C} - \widehat{B}^2}, \widehat{\beta} = -\frac{2\widehat{B}}{\widehat{A}\widehat{C} - \widehat{B}^2}, \text{and } \widehat{\gamma} = \frac{\widehat{A}}{\widehat{A}\widehat{C} - \widehat{B}^2}$$

With: $\widehat{A} = \widehat{\vartheta}^T\widehat{\Sigma}^{-1}\widehat{\vartheta}$, $\widehat{B} = 1_N{}^T\widehat{\Sigma}^{-1}\widehat{\vartheta}$ and $\widehat{C} = 1_N{}^T\widehat{\Sigma}^{-1}1_N$

$\widehat{\vartheta}$ and $\widehat{\Sigma}$ are the estimators of the vector of the mean and the covariance of returns of a Portfolio "P", and their estimations will be from Data.

3.3.5. Determination of the Boundary of the Estimated Markowitz Set of Portfolio "P"

We can wright $\widehat{\sigma}^2{}_{Mz}(\mu)$ under its canonical form:

$$\widehat{\sigma}^2{}_{Mz}(\mu) = \widehat{\alpha}\left[\left(\mu + \frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2\right] + \widehat{\gamma} \quad (15)$$

We denote that $\widehat{\sigma}^2{}_{Mz}(\mu)$ must to be positive for each $\mu \in IR$.

In this paper we will suppose that $\widehat{\alpha} > 0$, then:

$$\widehat{\alpha}\left[\left(\mu + \frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2\right] + \widehat{\gamma} \geq 0$$

$$<==> \left(\mu + \frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 \geq \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}}$$

And we conclude that:

If $\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}} \leq 0$, then:

$$\forall\mu \in IR \ \left(\mu + \frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 \geq 0 \geq \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}}$$

else if $\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}} \geq 0$, then :

$$\mu \in \left]-\infty; -\frac{\widehat{\beta}}{2\widehat{\alpha}} - \sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}}}\right[\cup \left]-\frac{\widehat{\beta}}{2\widehat{\alpha}} + \sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}}}; +\infty\right[$$

We have $\widehat{\sigma}^2{}_{Mz}(\mu)$ is continuous and strictly increasing function, then it's a bijective function from I to IR^+, with:

$$I = IR, \quad \text{if } \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}} \leq 0$$

or

$$I = \left]-\infty; -\frac{\widehat{\beta}}{2\widehat{\alpha}} - \sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}}}\right] \cup \left[-\frac{\widehat{\beta}}{2\widehat{\alpha}} + \sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}}}; +\infty\right[$$

$$\text{if } \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \frac{\widehat{\gamma}}{\widehat{\alpha}} \geq 0$$

3.3.5.1. Determination of Reciprocal Function of $\boldsymbol{\widehat{\sigma}^2{}_{Mz}}$

We consider a bijective function $f: I \to IR^+$, such that: $f(\mu) = \widehat{\sigma}^2{}_{Mz}(\mu)$ for each $\mu \in I$. And we search the reciprocal function of f.

We have: $\forall y \in IR^+, \exists!\, \mu \in I \ / \ y = f(\mu)$, then: $\forall y \in IR^+, \exists!\, \mu \in I \ / \ \mu = f^{-1}(y)$.

With f^{-1} is the reciprocal function of f, that we have to search its form.

We have:

$$y = f(\mu) = \widehat{\alpha}\left[\left(\mu + \frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 - \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2\right] + \widehat{\gamma}$$

Then:

$$\left(\mu + \frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 = \frac{y - \widehat{\gamma}}{\widehat{\alpha}} + \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2$$

We have for each $\mu \in IR, \left(\mu + \frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 \geq 0$.

Then we conclude that:

$$\frac{y - \widehat{\gamma}}{\widehat{\alpha}} + \left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2 \geq 0 <==> \ y \geq \widehat{\gamma} - \frac{\widehat{\beta}^2}{4\widehat{\alpha}}$$

And we have:

$$f^{-1}(y) = -\frac{\hat{\beta}}{2\hat{\alpha}} \mp \sqrt{\frac{y-\hat{\gamma}}{\hat{\alpha}} + \left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2} = \mu$$

Therefore, f^{-1} is a continuous function from $\left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[\cap IR^+$ to I and we denote:

$$f^{-1}: \left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[\cap IR^+ \longrightarrow I$$
$$y = \hat{\sigma}^2{}_{Mz}(\mu) \longrightarrow \quad \mu = f^{-1}(y)$$

Finally, we can conclude that:

$$\min_{\mu\in I} \hat{\sigma}^2{}_{Mz}(\mu) = \min\left(\left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[\cap IR^+\right) \qquad (16)$$

If $\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}} > 0$, then:

$$\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}} \leq 0$$

And we conclude that:

$$\left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[\cap IR^+ = \left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[$$

we denote:

$$f^{-1}: \left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[\longrightarrow I = IR$$
$$y = \hat{\sigma}^2{}_{Mz}(\mu) \longrightarrow \mu = f^{-1}(y) = -\frac{\hat{\beta}}{2\hat{\alpha}} \mp \sqrt{\frac{y-\hat{\gamma}}{\hat{\alpha}} + \left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2}$$

And the Markowitz set becomes:

$$\hat{H}(\mu,\sigma^2) = \{(\mu,\sigma^2) \in IR \times \left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[/ \ \hat{\alpha}\mu^2 + \hat{\beta}\mu + \hat{\gamma} - \sigma^2 \leq 0\}$$

Then:

$$\begin{cases} \min\limits_{\mu\in I} \hat{\sigma}^2{}_{Mz}(\mu) = \hat{\sigma}^2{}_{Mz}(\mu_0) = \hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}} \\ \mu_0 = f^{-1}\left(\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}\right) \end{cases}$$

We replace y by $\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}$ in the formula of f^{-1}, we get :

$$\mu_0 = f^{-1}\left(\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}\right) = -\frac{\hat{\beta}}{2\hat{\alpha}} \mp \sqrt{\frac{\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}} - \hat{\gamma}}{\hat{\alpha}} + \left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2} = -\frac{\hat{\beta}}{2\hat{\alpha}}$$

Then:

$$\inf\left(\hat{H}(\mu,\sigma^2)\right) = \inf\{(\mu,\sigma^2) \in IR \times \left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[/ \ \hat{\alpha}\mu^2 + \hat{\beta}\mu + \hat{\gamma} - \sigma^2 \leq 0\} = \left(-\frac{\hat{\beta}}{2\hat{\alpha}}, \hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}\right) \qquad (17)$$

And the boundary of our estimable Markowitz set denoted $\inf(\hat{M}z)$ is the couple $\left(\mu_0, \hat{\sigma}^2{}_{Mz}(\mu_0)\right) = \left(-\frac{\hat{\beta}}{2\hat{\alpha}}, \hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}\right)$ which is the efficient frontier.

If $\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}} \leq 0$, then $\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}} \geq 0$

We conclude that $\left[\hat{\gamma} - \frac{\hat{\beta}^2}{4\hat{\alpha}}; +\infty\right[\cap IR^+ = IR^+$ and we denote:

$$f^{-1}: IR^+ to \left]-\infty; -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}\right] \cup \left[-\frac{\hat{\beta}}{2\hat{\alpha}} + \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}; +\infty\right[$$

Then the formula of f^{-1} is defined by:

$$f^{-1}(y) = \begin{cases} -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\frac{y-\hat{\gamma}}{\hat{\alpha}} + \left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2} \ \ if\ \mu \in \left]-\infty; -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}\right] \\ -\frac{\hat{\beta}}{2\hat{\alpha}} + \sqrt{\frac{y-\hat{\gamma}}{\hat{\alpha}} + \left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2} \ \ if\ \mu \in \left[-\frac{\hat{\beta}}{2\hat{\alpha}} + \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}; +\infty\right[\end{cases}$$

1^{st} Case:

If $I = \left]-\infty; -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}\right]$ then the Markowitz set becomes:

$$\hat{H}(\mu,\sigma^2) = \{(\mu,\sigma^2) \in \left]-\infty; -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}\right] \times IR^+ / \ \hat{\alpha}\mu^2 + \hat{\beta}\mu + \hat{\gamma} - \sigma^2 \leq 0\}$$

Then:

$$\begin{cases} \min\limits_{\mu\in I} \hat{\sigma}^2{}_{Mz}(\mu) = \hat{\sigma}^2{}_{Mz}(\mu_0) = 0 \\ \mu_0 = f^{-1}(0) \end{cases}$$

We replace y by 0 in the formula of f^{-1}, we get :

$$\mu_0 = f^{-1}(0) = -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\frac{0-\hat{\gamma}}{\hat{\alpha}} + \left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2} = -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}$$

Then:

$$\inf\left(\hat{H}(\mu,\sigma^2)\right) = \inf\{(\mu,\sigma^2) \in \left]-\infty; -\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}\right] \times IR^+ / \ \hat{\alpha}\mu^2 + \hat{\beta}\mu + \hat{\gamma} - \sigma^2 \leq 0\} = \left(-\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}; 0\right) \qquad (18)$$

And the boundary of our estimable Markowitz set denoted $\inf\left(\widehat{H}(\mu,\sigma^2)\right)$ is the couple

$$\left(\mu_0\,,\widehat{\sigma}^2{}_{Mz}(\mu_0)\right)=\left(-\frac{\widehat{\beta}}{2\widehat{\alpha}}-\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}}\,,0\right)$$

which is the efficient frontier.

2nd Case:

If $\mu\in\left[-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}};+\infty\right[$ then the Markowitz set becomes:

$$\widehat{H}(\mu,\sigma^2)=\{(\mu,\sigma^2)\in\left[-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}};+\infty\right[\times IR^+/\ \widehat{\alpha}\mu^2+\widehat{\beta}\mu+\widehat{\gamma}-\sigma^2\leq 0\ \}$$

Then:

$$\begin{cases}\min\limits_{\mu\in I}\widehat{\sigma}^2{}_{Mz}(\mu)=\widehat{\sigma}^2{}_{Mz}(\mu_0)=0\\ \mu_0=f^{-1}(0)\end{cases}$$

We replace y by 0 in the formula of f^{-1}, we get:

$$\mu_0=f^{-1}(0)=-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\frac{0-\widehat{\gamma}}{\widehat{\alpha}}+\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2}=-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}}$$

Then:

$$\inf\left(\widehat{H}(\mu,\sigma^2)\right)=\inf\{(\mu,\sigma^2\in\left[-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}};+\infty\right[\times IR^+/\ \widehat{\alpha}\mu^2+\widehat{\beta}\mu+\widehat{\gamma}-\sigma^2\}\leq 0\ \}$$

$$=\left(-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}}\,,0\right)\qquad(19)$$

And the boundary of our estimable Markowitz set denoted $\inf\left(\widehat{H}(\mu,\sigma^2)\right)$ is the couple $\left(\mu_0\,,\widehat{\sigma}^2{}_{Mz}(\mu_0)\right)=\left(-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}}\,,0\right)$ which is the efficient frontier.

3.3.5.2. Conclusion

The table below gives a synthesis of the different results found:

Table 1. Synthesis of the different results found

$\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}$	I	$\widehat{H}(\mu,\sigma^2)$	$\inf\left(\widehat{H}(\mu,\sigma^2)\right)$
$\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}\leq 0$	IR	$\{(\mu,\sigma^2)\in IR\times\left[\widehat{\gamma}-\frac{\widehat{\beta}^2}{4\widehat{\alpha}};+\infty\right[/\ \widehat{\alpha}\mu^2+\widehat{\beta}\mu+\widehat{\gamma}-\sigma^2\leq 0\ \}$	$\left(-\frac{\widehat{\beta}}{2\widehat{\alpha}}\,,\widehat{\gamma}-\frac{\widehat{\beta}^2}{4\widehat{\alpha}}\right)$
$\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}\geq 0$	$\left]-\infty;-\frac{\widehat{\beta}}{2\widehat{\alpha}}-\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}}\right]$	$\{(\mu,\sigma^2)\in\left]-\infty;-\frac{\widehat{\beta}}{2\widehat{\alpha}}-\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}}\right]\times IR^+/\ \widehat{\alpha}\mu^2+\widehat{\beta}\mu+\widehat{\gamma}-\sigma^2\leq 0\ \}$	$\left(-\frac{\widehat{\beta}}{2\widehat{\alpha}}-\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}};0\right)$
$\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}\geq 0$	$\left[-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}};+\infty\right[$	$\{(\mu,\sigma^2)\in\left[-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}};+\infty\right[\times IR^+/\ \widehat{\alpha}\mu^2+\widehat{\beta}\mu+\widehat{\gamma}-\sigma^2\leq 0\ \}$	$\left(-\frac{\widehat{\beta}}{2\widehat{\alpha}}+\sqrt{\left(\frac{\widehat{\beta}}{2\widehat{\alpha}}\right)^2-\frac{\widehat{\gamma}}{\widehat{\alpha}}}\,,0\right)$

Source: Author

4. Application of Markowitz Random Set on the Returns of a Portfolio Composed by 200 Assets from the Paris Stock Market Prices

4.1. Numerical Estimation of Markowitz Set $\widehat{H}(\mu, \sigma^2)$

We use a Data base of returns of a portfolio composed by 200 assets from the Paris Stock Market Prices (All the data come from the site of Paris Stock Market Prices) [17].

The expression of estimable Markowitz set is given by:

$$\widehat{H}(\mu,\sigma^2) = \{(\mu,\sigma^2) \in (R \times R+) \cap K/ \ \widehat{\sigma}^2{}_{Mz}(\mu) - \sigma^2 \leq 0\}$$

With $K \subset IR^2$ is a compact convex subset, and:

$$\begin{cases} \widehat{\sigma}^2{}_{Mz}(\mu) = \widehat{\alpha}\mu^2 + \widehat{\beta}\mu + \widehat{\gamma} \\ \text{with} \\ \widehat{\alpha} = \dfrac{\widehat{C}}{\widehat{A}\widehat{C} - \widehat{B}^2};\ \widehat{\beta} = -\dfrac{2\widehat{B}}{\widehat{A}\widehat{C} - \widehat{B}^2};\ \widehat{\gamma} = \dfrac{\widehat{A}}{\widehat{A}\widehat{C} - \widehat{B}^2} \\ \text{and} \\ \widehat{A} = \widehat{\vartheta}^T\widehat{\Sigma}^{-1}\widehat{\vartheta};\ \widehat{B} = 1_N{}^T\widehat{\Sigma}^{-1}\widehat{\vartheta}\ ;\ \widehat{C} = 1_N\widehat{\Sigma}^{-1}1_N \end{cases}$$

4.1.1. Estimation of the Factors $\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}$

We use the code from "R" software for statistical calculation (see appendix 1), we get the following results:

$\widehat{\alpha} = 4450718$, $\widehat{\beta} = 119394.3$ and $\widehat{\gamma} = 0.0001435223$.

Then:

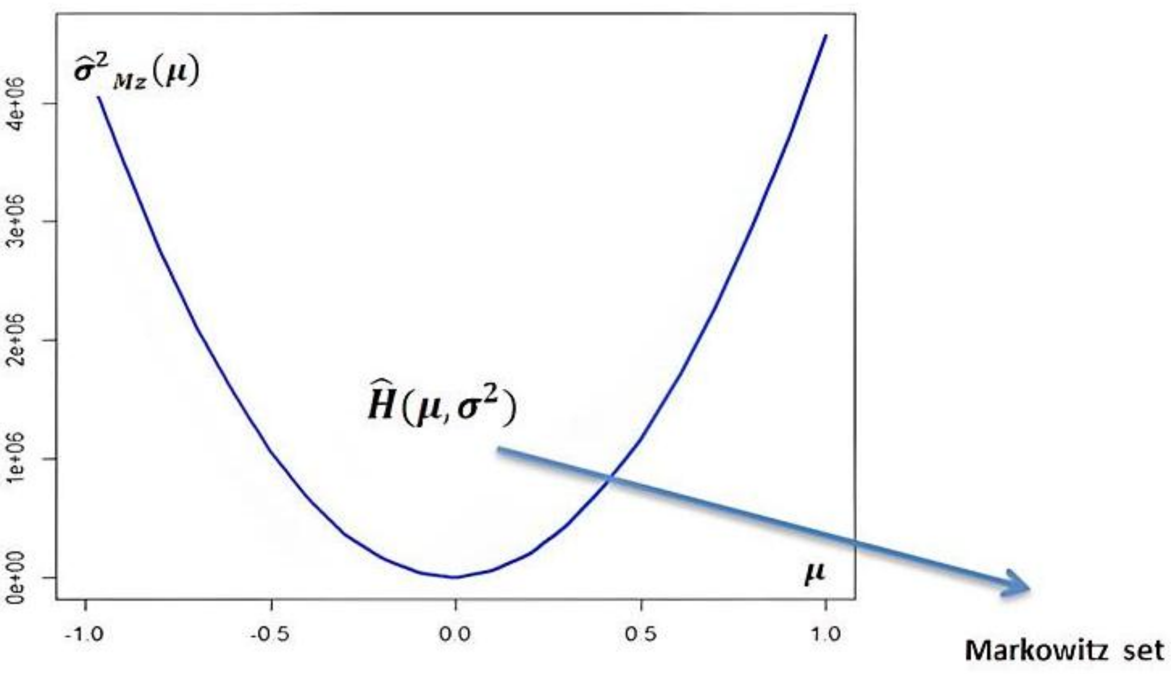
$$\widehat{\sigma}^2{}_{Mz}(\mu) = 4450718\,\mu^2 + 119394.3\mu + 0.0001435223$$

And our estimation of Markowitz set is given by:

$$\widehat{H}(\mu,\sigma^2) = \{(\mu,\sigma^2) \in (R \times R+) \cap K/\ 4450718\,\mu^2 + 119394.3\,\mu + 0.0001435223 - \sigma^2 \leq 0\} \quad (20)$$

The figure (Figure 1) below represents the curve of the equation $\widehat{\sigma}^2{}_{Mz}(\mu)$ = 4450718μ^2+119394.3μ+0.0001435223, which is a hyperbola, where the abscissa axis is the mean of the portfolio returns, and the coordinate axis is the variances of these returns.

The area above the hyperbola curve represents the Markowitz set.



Source: Author

Figure 1. Graphical representation of Markowitz Set $\widehat{H}(\mu,\sigma^2)$

4.1.2. Determination of the Efficient Frontier of $\widehat{\boldsymbol{H}}(\boldsymbol{\mu},\boldsymbol{\sigma^2})$

Let f a bijective function $f: I \rightarrow IR^{+}$, such that:
$f(\mu) = \hat{\sigma}^2{}_{Mz}(\mu)$ for each $\mu \in I$.
On the one hand, we have:

$$\begin{cases} \hat{\alpha} = 4450718 > 0 \\ \text{and} \\ \left(\dfrac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \dfrac{\hat{\gamma}}{\hat{\alpha}} = 0.0001799064 \geq 0 \end{cases}$$

Then there are two cases:

1st Case:

$$\mathbf{I} = \left]-\infty; -\frac{\hat{\boldsymbol{\beta}}}{\mathbf{2}\hat{\boldsymbol{\alpha}}} - \sqrt{\left(\frac{\hat{\boldsymbol{\beta}}}{\mathbf{2}\hat{\boldsymbol{\alpha}}}\right)^2 - \frac{\hat{\boldsymbol{\gamma}}}{\hat{\boldsymbol{\alpha}}}}\right] =]-\infty; -\mathbf{0.02682585}]$$

Following the results mentioned in the third row of the Table 1, we have:

$$\widehat{H}(\mu,\sigma^2) = \{(\mu,\sigma^2) \in]-\infty; -0.02682585] \times IR^{+} / \ 4450718\,\mu^2 + 119394.3\mu + 0.0001435223 - \sigma^2 \leq 0\}$$

And:

$$f^{-1}: \ [0; +\infty[\ \rightarrow]-\infty; -0.02682585]$$
$$y = \hat{\sigma}^2{}_{Mz}(\mu) \rightarrow \mu = f^{-1}(y)$$

with

$$f^{-1}(y) = -0.01341293 - \sqrt{\frac{y - 0.0001435223}{4450718} + 0.0001799064}$$

Then:

$$\min_{\mu \in I} \hat{\sigma}^2{}_{Mz}(\mu) = \hat{\sigma}^2{}_{Mz}(\mu_0) = 0 \text{ , with } \mu_0 = f^{-1}(0)$$

Finally:

$$\mu_0 = f^{-1}(0) = -0.02682585$$

Then:

$$\inf\left(\widehat{H}(\mu,\sigma^2)\right) = \inf\{(\mu,\sigma^2) \in]-\infty; -0.02682585] \times IR^{+} / \ 4450718\,\mu^2 + 119394.3\mu + 0.00014\} = (-0.02682585\,;\ 0) \tag{21}$$

And the boundary of our estimable Markowitz set denoted $\inf\left(\widehat{H}(\mu,\sigma^2)\right)$ is the couple

$$\left(-\frac{\hat{\beta}}{2\hat{\alpha}} - \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}\ , 0\right) = (-0.02682585\,;\ 0)$$

which is the efficient frontier.

2nd Case:

$$\mathbf{I} = \left[-\frac{\hat{\boldsymbol{\beta}}}{\mathbf{2}\hat{\boldsymbol{\alpha}}} + \sqrt{\left(\frac{\hat{\boldsymbol{\beta}}}{\mathbf{2}\hat{\boldsymbol{\alpha}}}\right)^2 - \frac{\hat{\boldsymbol{\gamma}}}{\hat{\boldsymbol{\alpha}}}}; +\infty\right[= [-\mathbf{1.205673exp}(-\mathbf{08}), +\infty[$$

Following the results mentioned in the fourth row of the Table 1, we have:

$$\widehat{H}(\mu,\sigma^2) = \{(\mu,\sigma^2) \in [-1.205673exp(-08), +\infty[\times IR^{+} / \ 4450718\,\mu^2 + 119394.3\mu + 0.0001435223 - \sigma^2 \leq 0\}$$

And:

$$f^{-1}: \ [0; +\infty[\ \rightarrow [-1.205673\exp(-08), +\infty[$$
$$y = \hat{\sigma}^2{}_{Mz}(\mu) \rightarrow \mu = f^{-1}(y)$$

$$f^{-1}(y) = -0.01341293 + \sqrt{\frac{y - 0.0001435223}{4450718} + 0.0001799064}$$

Then:

$$\min_{\mu \in I} \hat{\sigma}^2{}_{Mz}(\mu) = \hat{\sigma}^2{}_{Mz}(\mu_0) = 0 \text{ , with } \mu_0 = f^{-1}(0)$$

Finally:

$$\mu_0 = f^{-1}(0) = -1.205673\exp(-08)$$

Then:

$$\inf\left(\widehat{H}(\mu,\sigma^2)\right) = \inf\{(\mu,\sigma^2) \in [-1.205673\exp(-08), +\infty[\times IR^{+} / \ 4450718\,\mu^2 + 119394.3\mu + 0.0001435223 - \sigma^2 \leq 0\} = (-1.205673\exp(-08)\,;\ 0) \tag{22}$$

And the boundary of our estimable Markowitz set denoted $\inf\left(\widehat{H}(\mu,\sigma^2)\right)$ is the couple

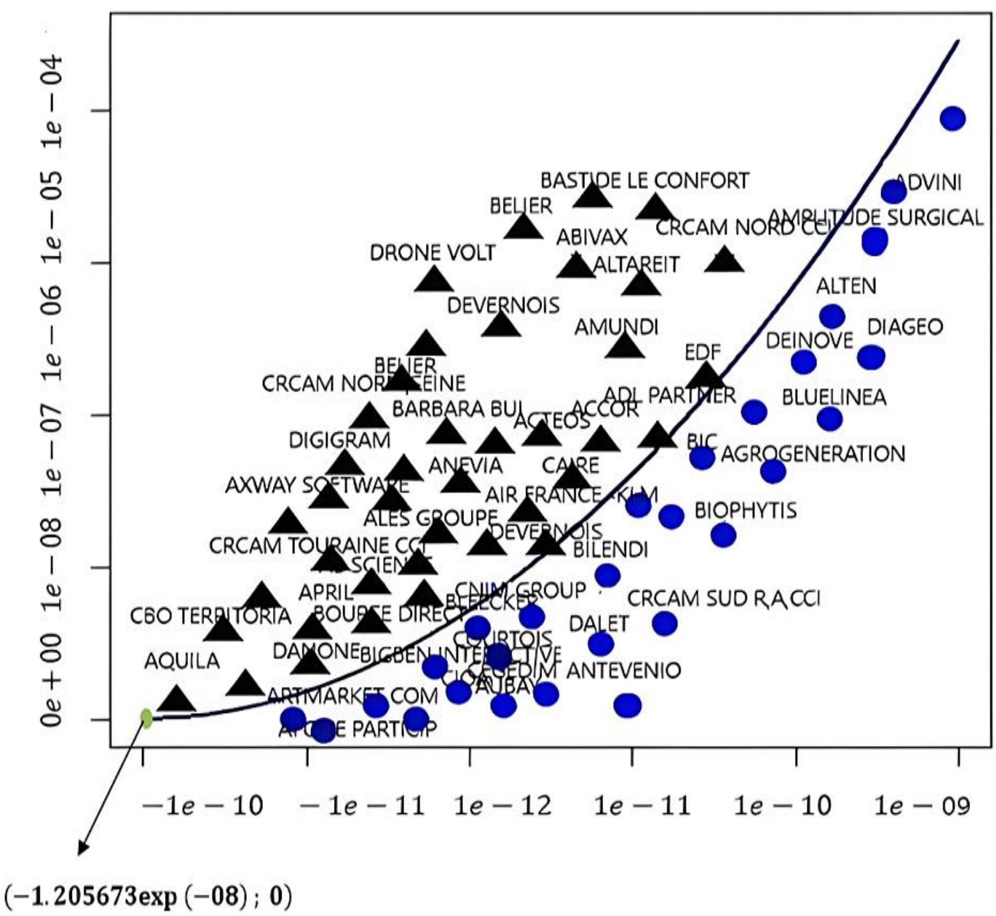
$$\left(-\frac{\hat{\beta}}{2\hat{\alpha}} + \sqrt{\left(\frac{\hat{\beta}}{2\hat{\alpha}}\right)^2 - \frac{\hat{\gamma}}{\hat{\alpha}}}\ , 0\right) = (-1.205673\exp(-08)\,;\ 0)$$

which is the efficient frontier.

4.2. Graphical Representation of the Markowitz Set $\widehat{\boldsymbol{H}}(\boldsymbol{\mu},\boldsymbol{\sigma^2})$, and the Efficient Frontier

In our case, the mean of the returns of the stocks in a portfolio operated from the Paris Stock Market Prices [17] are higher than -1.205673exp (-08), so we will be in the second case of the efficient frontier.

Using the code from "R" software for statistical calculation (see appendix 2), we will get the following repartition of the stocks in figure 2, which allows us to have an idea on the stocks that the investor can choose to make his investment.



Source: Author

Figure 2. Distribution of the stocks on the Markowitz set

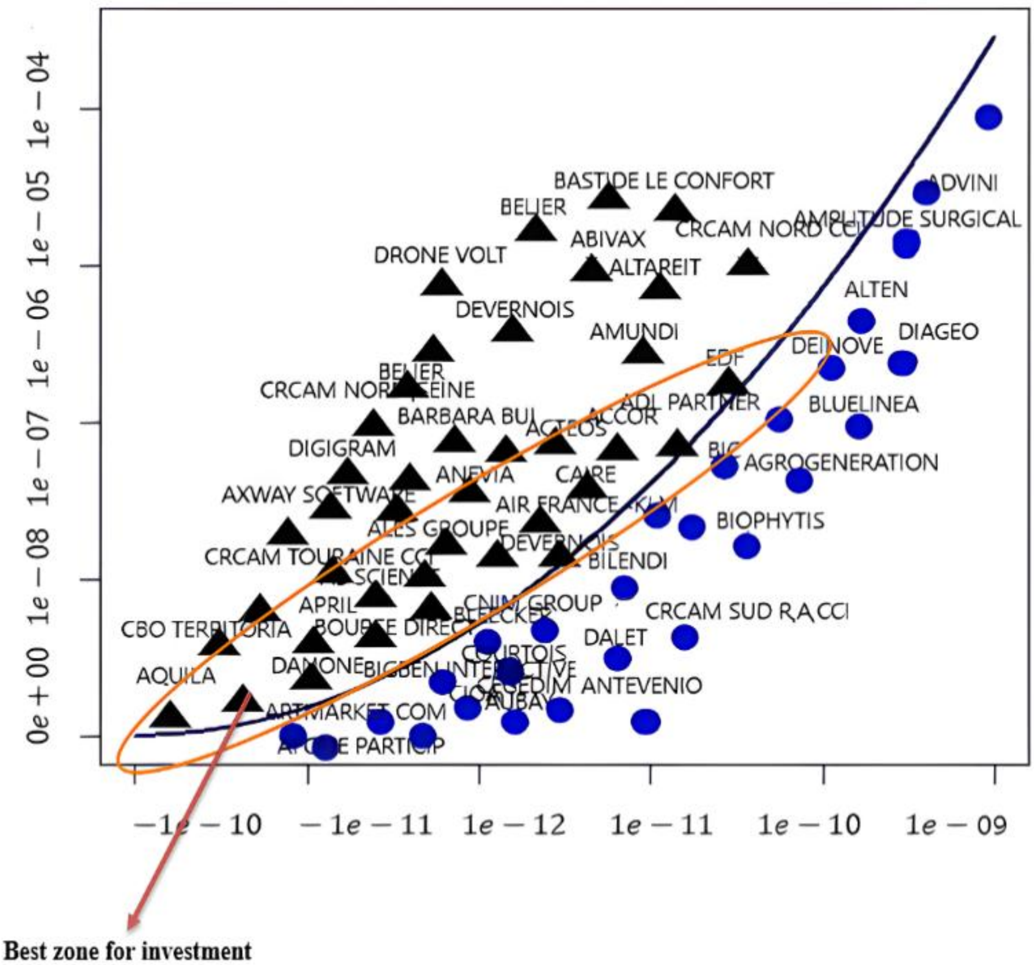
The abscissa axis and the coordinate axis in the figure 2 represent the expectation and variance of stock returns, respectively.

Stocks above the hyperbola curve have returns whose means and variances are peers belonging to the Markowitz set.

Also, we remark that in figure 2, the couple $(-1.205673\ \exp(-08)\ ;\ 0)$ belongs to the hyperbola curve representing the efficient frontier.

The stocks located between the curve of the function $\hat{\sigma}^2_{Mz}(\mu) = 4450718\mu^2 + 119394.3\mu + 0.0001435223$, and the vertical line $\mu = -1.205673exp\,(-08)$ belong to the Markowitz set (figure 3). This zone of stocks is considered as the good investment zone for an investor, which allows him to optimize his choices.

As we remark in figure 3 below, when the variance of the returns of the assets located in the Markowitz set, are very near to the curve of the function $\hat{\sigma}^2_{Mz}$, we will be in the best investment zone.



Source: Author

Figure 3. Graphical representation of best zone of investment

Then we conclude that the stocks located within the best zone for investment, are considered the best choice for a rational investor.

5. Conclusions

This article has shown that the theory of random sets is very important. This is manifested by the use of these concepts in the financial analysis of portfolio returns, based on the Markowitz set, which is composed by the pairs of means and variances of returns.

The combination of these two theories, allowed us to have new techniques to search the efficient frontier of such a portfolio.

Mathematical optimization techniques and statistical estimation tools used, as well as the manipulation and processing of data from the Paris Stock Market Prices with the statistical calculation software "R", give a meaning to the theoretical results found, especially since many applications in finance are based on Markowitz approaches, and our goal in this sense is to push investors to use this new technique for having an idea about the decision-making behavior in the investment on the stocks that should belong to the Markowitz set, in order to help them to optimize their choices.

In another way, the investors should search for the variances of the returns of the assets located in the Markowitz set that are very near the hyperbola curve that represents the bound of this set.

Appendices

Appendix 1

>data=read.csv("C:/Users/BOURAKADI/Desktop/R200.csv",sep=";")
> $\hat{\boldsymbol{\vartheta}}$ *=colMeans(data)*
>$\hat{\Sigma}$*=cov(data)*
>$\hat{\Sigma}^{-1}$*=solve(*$\hat{\Sigma}$*)*
>*A=t(*$\hat{\boldsymbol{\vartheta}}$*)%*%* $\hat{\Sigma}^{-1}$*%*%* $\hat{\boldsymbol{\vartheta}}$
>*I=matrix(1,nrow=200,ncol=1)*
>*B=t(I)%*%* $\hat{\Sigma}^{-1}$*%*%* $\hat{\boldsymbol{\vartheta}}$
>*C=t(I)%*%* $\hat{\Sigma}^{-1}$*%*%I*

```
>alpha=C/(A*C-B*B)
>beta=-2*B/(A*C-B*B)
>gamma=A/(A*C-B*B)
```

Appendix 2

```
> x=seq(-1.205673*exp(-08),10,0.00001)
> y=4450718*x*x+119394.3*x+0.0001435223
> plot(x,y, type="l", col="blue", lwd=2)
>y1Color <- "#000000"
>y2Color <- rgb(0.2, 0.6, 0.9, 1)
coeff <- 1
>ggplot(data, aes(x=log(X))) + geom_point( aes(y=log(Y1)), size=2, color=y1Color) + geom_line( aes(y=log(Y2)), size=2, color=y2Color) + scale_y_continuous( name = "Y1", sec.axis = sec_axis(~.*coeff, name="Y2")) +theme(axis.title.y = element_text(color = "red", size=13), axis.title.y.right = element_text(color = "blue", size=13) +ggtitle("point cloud").
```

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