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The f-prime Radicals in Posets

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Abstract A relation is a mathematical tool for describing set relationships. Relationships are common in databases and scheduling applications. Science and engineering are designed to help humans make better decisions. To make these choices, we must first understand human expectations, the outcomes of various options, and the degree of confidence. With all of these data, partial orders will be generated.

In several fields of engineering and computer science, partial order and lattice theory are now widely used. To mention a few, they are used in cloud computing (vector clocks, global predicate detection), concurrency theory (pomsets, occurrence nets), programming language semantics (fixed-point semantics), and data mining (concept analysis). Other theoretical disciplines benefit from them as well, such as combinatorics, number theory, and group theory. Partially ordered sets emerge naturally when dealing with multidimensional systems of qualitative ordinal variables in social science, especially to solve ranking, prioritising, and assessment concerns. As an alternative to standard techniques, partial order theory and partially ordered sets can be used to generate composite indicators for evaluating well-being, quality of life, and multidimensional poverty. They can be applied in multi-criteria analysis or for decision-making purposes in the study of individual and social desires, including in social choice theory. They're also valuable in social network analysis, where they may be utilized to apply mathematics to explore network topologies and dynamics. The Hasse diagram method, for example, produces a partial order with multiple incomparabilities (lack of order) between pairs of items. This is a common problem in ranking studies, and it can often be avoided by combining object attributes that lead to a complete order. However, such a mix introduces subjectivity and prejudice into the rating process. This work discusses the notion of a f-prime radical of a partially ordered set with respect to ideal. In posets, we investigated the concept of f-primary ideals. It is investigated how to characterise f-primary ideals in relation to f-prime radicals. In addition, an ideal's f-primary decomposition is constructed.

Keywords Strongly Prime Ideal, Strongly m-system, Poset, Ideals, f-prime Ideal, f-system, f-prime Radical, f-primary, f-primary Decomposition.

1 Introduction

In algebraic structures, radicals play a significant function. The Jacobson radical is the intersection of all maximal ideals in a commutative ring with unity, while the prime radical of the ring is the intersection of all prime ideals. The radical notion was used to launch the primary ideal, which was a development of prime ideal principles[1].

Van der Walt [2] defined s-prime ideals in non-commutative rings and derived McCoy [3] findings for s-prime ideals. Many authors confirmed several of Van der Walt's earlier results to near-rings. Murata et al.[4] defined f-prime ideals and obtained findings comparable to Van der Walt. Some of these conclusions have been extended to Γ-rings by Hsu [5].

Sambasiva Rao and Satyanarayana [6] defined the prime radical in terms of strongly nilpotent elements of near-rings and extended certain results of Hsu [5] to f-prime and f-semiprime ideals in near-rings.

The notion of primary ideals is important in mathematics, particularly in abstract algebra, because the deconstruction of an ideal into primary ideals is a key pillar of ideal theory. It establishes an algebraic foundation for breaking down an algebraic variety into irreducible parts.

The concept of primary ideals was important in commutative ring theory but was subsequently used to commutative semigroups[7].

In the arbitrary semi-group, A.Anjaneyulu[8] proposed the establishment of basic ideals. According to [9], M. Satyanarayana established commutative primary semi-groups wherein every ideal in the semi-group is the primary ideal. He separates it from commutative primary rings in terms of structure. He also defined a threshold for semi-group to primary semi-group conversion that is both necessary and sufficient.

Badawi[10] defined a 2-absorbing primary ideal and presented certain 2-absorbing primary ideal features for a commutative ring R. He further constructed some of 2-absorbing primary ideals. Kim[11] and Ze Gu[12] studied f-primary ideals in semigroups and ordered semigroups, respectively.

Murata [13] used the m-system to apply the concept of a primary ideal to a compactly built multiplicative lattice. He also devised a basic decomposition theorem for lattice ideals. Many authors studied the prime ideal concepts in posets [14, 15, 16, 17, 18, 19, 20]. Joshi [21] and John [22] later extended the primary ideal concept to posets. Considering [23, 24], in this work, we investigate the concept of f-primary ideals in posets.

2 Preliminaries

During all of this work, a poset including 0 as its least element is indicated as $(\mathbb{Q}, \leqslant)$. We refer to [14] and [15] for basic concepts and notations of posets.

For $S \subseteq \mathbb{Q}$, $(S)^\ell = \{q \in \mathbb{Q} : q \leqslant s \ \forall s \in S\}$ indicates the lower cone of S in $\mathbb{Q}$ and $(S)^u = \{q \in \mathbb{Q} : s \leqslant q \ \forall s \in S\}$ indicates the upper cone of S in $\mathbb{Q}$. For any subsets S, T of $\mathbb{Q}$, we represent $(S, T)^\ell$ rather than $(S \cup T)^\ell$ and $(S, T)^u$ instead of $(S \cup T)^u$.

For a finite subset $S = \{s_1, s_2, ..., s_n\}$ of $\mathbb{Q}$, we write $(s_1, s_2, ..., s_n)^l$ instead of $(\{s_1, s_2, ..., s_n\})^l$ and dually. Clearly for a subset S of $\mathbb{Q}$, $S \subseteq (S)^{u\ell}$ and $S \subseteq (S)^{\ell u}$. If $S \subseteq T$, then $(T)^\ell \subseteq (S)^\ell$ and $(T)^u \subseteq (S)^u$. Also, $(S)^{u\ell u} = (S)^\ell$ and $(S)^{\ell u \ell} = (S)^u$.

Considering [16], a subset $B(\neq \phi)$ of $\mathbb{Q}$ is termed as semi-ideal if $q \in B$ with $s \leqslant q$, then $s \in B$. Also B is termed as ideal, if whenever $s, k \in B$ implies $(s, k)^{u\ell} \subseteq B$[15].

For ideals $B_i's$ of $\mathbb{Q}$, $\bigcap_i B_i$ is an ideal of $\mathbb{Q}$. In general, $\bigcup_i B_i$ is not essentially to be an ideal of $\mathbb{Q}$.

A semi-ideal (resp., ideal) $B(\neq \mathbb{Q})$ of $\mathbb{Q}$ is referred to as prime if whenever $(s, d)^\ell \subseteq B$ implies either $s \in B$ or $d \in B$ [14]. We have dually filter and prime filter concepts.

An ideal B of $\mathbb{Q}$ is termed as semiprime whenever $(r, s)^\ell \subseteq B$ and $(r, t)^\ell \subseteq B$ together imply $(r, (s, t)^u)^\ell \subseteq B$ for all $r, s, t \in \mathbb{Q}$ [15].

For $s \in \mathbb{Q}$, the principal ideal (resp., filter) of $\mathbb{Q}$ generated by s is $(s] = (s)^\ell = \{q \in \mathbb{Q} : q \leqslant s\}$ (resp., $[s) = (s)^u = \{q \in \mathbb{Q} : q \geq s\}$). A subset $S(\neq \phi)$ of $\mathbb{Q}$ is known as an up directed set if $S \cap (r, s)^u \neq \phi \ \forall r, s \in S$. We also have dually the notion of a down directed set. If an ideal B (resp., filter F) of $\mathbb{Q}$ is an up (resp., down) directed set of $\mathbb{Q}$, then it is referred to as u-ideal (resp., l-filter).

Considering [17], for a subset K and a semi-ideal J of $\mathbb{Q}$, we indicate

$$\langle K, J \rangle = \{t \in \mathbb{Q} : (a, t)^\ell \subseteq J \ \forall a \in K\} = \bigcap_{a \in K} \langle a, J \rangle.$$

We write $\langle s, J \rangle$ instead of $\langle \{s\}, J \rangle$ while $K = \{s\}$. It is evident $K \subseteq \langle \langle K, J \rangle, J \rangle$ and $t \in \langle \langle t, J \rangle, J \rangle$ for a semi-ideal J of $\mathbb{Q}$. Furthermore, if $K \subseteq C$, then $\langle C, J \rangle \subseteq \langle K, J \rangle$.

For any subset Q_1 of $\mathbb{Q}$ and a semi-ideal I_1 of $\mathbb{Q}$, it is trivial to establish that $\langle \langle \langle Q_1, I_1 \rangle, I_1 \rangle, I_1 \rangle = \langle Q_1, I_1 \rangle$. If I_1 is a semi-prime ideal of $\mathbb{Q}$, then $\langle y, I_1 \rangle$ is an ideal of $\mathbb{Q} \ \forall y \in \mathbb{Q}$. Minimal prime ideals (resp., minimal prime u-ideals) of $\mathbb{Q}$ are the minimal elements of all prime ideals (resp., prime u-ideals) of $\mathbb{Q}$. Dually, we have the definitions of maximal filters and maximal l-filters[15].

Following [18], an ideal J of $\mathbb{Q}$ is called strongly prime if whenever $(I_1^*, I_2^*)^\ell \subseteq J$ implies either $I_1 \subseteq J$ or $I_2 \subseteq J$ for proper different ideals I_1, I_2 of $\mathbb{Q}$, where $I_1^* = I_1 \backslash \{0\}$.

Let G be a semi-ideal of $\mathbb{Q}$. Then G satisfies (*) condition if whenever $(T, S)^\ell \subseteq G$, then $T \subseteq \langle S, G \rangle$ for any subsets T and S of $\mathbb{Q}$.

A subset $F(\neq \phi)$ of $\mathbb{Q}$ is called a m-system if for $t_1, t_2 \in F$, $\exists t \in (t_1, t_2)^\ell : t \in F$. A subset $F(\neq \phi)$ of $\mathbb{Q}$ is referred to as strongly m-system if for different proper ideals I_1, I_2 of $\mathbb{Q}$, whenever $I_1 \cap N \neq \emptyset$ and $I_2 \cap N \neq \emptyset$ imply $(I_1^*, I_2^*)^\ell \cap N \neq \emptyset$.

It is obvious that for an ideal I_1 of $\mathbb{Q}$, $\mathbb{Q} \backslash I_1$ is a strongly m-system of $\mathbb{Q}$ if and only if I_1 is strongly prime. Any strongly m-system of $\mathbb{Q}$ is also a m-system of $\mathbb{Q}$. However, the reverse does not have to be true in all cases, see Example 1.1 of [17].

For a poset $\mathbb{Q}$, $spec(\mathbb{Q})$, $Sspec(\mathbb{Q})$ and $Smin(\mathbb{Q})$ represents the gathering of all prime, strongly prime and minimal strongly prime ideals of $\mathbb{Q}$, respectively.

3 f-prime radicals and f-primary ideals

For every element $q \in \mathbb{Q}$, we associate a unique ideal $f(q)$, which meets the following criteria:

(i) $q \in f(q)$ and

(ii) $w \in f(q)$ implies that $f(w) \subseteq f(q)$, for $w \in \mathbb{Q}$.

The collection of all such mappings from $\mathbb{Q}$ into collection of all ideals of $\mathbb{Q}$ is indicated by $\mathbb{F}(\mathbb{Q})$.

Example 3.1. *In a poset $\mathbb{Q}$, for each element q of $\mathbb{Q}$, if $f(q) = (q)^\ell$, the principal ideal is generated by q, then it is obvious that f meets the previous requirements..*

Definition 3.2. *For $f \in \mathbb{F}(\mathbb{Q})$, a set $S(\subseteq \mathbb{Q})$ is referred to as an f-system if and only if it consists a strongly m-system K_1 such that $K_1 \cap f(q) \neq \phi$ for each $q \in S$. In each case K_1 will be called a kernal of S.*

Definition 3.3. *An ideal G of $\mathbb{Q}$ is called a f-prime if and only if its complement G^c is a f-system of $\mathbb{Q}$.*

It is clear that every strongly m-system is a f-system and every strongly prime ideal of $\mathbb{Q}$ is a f-prime ideal of $\mathbb{Q}$. However, as shown in the following illustration, the reverse is not always valid.

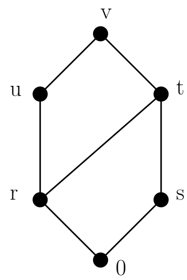

Figure 1. Example of a f-system which is not a strongly m-system

Example 3.4. *For a poset $(\mathbb{Q}, \leqslant)$, where $\mathbb{Q} = \{0, r, s, t, u, v\}$ and a relation $\leqslant$ defined on $\mathbb{Q}$ as follows.*

Consider a mapping f from $\mathbb{Q}$ into collection of ideals of $\mathbb{Q}$ and defined by $f(0) = \{0\}$, $f(r) = \{u, r, 0\}$, $f(s) = \{0, s\}$, $f(u) = \{0, r, u\}$, $f(t) = \{0, r, s, t, u, v\}$ and $f(v) = \{0, r, s, t, u, v\}$. Then $f \in \mathbb{F}(\mathbb{Q})$. Here $M_1 = \{u, v, t, r\}$ is a f-system and contains the strongly m-system $M_2 = \{u, v\}$, but M_1 is not a strongly m-system as for the ideals $G_1 = \{0, r, u\}$, $G_2 = \{0, r, s, t\}$, we have $G_1 \cap M_1 \neq \phi$, $G_2 \cap M_1 \neq \phi$ and $(G_1^*, G_2^*)^\ell \cap M_1 = \phi$.

Theorem 3.5. *For any f-prime ideal G of $\mathbb{Q}$, $(f(w_1)^*, f(w_2)^*)^\ell \subseteq G$ implies that either $w_1 \in G$ or $w_2 \in G$ for different proper ideals $f(w_1)$, $f(w_2)$ of $\mathbb{Q}$, where $f(w_1)^* = f(w_1) \backslash \{0\}$.*

Proof: Suppose not, $w_i \in \mathbb{Q}\backslash G$ for $i = 1, 2$. As $\mathbb{Q}\backslash G$ is a f-system, $\exists$ a strongly m-system $F \subseteq \mathbb{Q}\backslash G : F \cap f(w_i) \neq \phi$ for $i = 1, 2$. Since F is a strongly m-system of $\mathbb{Q}$, we have $(f(w_1)^*, f(w_2)^*)^\ell \cap F \neq \phi$ which implies $(f(w_1)^*, f(w_2)^*)^\ell \cap \mathbb{Q}\backslash G \neq \phi$, a contradiction.

Remark 3.6. *According to the preceding results, every f-prime ideal of $\mathbb{Q}$ is prime ideal, and every strongly prime ideal of $\mathbb{Q}$ is f-prime ideal. In general, the converse is not true in every case.*

Definition 3.7. *For an ideal G of $\mathbb{Q}$, f-prime radical $r_f(G)$ of G has been constructed as the collection of all $q \in \mathbb{Q}$ such that each f-system of $\mathbb{Q}$ which contains q has a non-empty intersection with G.*

Remark 3.8. *The condition (α) is shown to be satisfied by a poset $\mathbb{Q}$ if for any f-system F with kernal M and for any ideal G, $F \cap G$ is non-empty, then so is $M \cap G$.*

Theorem 3.9. *Let G_1 and G_2 be ideals of $\mathbb{Q}$. Then the preceding statements remain true.*

(i) $G_1 \subseteq r_f(G_1)$.

(ii) $r_f(r_f(G_1)) = r_f(G_1)$.

(iii) If $G_2 \subseteq G_1$, then $r_f(G_2) \subseteq r_f(G_1)$.

(iv) $r_f((G_2^*, G_1^*)^\ell) = r_f(G_2 \cap G_1) = r_f(G_2) \cap r_f(G_1)$, provided $\mathbb{Q}$ satisfies the condition (α).

(v) If G_1 is a f-prime ideal of $\mathbb{Q}$, then $r_f(G_1) = G_1$.

(vi) If G_1 is f-prime ideal and $G_2 \subseteq G_1$, then $r_f(G_2) \subseteq G_1$.

Proof: (i) Let $q \in G_1$. Then, clearly, each f-system having q has a non-empty intersection with G_1. Therefore $q \in r_f(G_1)$.

(ii) Let $q \in r_f(r_f(G_1))$ and assume that $q \notin r_f(G_1)$. Then a f-system F_q exists with $q \in F_q$ and $F_q \cap G_1 = \phi$. Since $q \in r_f(r_f(G_1))$, we get $F_q \cap r_f(G_1) \neq \phi$. Let $p \in F_q \cap r_f(G_1)$. As $p \in r_f(G_1)$, then every f-system having p must intersect G_1. Specifically, $F_q \cap G_1 \neq \phi$, is a contradiction.

(iii) it is trivial.

(iv) For any ideals G_1 and G_2 of $\mathbb{Q}$, we have $(G_2^*, G_1^*)^\ell \subseteq G_2 \cap G_1 \subseteq G_2$. Then by (iii), we have $r_f((G_2^*, G_1^*)^\ell) \subseteq r_f(G_2 \cap G_1) \subseteq r_f(G_2)$ which implies $r_f((G_2^*, G_1^*)^\ell) \subseteq r_f(G_2 \cap G_1) \subseteq r_f(G_2) \cap r_f(G_1)$.

Let $s \in r_f(G_2) \cap r_f(G_1)$ and H be a f-system of $\mathbb{Q}$ having s. Now we get $H \cap G_1 \neq \phi$ and $H \cap G_2 \neq \phi$. Since $\mathbb{Q}$ satisfies the condition (α), an strongly m-system M which is kernal of H gives $M \cap G_1 \neq \phi$ and $M \cap G_2 \neq \phi$ which implies $(G_1^*, G_2^*)^\ell \cap H \neq \phi$, so $s \in r_f((G_1^*, G_2^*)^\ell)$.

(v) Let G_1 be a f-prime ideal of $\mathbb{Q}$ and $r_f(G_1) \nsubseteq G_1$. Then $\exists\ q \in r_f(G_1) : q \notin G_1$. As G_1 is f-prime, we get $\mathbb{Q}\backslash G_1$ is a f-system of $\mathbb{Q}$ containing q and $(\mathbb{Q}\backslash G_1) \cap G_1 = \phi$, which is a contradiction to the fact that $q \in r_f(G_1)$. Hence $G_1 = r_f(G_1)$.

(vi) Let G_1 and G_2 be ideals of $\mathbb{Q}$ and G_1 be f-prime such that $G_2 \subseteq G_1$. Then by (iii) and (v), we have $r_f(G_2) \subseteq r_f(G_1) = G_1$. □

Theorem 3.10. *([18], Theorem 2.1) Let $F(\neq \phi)$ be a strongly m-system of $\mathbb{Q}$ and G be an ideal of $\mathbb{Q}$ and $G \cap F = \phi$. Then G is contained in a strongly prime ideal R of $\mathbb{Q}$ and $R \cap F = \phi$.*

Theorem 3.11. *For an ideal T of $\mathbb{Q}$, We now possess $\{\bigcap_i R_i :$ R_i is f-prime ideal of $\mathbb{Q}$ and $R_i \supseteq T\ \} = \{q \in \mathbb{Q} : F$ is a f-system with $q \in F$ and $F \cap T \neq \phi\}$.*

Proof: Let $K = \{w \in \mathbb{Q} : F$ is a f-system with $w \in F$ and $F \cap T \neq \phi\}$ and $q \notin K$. There is then, a f-system F of $\mathbb{Q}$ which contains q and $F \cap T = \phi$. Together with help of Theorem 3.10, $\exists$ a f-prime ideal R of $\mathbb{Q}$ such that $T \subseteq R$ together with $R \cap F = \phi$ imply that $q \notin \cap R_i$. So $\cap R_i \subseteq K$.

On the other hand, let $q \notin \cap R_i$. Consequently, there has been a f-prime ideal R_i of $\mathbb{Q}$ with $q \notin R_i$ which implies $q \in \mathbb{Q}\backslash R_i$ and $\mathbb{Q}\backslash R_i$ is a f-system in $\mathbb{Q}$. Since $(\mathbb{Q}\backslash R_i) \cap T = \phi$, we have $q \notin K$. Hence $K \subseteq \cap R_i$. □

Definition 3.12. *For an ideal G of $\mathbb{Q}$ and a f-prime ideal T of $\mathbb{Q}$ with $G \subseteq T$, T is also known as minimal f-prime ideal of G if there is no f-prime ideal U of $\mathbb{Q}$ with $G \subset U \subset T$.*

$Fspec(\mathbb{Q})$ denotes the collections of all f-prime ideals in $\mathbb{Q}$ and $Fmin(\mathbb{Q})$ represents the collection of all minimal f-prime ideals in $\mathbb{Q}$. For any ideal G of $\mathbb{Q}$, $FP(G) = \bigcap_{G_i \supseteq G} G_i$ and $FP(\mathbb{Q}) = \bigcap G_i$, where G_i's are f-prime ideal of $\mathbb{Q}$. We also have $FP(G) = FP(\mathbb{Q})$ if $G = \{0\}$.

Remark 3.13. *For an ideal R of $\mathbb{Q}$, we have $FP(R) = r_f(R)$. Moreover, $r_f(R)$ becomes an ideal of $\mathbb{Q}$ as $\bigcap_{R \in Id(\mathbb{Q})} R$ is again* *an ideal in* $\mathbb{Q}$*. Following [14], for an ideal* R *of* $\mathbb{Q}$, $\bigcap_{K_i \supseteq R} K_i = R$*, where* K_i*'s are prime ideals in* $\mathbb{Q}$*.*

However, here's an example for $\bigcap_{R_i \supseteq R} R_i \neq R$, where R_i's are f-prime ideals in $\mathbb{Q}$.

Example 3.14. *For a poset* $(\mathbb{Q}, \leq)$*, where* $\mathbb{Q} = \{g, h, r, q, 0\}$ *with a partial relation* $\leq$ *on* $\mathbb{Q}$ *defined like the below.*

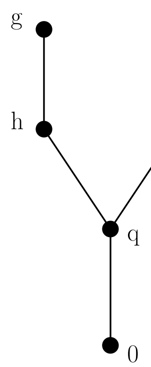

Figure 2. Example of intersection of all f-prime ideals having an ideal D is not equal to D

Define a map $f : \mathbb{Q} \to Id(\mathbb{Q})$ *by* $f(a) = (a]$*. Here* $I_1 = \{0, q, r\}$*;* $I_2 = \{g, h, q, 0\}$ *have been* f*-prime ideals of* $\mathbb{Q}$*. For an ideal* $D_1 = \{0, q\}$*,* $FP(D_1) = I_1 \cap I_2 = \{0, q\} = D_1$*.*

Furthermore, for the ideal $D_2 = \{0, q, h\}$*, We now also have* $FP(D_2) = I_2 = \{0, q, h, g\} \neq D_2$*.* □

Definition 3.15. *An ideal* $T(\subsetneq \mathbb{Q})$ *of* $\mathbb{Q}$ *is also known as* f*-primary, if* $(U^*, G^*)^\ell \subseteq T$ *implies that* $U \subseteq T$ *or* $G \subseteq r_f(T)$ *for any different proper ideals* U, G *of* $\mathbb{Q}$ *where* $U^* = U \backslash \{0\}$*.*

Each f-prime ideal in a poset $\mathbb{Q}$ is also a f-primary ideal, which is a significant finding. However, in the below example, the converse does not have to be true in some cases.

Example 3.16. *For a poset* $(\mathbb{Q}, \leq)$*, where* $\mathbb{Q} = \{w, v, u, t, s, h, 0\}$ *and* $\leq$*, a partial relation on* $\mathbb{Q}$ *defined as follows:*

Define a map $f : \mathbb{Q} \to Id(\mathbb{Q})$ *by* $f(a) = (a]$*. Here* $G = \{t, h, 0\}$ *is a* f*-primary ideal of* $\mathbb{Q}$*, but not a* f*-prime ideal as* $((u]^*, (s]^*) \subseteq G$ *with* $(u] \nsubseteq G$ *and* $(s] \nsubseteq G$*.* □

Definition 3.17. *Let* G *be an ideal of* $\mathbb{Q}$*.Then* G *is said to be* f_K*-primary if* G *is* f*-primary and* $r_f(G) = K$*, for some* f*-prime ideal* K *of* $\mathbb{Q}$*. Also representation* $G = U_1 \cap U_2 \cap ... \cap U_n$*, where each* U_i *is a* f_{K_i}*-primary is referred to as* f*-primary decomposition of* G*.*

Definition 3.18. *For an ideal* G *of* $\mathbb{Q}$ *and a* f*-primary decomposition* $G = U_1 \cap U_2 \cap ... \cap U_p$*, the* f*-primary decomposition of* G *is regarded as minimal if* $U_i \nsupseteq \bigcap_{l \neq i} U_l$ *for all* $i = 1, 2, 3, ..., p$ *and all* U_i'*s are distinct.*

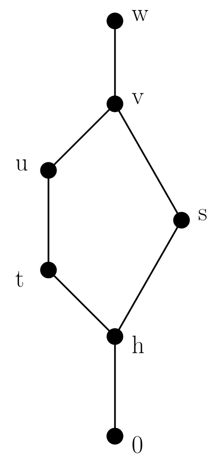

Figure 3. Example of a f-primary ideal which isn't a f-prime ideal

Definition 3.19. *For an ideal* G *of* $\mathbb{Q}$*,* $G = \bigcap_{i=1}^{p} U_i$ *and* $r_f(U_i) = G_i$*,* $i = 1, 2, ..., p$ *be a minimal* f*-primary decomposition of* G *in* $\mathbb{Q}$*. Also the* f*-prime ideals* G_i*,* $i = 1, 2, 3, ..., p$*, are called associated* f*-prime ideals of the decomposition.*

Furthermore, G *is also known as decomposable if it has a* f*-primary decomposition.*

Example 3.20. *For a poset* $(\mathbb{Q}, \leq)$*, where* $\mathbb{Q} = \{w, v, u, k, 0\}$ *and* $\leq$*, a partial relation on* $\mathbb{Q}$ *defined as below:*

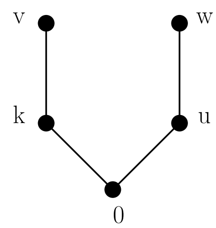

Figure 4. Example of a f_A-primary ideal

Define a map $f : \mathbb{Q} \to Id(\mathbb{Q})$ *by* $f(a) = (a]$*. Here* $G = \{0, k\}$ *is* f_A*-primary ideal as* $r_f(G) = (v] = A$*.* □

Theorem 3.21. *Let* G *be a* f_U*-primary ideal of* $\mathbb{Q}$ *for some* f*-prime ideal* U *of* $\mathbb{Q}$ *and* $w \in \mathbb{Q}$*. Then*

(i) $w \notin G$ *and the ideal* $< w, G >$ *has* $(*)$ *condition propose that* $< w, G >$ *is a* f_U*-primary ideal of* $\mathbb{Q}$*.*

(ii) $w \notin U$ *implies* $< w, G >= G$*.*

Proof: (i) Let $p \in < w, G >$. Then $(((w)^\ell)^*, ((p)^\ell)^*)^\ell \subseteq (w, p)^\ell \subseteq G$. As G is f_U-primary and $(w)^\ell \nsubseteq G$, we get $p \in (p)^\ell \subseteq r_f(G) = U$ for some f-prime ideal U of $\mathbb{Q}$. So, $G \subseteq < w, G > \subseteq U$. By Theorem 3.9, $U = r_f(G) \subseteq r_f(< w, G >) \subseteq U$. Thus $U = r_f(< w, G >)$.

Now,we have to prove that $< w, G >$ is a f-primary ideal.

Assume $(((p)^\ell)^*, ((r)^\ell)^*)^\ell \subseteq < w, G >$. If $(p)^\ell \nsubseteq < w, G >$, then $(w, p)^\ell \nsubseteq G$. As a result, $\exists\ t \in (w, p)^\ell$ with $t \notin G$. Additionally, $r_f(< t, G >) = U$. Since $(t, r)^\ell \subseteq (p, r)^\ell \subseteq < w, G >$, we have $(w, t, r)^\ell \subseteq G$. As

$t \leq w$, we have $(t,r)^\ell \subseteq G$. Accordingly $r \in < t, G > \subseteq r_f(< t, G >) = U = r_f(< w, G >)$. Thus $r \in r_f(< w, G >)$ and $(r)^\ell \subseteq r_f(< w, G >)$.

On the other side, if $(p)^\ell \nsubseteq r_f(< w, G >) = U$, then we now show that $(r)^\ell \subseteq < w, G >$. Let $t \in (w,r)^\ell$. So $(t,p)^\ell \subseteq (w,p,r)^\ell \subseteq G$. Since G is f-primary and $p \notin U = r_f(G)$, We get $t \in G$.

(ii) Assume that $w \notin U$. Suppose that $< w, G > \nsubseteq G$. Then there is a $t \in < w, G >$ with $t \notin G$. As G is f_U-primary and $w \notin U = r_f(G)$, we have $t \in G$, a contradiction. So, $< w, G > = G$. □

Theorem 3.22. *For a decomposable ideal R of $\mathbb{Q}$ and $\mathbb{Q}$ satisfies (α) condition, if $R = \bigcap_{i=1}^{p} G_i$ is a minimal f-primary decomposition of R, where $U_i = r_f(G_i)$, $i = 1,2,3,...p$ be associated f-prime ideals of the decomposition, then every f-prime ideal of the form $r_f(< s, R >)$ for some $s \in \mathbb{Q}$ is one of the associated f- prime ideals U_i for some i and moreover, for each associated f-prime ideal U_i, $\exists\, s_i \in \mathbb{Q} : r_f(< s_i, R >) = U_i$.*

Proof: Consider $R = G_1 \cap G_2 \cap ... \cap G_p$. For $s \notin R$, $< s, R > = < s, \bigcap_{i=1}^{p} G_i > = \bigcap_{i=1}^{p} < s, G_i >$. Hence $< s, R > = \bigcap_{s \notin G_k} (< s, G_k >), 1 \leq k \leq p$. So, $r_f(< s, R >) = r_f(\bigcap_{s \notin G_k} < s, G_k >) = \bigcap_{s \notin G_k} r_f < s, G_k >, 1 \leq k \leq p$ by Theorem 3.9.

If $r_f(< s, R >)$ is f-prime, then, it should be stated that $r_f(< s, R >) = U_k$, for some k, $1 \leq k \leq p$. Hence every f-prime ideal of the form $r_f(< s, R >)$ is one of the U_k's for some k, $1 \leq k \leq p$.

Consider the associated f-prime ideal U_k, $1 \leq k \leq p$. We should still take note out $s \in \mathbb{Q} : r_f(< s, R >) = U_k$. As the decomposition of R is minimal, we have $G_k \nsupseteq \bigcap_{i \neq k} G_i$ for all $k \in \{1,2,3,...,p\}$. This provides $\exists\, s_k \in \bigcap_{i \neq k} G_i$ and $s_k \notin G_k$.

Now, $< s_k, R > = < s_k, \bigcap_{i=1}^{n} G_k > = \bigcap_{k=1}^{p} < s_k, G_k >$. Since $s_k \in \bigcap_{i \neq k} G_i$ we get that $< s_k, R > = < s_k, G_k >$. This implies that $r_f(< s_k, R >) = r_f(< s_k, G_k >) = U_k$, by Theorem 3.21. □

Example 3.23. *In Example 3.14, for the ideal $R = (q]$, take note of this: $(q] = (g] \cap (h] \cap (r]$ is a f-primary decomposition of $R = (q]$ and a minimal f-primary decomposition of $(q]$ is $(h] \cap (r] = (q]$.*

Moreover, $(r] = r_f((r])$ and $(g] = r_f((h])$. Thus $(g]$ and $(r]$ are associated f-prime ideals of the minimal f-primary decomposition of R.

For the associated f-prime ideal $(r]$, $\exists\, h \in \mathbb{Q} : r_f(< h, R >) = (r]$ and for that associated f-prime ideal $(g]$ $\exists\, r \in \mathbb{Q} : r_f(< r, R >) = (g]$.

Remark 3.24. *Let T be an ideal of $\mathbb{Q}$ and U be a f-prime ideal of $\mathbb{Q}$. Then we indicated, $T_U = \{r \in \mathbb{Q} : (r,s)^\ell \subseteq T$ for some $s \in \mathbb{Q} \backslash U\} = \bigcup_{s \in \mathbb{Q} \backslash U} \langle s, T \rangle$.*

Theorem 3.25. *For an ideal T of $\mathbb{Q}$ and $\mathbb{Q}$ having (α) condition with $T = \bigcap_{i=1}^{p} G_i$, a minimal f-primary decomposition of T, where $r_f(G_i) = U_i$ is the associated f-prime ideals of the decomposition, The following statements are true.*

(i) If U is f-prime and $T \subseteq U$ and $U \supseteq U_i$ $(1 \leq i \leq k)$, but does not contain $U_{k+1}, U_{k+2}, ..., U_p$, then $T_U = G_1 \cap G_2 \cap ... \cap G_k$.

(ii) If $U \nsupseteq U_i$ for all i, then $T_U = \mathbb{Q}$.

Proof: (i) Assume that U contains $U_1, U_2, ..., U_k$, but does not contain $U_{k+1}, U_{k+2}, ..., U_p$. Let $v \in T_U$, consequently, for some $c \notin U$, $(v,c)^\ell \subseteq T$. It indicates that $(v,c)^\ell \subseteq G_i \subseteq r_f(G_i)$, for each $i = 1,2,...,p$. It is simply obtained that c is precisely not in $U_1, U_2, ..., U_k$. For otherwise, if $c \in U_i$, for some $i \in \{1,2,...,k\}$, Consequently, by hypothesis, $c \in U$, is a contradiction. Hence, $c \notin r_f(G_1), r_f(G_2), ..., r_f(G_k)$ which gives $(c)^\ell \nsubseteq r_f(G_1), r_f(G_2), ..., r_f(G_k)$. As $G_1, G_2, ..., G_k$ are f-primary, we have $(v)^\ell \subseteq G_1, G_2, ..., G_k$. So, $v \in (v)^\ell \subseteq G_1 \cap G_2 \cap ... \cap G_k$.

Conversely, let $w \in G_1 \cap G_2 \cap \cap G_k$. As $U_{k+1}, U_{k+2}, ..., U_p \nsubseteq U$, there exist $v_{k+1} \in U_{k+1} \backslash U, v_{k+2} \in U_{k+2} \backslash U, ..., v_p \in U_p \backslash U$. Since $v_j \in U_j = r_f(G_j)$, $j = k+1, ..., p$, so all f-system including v_j intersects with G_j.

Particularly, $\mathbb{Q} \backslash U$ is a f-system having v_j and $\mathbb{Q} \backslash U$ intersects G_j for all $j = k+1, ..., p$. Choose $h_j \in G_j \cap (\mathbb{Q} \backslash U)$, $j = k+1, ..., p$. As U is f-prime, we get $(\{h_{k+1}, h_{k+2}, ..., h_p\})^\ell \nsubseteq U$. So, $\exists\; h \in (\{h_{k+1}, h_{k+2}, ..., h_p\})^\ell$ with $h \notin U$. So, $(w,h)^\ell \subseteq \bigcap_{i=1}^{p} G_i = T$ and $h \notin U$. Thus $w \in T_U$.

(ii) Suppose $U_i \nsubseteq U$ for all $i = 1,2,3,...,p$. Consequently, there exists $v_i \in U_i \backslash U$ for every each $i = 1,2,3,...,p$. As $\mathbb{Q} \backslash U$ is a f-system of $\mathbb{Q}$, using the same technical approach as described previously, we obtain $h_i \in G_i \cap (\mathbb{Q} \backslash U)$, $i = 1,2,3,...,p$. As U is f-prime ideal, we have $(\{h_1, h_2, ..., h_p\})^\ell \nsubseteq U$. Therefore there exists $h \in (\{h_1, h_2, ..., h_p\})^\ell : h \notin U$. It suffices to say that $h \in \bigcap_{i=1}^{p} G_i = T$ and hence for all $w \in \mathbb{Q}$, we have $(w,h)^\ell \subseteq T$, which gives $w \in T_U$. Hence $T_U = \mathbb{Q}$. □

Theorem 3.26. *([18], Theorem 2.4) For an ideal T of $\mathbb{Q}$, T has the property below that for $n > 2$, if pairwise distinct ideals $K_1, K_2, ..., K_n$ of $\mathbb{Q}$ with $(K_1^*, K_2^*, ..., K_n^*)^\ell \subseteq T$, then at least $(n-1)$ of n subsets $(K_2^*, K_3^*, ..., K_n^*)^\ell, (K_1^*, K_3^*, ..., K_n^*)^\ell, ..., (K_1^*, K_2^*, ..., K_{n-1}^*)^\ell$ are subsets of T.*

Theorem 3.27. *Let $\mathbb{Q}$ be a poset which satisfies the (α) condition and S be an ideal of $\mathbb{Q}$. If S has two minimal primary decompositions $H_1 \cap H_2 \cap ... \cap H_c = G_1 \cap G_2 \cap ... \cap G_d$, where H_i is f_{D_i}-primary and G_j is f_{T_j}-primary and each D_i and T_j are associated f-prime, then $c = d$.*

Proof: Suppose that $H_1 \cap H_2 \cap ... \cap H_c = G_1 \cap G_2 \cap ... \cap G_d$ where H_i is f_{D_i}-primary and G_j is f_{T_j}-primary. Then $D_1 \cap D_2 \cap ... \cap D_c = r_f(H_1) \cap r_f(H_2) \cap ... \cap r_f(H_c) = r_f(H_1 \cap H_2 \cap ... \cap H_c) = r_f(G_1 \cap G_2 \cap ... \cap G_d) = r_f(G_1) \cap r_f(G_2) \cap ... \cap r_f(G_d) = T_1 \cap T_2 \cap ... \cap T_d$.

Now $L(D_1^*, D_2^*, ..., D_c^*) \subseteq D_1 \cap D_2 \cap ... \cap D_c \subseteq T_j \ \forall j$. Since T_j is f-prime ideal and Theorem 3.26, we get $D_i \subseteq T_j$ for some i.

Also $L(T_1^*, T_2^*, ..., T_d^*) \subseteq T_1 \cap T_2 \cap ... \cap T_d \subseteq D_i \ \forall i$. As D_i is f-prime ideal and Theorem 3.26, we get $T_r \subseteq D_i$ for some r. So $T_r \subseteq D_i \subseteq T_j$. Since T_j is an associated f-prime, we have $T_r = T_j$ which gives $T_j = D_i$, hence $c = d$. □

Conclusion

The prime ideal's concept and generalisation hold a distinct place in algebraic geometry and commutative algebra. All of those are beneficial methods for determining algebraic structural qualities. This article provided a broader generalisation of primary ideals in posets as well as certain features of those f-primary ideals. Further, decomposition of an ideal in posets with respect to f-primary decomposition in posets had been discussed. Using the approach described in this study, these conclusions may be extended to 0-distributive posets, lattices, near lattices, semilattices, and 0-distributive near lattices.

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