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Statistical Inference of Modified Kies Exponential Distribution Using Censored Data

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Abstract This paper deals with obtaining the interval and point estimation to Modified Kies exponential distribution in case of progressive first failure (PFF) censored d ata. It uses two approaches, classical and non-classical methods of estimation, including the highest posterior density (HPD). We obtained the Maximum Likelihood Estimation of the parameters and the logarithm likelihood function, and we used the maximum likelihood estimation of the parameters as a classical approach. We calculated the confidence intervals for the parameters and the Bootstrap confidence Intervals. We employed the posterior distribution and the Bayesian estimation (BE) under different loss functions (Symmetric loss function, The MCMC usage, and The M-H algorithm). Some results depending on simulation data are adopted to explain estimation methods. We used various censoring schemes and various sample sizes to determine whether the sample size affects the estimation measures. We used different confidence intervals to determine the best and shortest intervals. Also, the major findings in the paper are remarked on in the conclusion section.

Keywords Modified Kies Exponential Distribution, Bayesian Estimation, Progressive First Failure, Maximum Likelihood Estimation

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1 Introduction

Type-II censoring is considered one of the most popular tools used in statistical inference. It is observed that Type-II censoring can be used to save time and money. A progressive Type-II censoring is a generalization of Type-II censoring. [1] explained that the experimenter could be decided to group the test units into different sets of a life test, each set as an assembly of test units. After that, the test units run simultaneously until each group's first failure occurs. This censoring technique is named a Progressive first-failure censoring (PFFC). In FFC, sets will be removed from the test at the ultimate termination point, and this allowance will be desirable in practice. To develop and make a new life test plan known as a PFFC method, the concepts of PFFC and progressive censoring are combined in [6].[2] and [3] achieved confidence intervals for the unknown parameters of the Gompertz and Burr Type-XII distributions using PFFC sampling, respectively. More details can be found in [4] and [5]. Many authors worked on the Gompertz distribution under PFFC. See for instance [7]. [9]and [8]. They explained Bayesian estimation and Maximum Likelihood Estimation. For the parameters of the Burr Type-XII and Gompertz distributions under PFFC sampling, we compute exact confidence intervals and exact confidence regions.

This research paper comprises the following sections: Section 2 presents the model description Test assumptions. Section 3 shows the modified exponential distribution of Kies. In Section 4, MLEs are obtained for model parameters under stress. In Section 5, Interval confidence was obtained using Approximate confidence intervals- Bootstrap confidence Intervals Section 6 obtained

BEs for model parameters useing The MCMC method; asymptotic and reliable confidence limits of the model Parameters are generated. Section 7 demonstrates simulation studies. Section 8 displays the conclusion.

2 The Model description

So, let's consider the PFFC scheme in which k is the number of groups with n within each category, these items or units of goods put to the test throughout their lives are examined. Let R_1, (R is an integer number greater than zero) groups and the group wherein the occurrence of the first failure is observed, then randomly removed from the test once as the first failure $Y^R_{1;m,k,n}$ has taken place. In the next step, R_2 groups and the group wherein the second failure is observed are randomly deleted eliminated from the test as soon as the second failure takes place $Y^R_{2;m,k,n}$, and so on. When the m^th failure $Y^R_{m;m,k,n}$ happens, the residual groups $R_m, (m \leq k)$ are extracted from the test. After that $Y^{\ R}_{1;m,k,n} < ... < Y^{\ R}_{m;m,k,n}$ are called PFFC order statistics with the progressive censored scheme $R = (R_1, R_2, ..., R_m)$, where $k = m + \sum_{i=1}^{m} R_i$. Now the joint probability density function for $Y^{\ R}_{1;m,k,n}, Y^{\ R}_{2;m,k,n}, ..., Y^{\ R}_{m;m,k,n}$ is defined below:

$$f_{1,2,...,m}(Y^{\ R}_{1;m,k,n}, Y^{\ R}_{2;m,k,n}, ..., Y^{\ R}_{m;m,k,n})= B(k,m-1)n^m \prod_{r=1}^{m} f(Y^{\ R}_{r;m,k,n}) \left[1 - F(Y^{\ R}_{r;m,k,n})\right]^{n(R_r+1)-1} \tag{2.1}$$

$, 0 < y_1 < y_2 < ... < y_m < \infty$, where$B(k, m-1) = k(k - R_1 - 1)...(k - R_1 - R_2 - ... - R_{m-1} - (m-1))$

Special cases:

1. By setting $n = 1$ in the likelihood equation, we are talking about the progressively Type-II censored sample.
2. Obtaining the whole sample instance was accomplished by adding $R = \{0, 0, ..., 0\}$ and $n = 1$.

3 Modified Kies exponential Distribution

The probability density function and distribution function of the modified kies exponential distribution are defined as follows: PDF

$$f(y) = \xi\beta e^{\xi\beta y}\left(1 - e^{-\beta y}\right)^{\xi-1} e^{-\left(e^{\beta y}-1\right)^{\xi}}, \quad y > 0, \tag{3.2}$$

CDF is

$$F(y) = 1 - e^{-\left(e^{\beta y}-1\right)^{\xi}}, \quad y > 0, \tag{3.3}$$

4 Maximum Likelihood Estimation

Here, we used the maximum likelihood (ML) to make a point and interval estimation for the unknown parameters of modified kies exponential distribution on the basis of the PFFC scheme.

To define the point estimation, let assume that $\underline{Y} = y^R_{1:m,k,n}, y^R_{2:m,k,n}, \cdots, y^R_{m:m,k,n}$ be the PFF censored order statistics from MKiEx with censored scheme R . Using (3.2) and (3.3) in (2.1) the likelihood function is defined as follow :

$$\begin{aligned} L(\xi, \beta|\underline{Y}) &= An^m \prod_{r=1}^{m} \xi\beta e^{\xi\beta y_r}\left(1 - e^{-\beta y_r}\right)^{\xi-1} e^{-\left(e^{\beta y_r}-1\right)^{\xi}} \left[1 - \left\{1 - e^{-\left(e^{\beta y_r}-1\right)^{\xi}}\right\}\right]^{n(R_r+1)-1} \\ &= An^m\alpha^m\beta^m \exp\left\{\sum_{r=1}^{m}\left[\xi\beta y_r - \left(e^{\beta y_r}-1\right)^{\xi} - (n(R_r+1)-1)\left(e^{\beta y_r}-1\right)^{\xi}\right]\right\} \\ &\quad \prod_{r=1}^{m}\left(1 - e^{-\beta y_r}\right)^{\xi-1} \end{aligned} \tag{4.4}$$

The logarithm likelihood function is :

$$\begin{aligned} \ell(\xi, \beta|\underline{Y}) &= \log A + m\ln n + m\log\xi + m\log\beta \\ &+ \sum_{r=1}^{m}\left[\xi\beta y_r - \left(e^{\beta y_r}-1\right)^{\xi} - (k(R_r+1)-1)\left(e^{\beta y_r}-1\right)^{\xi}\right] \\ &\times (\xi-1)\sum_{r=1}^{m}\ln\left[1 - e^{-\beta y_r}\right] \end{aligned} \tag{4.5}$$

We compute the parameters first derivation with regard to ξ and β and afterwards equal to zero, we get the likelihood equations

$$\begin{aligned}\frac{\partial \ell}{\partial \xi} &= \frac{m}{\xi} + \beta \sum_{r=1}^{m} y_i + \sum_{r=1}^{m} n(R_r+1)\left[e^{\beta y_r}-1\right]^{\xi} \ln\left[e^{\beta y_r}-1\right] \\ &+ \sum_{r=1}^{m} \ln\left[1-e^{-\beta y_r}\right] = 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= \frac{m}{\beta} + \xi \sum_{r=1}^{m} y_r + \sum_{r=1}^{m} n(R_r+1)\xi y_r e^{\beta y_r}\left[e^{\beta y_r}-1\right]^{\xi-1} \\ &+ (\xi-1)\sum_{r=1}^{m} \frac{y_r e^{-\beta y_r}}{1-e^{-\beta y_r}} = 0. \end{aligned} \tag{4.7}$$

analytical solution for equations (4.6) and (4.7) can't be obtained. So, a numerical solution using Newton-Raphson method will be used.

5 Interval estimation

Here, approximate Bootstrap CI for the ML method and credible intervals (CIs) of the parameters ξ and β will be obtained.

5.1 Approximate confidence intervals

Here, the confidence intervals for the parameters which use the asymptotic distribution of the MLEs of the unknown parameters ξ and β will be derived. The asymptotic variances and cov-matrix of the MLE for the following parameters ξ and β can give the Fisher information matrix. It is not possible to obtain the exact mathematical expressions for the equations (4.6) and (4.7). Hence, the inverse matrix I^{-1} of the Fisher information matrix can be expressed as :

$$I_{ij}^{-1}(\xi,\beta) = \begin{pmatrix} -\frac{\partial^2 \ell}{\partial \xi^2} & \frac{\partial^2 \ell}{\partial \xi \partial \beta} \\ \frac{\partial^2 \ell}{\partial \beta \partial \xi} & -\frac{\partial^2 \ell}{\partial \beta^2} \end{pmatrix}^{-1} = \begin{pmatrix} var(\widehat{\xi}) & cov(\widehat{\xi},\widehat{\beta}) \\ cov(\widehat{\beta \xi}) & var(\widehat{\beta}) \end{pmatrix} \tag{5.8}$$

where

$$\frac{\partial^2 \ell}{\partial \xi^2} = -\frac{m}{\xi^2} + \xi \sum_{r=1}^{m} y_r + \sum_{r=1}^{m} n\left(R_r+1\right)\left[e^{\beta y_r}-1\right]^{\xi}\left[\ln\left(e^{\beta y_r}-1\right)\right]^2 \tag{5.9}$$

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \beta^2} &= -\frac{m}{\beta^2} + \xi n \sum_{r=1}^{m} \left(R_r+1\right) e^{\beta y_r}\left[e^{\beta y_r}-1\right]^{\xi-2}\left[\xi e^{\beta y_r}-1\right] \\ &+ (\xi+1)\sum_{r=1}^{m} \frac{y_r^2 e^{-\beta y_r}}{\left[1-e^{-\beta y_r}\right]^2} \end{aligned} \tag{5.10}$$

and

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \xi \partial \beta} &= \sum_{r=1}^{m} y_r + \sum_{r=1}^{m} n\left(R_r+1\right) y_r e^{\beta y_r}\left[e^{\beta y_r}-1\right]^{\xi-1}\left[\xi \ln\left(e^{\beta y_r}-1\right)+1\right] \\ &+ \sum_{r=1}^{m} \frac{y_r e^{-\beta y_r}}{1-e^{-\beta y_r}} \end{aligned} \tag{5.11}$$

where $Z_{\frac{\gamma}{2}}$ is the percentile of the standard of the normal distribution with right-tail probability.

5.2 Bootstrap confidence Intervals

In this subsection, the percentile bootstrap (Boot-p) is constructed (see, [19]). It should be used to calculate confidence intervals and to determine an estimator's bias and variance, as well as to re-calibrate hypothesis tests. The following algorithm explains the method of the bootstrap.

Algorithm

1. Using the generated sample or data of size n $Y=(y^R_{1,m,n,k}, y^R_{2,m,k,n}, y^R_{3,m,k,n}, \cdots, y^R_{m,m,k,n})$, calculate the maximum likelihood estimates of the following parameters ξ and β from equations (4.6)and (4.7).

2. We use $\widehat{\xi}$ and $\widehat{\beta}$ to generate a bootstrap sample $\underline{Y}^*$ with the same values of $R_r,(r=1.2.\cdots,m)$ using the algorithm made by [20] .

3. Same step 1 on the basis of $\underline{Y}^*$ we calculate the bootstrap sample estimates of ξ and β say $\widehat{\xi}^*$ and $\widehat{\beta}^*$.

4. Rerun steps 2 and 3 where K times which represent K bootstrap ML estimates of ξ and β on the basis of K different bootstrap samples.

5. Arrange a $\widehat{\xi}^{*r}$ and $\widehat{\beta}^{*r}$ in an ascending order to get bootstrap sample $(\psi_t^{(1)}, \psi_t^{(2)}, \cdots, \psi_t^{(K)}), t=1,2$ where $\psi_2=\widehat{\xi}^*, \psi_3=\widehat{\beta}^*$ Let $G(y)=P(\psi_t\leq y)$ the CDF of ψ_t. Define $\psi_{tboot}=G^{-1}(y)$ for given y. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of ψ_t given by:
$$\left[\psi_{tboot}\left(\frac{\gamma}{2}\right), \psi_{tboot}\left(1-\frac{\gamma}{2}\right)\right] \tag{5.12}$$

6 Bayesian Approach under different loss functions

This section will demonstrate how to derive Bayesian estimate for the FD parameters under various loss functions. Additionally, a Bayesian research is conducted by taking into account the preceding independent gamma priors for ξ and β :

$$g_1(\xi)\propto \xi^{\mu-1}e^{-\lambda\xi}, \xi>0, \tag{6.13}$$

$$g_2(\beta)\propto \beta^{\mu_1-1}e^{-\lambda_1\beta}, \beta>0 \tag{6.14}$$

Where $\mu_1,\mu,\lambda,\lambda_1$ are hyperparameters and $\mu_1,\mu,\lambda,\lambda_1>0$.

The joint prior PDF of ξ and β under the condition of independence can be written as:

$$\pi(\alpha,\beta)\propto \beta^{\mu_1-1}\xi^{\mu-1}e^{-(\xi\lambda+\beta\lambda_1)}, \quad \xi,\beta>0. \tag{6.15}$$

6.1 Symmetric loss function

In this part, Bayesian estimation is considered by using Squared Error Loss SEL function (SELF).

From (4.4) and (6.15), the joint posterior density function for following parameters ξ,β, a and b can be given by:

$$\pi^*(\xi,\beta\mid y)\propto L(\xi,\beta)\pi(\xi,\beta) \tag{6.16}$$

$$\begin{aligned} &= An^m\xi^m\beta^m\exp\left\{\sum_{r=1}^m\left[\xi\beta y_r-\left(e^{\beta y_r}-1\right)^\xi-(n(R_r+1)-1)\left(e^{\beta y_r}-1\right)^\xi\right]\right\} \\ &\times \prod_{r=1}^m\left[1-e^{-\beta y_r}\right]^{\xi-1} \end{aligned} \tag{6.17}$$

The posterior density function is:

$$\begin{aligned} \pi^* &= \frac{L(\xi,\beta \mid y) g(\xi,\beta \mid y)}{\int_0^\infty \int_0^\infty L(\xi,\beta \mid y) g(\xi,\beta \mid y) d\xi d\beta} \\ &= \frac{A n^m \xi^m \beta^m \exp\left\{\sum_{r=1}^m \left[\xi\beta y_r - \left(e^{\beta y_r}-1\right)^\xi - \left(n(R_r+1)-1\right)\left(e^{\beta y_r}-1\right)^\xi\right]\right\}}{\int_0^\infty \int_0^\infty A n^m \xi^m \beta^m \exp\left\{\sum_{r=1}^m \left[\xi\beta y_r - \left(e^{\beta y_r}-1\right)^\xi - \left(n(R_r+1)-1\right)\left(e^{\beta y_r}-1\right)^\xi\right]\right\}} \\ & \frac{\prod_{r=1}^m \left(1-e^{-\beta y_r}\right)^{\xi-1} \beta^{\mu_1-1}\xi^{\mu-1} e^{-(\xi\lambda+\beta\lambda_1)}}{\prod_{r=1}^m \left(1-e^{-\beta y_r}\right)^{\xi-1} \beta^{\mu_1-1}\xi^{\mu-1} e^{-(\xi\lambda+\beta\lambda_1)} d\xi d\beta} \end{aligned} \quad (6.18)$$

based on SELF, the Bayes estimator of the function of parameters $U = U(H)$, $H = (\xi, \beta)$ is given by:

$$\hat{U}_{SE} = \int_H U \, \pi^*(H) \, dH, \quad (6.19)$$

where $\pi^{HPD}(H)$ is given by equation (6.17).

The Bayes estimate of any function of H say $h(H)$ under SELF is given by:

$$\hat{H}_{SE} = E_{(H|data)}\left[h(H)\right] = \frac{\int_0^\infty h(H) L(H) g(H \mid y) dH}{\int_0^\infty L(H) g(H \mid y) dH.} \quad (6.20)$$

Analytical solution for equation (6.20) is very hard to obtain. And the posterior density function is not reduced to any familiar distributions. However, graphing it shows that its graph is so close to the normal distribution. The exact calculation of the integral cannot be obtained. Therefore, the Metropolis-Hasting Algorithm normal as a proposal distribution is a suitable approximation to use.

6.2 The MCMC usage

The MCMC method is used to generate samples from the posterior distribution in this case. From (6.20), the conditional posterior distributions of ξ and β are shown respectively by:

$$\begin{aligned} \pi^*(\xi|\beta,y) &= \xi^{\mu+m-1} e^{-\lambda\xi} \exp\left\{\sum_{r=1}^m \left[\xi\beta y_r - \left(e^{\beta y_r}-1\right)^\xi - \left(n(R_r+1)-1\right)\left(e^{\beta y_r}-1\right)^\xi\right]\right\} \\ & \prod_{r=1}^m \left(1-e^{-\beta y_r}\right)^{\xi-1} \end{aligned} \quad (6.21)$$

$$\begin{aligned} \pi^*(\beta|\xi,y) &= \beta^{\mu_1+m-1} e^{-\lambda\beta} \exp\left\{\sum_{r=1}^m \left[\xi\beta y_r - \left(e^{\beta y_r}-1\right)^\xi - \left(n(R_r+1)-1\right)\left(e^{\beta y_r}-1\right)^\xi\right]\right\} \\ & \prod_{r=1}^m \left(1-e^{-\beta y_r}\right)^{\xi-1} \end{aligned} \quad (6.22)$$

The eqs. (6.21) and (6.22) Analytically, this cannot be reduced from a familiar distribution. As a result, the Metropolis-Hastings (M-H) technique is used to generate random samples from such distributions; see [16]. The next algorithm is introduced to get BE of $U = U(\xi, \beta)$ under SE, GE, and LINEX loss functions.

6.3 The M-H algorithm

The algorithm was developed as a result of the efforts of [17]. Consider the scenario in which the goal was to produce samples from the distribution $f(y|\xi) = \nu g(\xi)$, ν is the normalizing constant. The M-H algorithm presented a way of sampling from $f(y|\xi)$ .Now, let suppose that $g\left(\xi^{(b)}|\xi^{(a)}\right)$ be an arbitrary transition kernel: that is the probability of jumping, from current state $\xi^{(a)}$ to $\xi^{(b)}$. This is also known as the proposal distribution. The next algorithm generated a series of values $\xi^{(1)}, \xi^{(2)}, ..., \xi^{(k)}$ which form a Markov chain with stationary distribution given by $f(y|\xi)$. The algorithm can be stepped in the following:

1. $\xi^{(0)} = \hat{\xi}_{MLE}, , \beta^{(0)} = \hat{\beta}_{MLE}$.

2. Set r=1.
3. Generate $\xi^{(*)}$ from the proposal distribution $N(\xi^{(r-1)}, var\xi^{(r-1)})$.
4. Compute the acceptance probability $re(\xi^{(r-1)}, \xi^{(*)}) = \min\left[1, \frac{\pi(\xi^{(*)})}{\pi(\xi^{(r-1)})}\right]$.
5. Generate U from the standard uniform distribution.
6. Let us have the assumption that $U < r(\xi^{(r-1)}, \xi^{(*)})$ then the proposal distribution can be accepted and the value is considered, and we can set $\xi^{(r)} = \xi^{(*)}$. Otherwise, reject the proposal distribution and set $\xi^{(r)} = \xi^{(r-1)}$.
7. Set $r = r + 1$.
8. Repeat Steps (1-7) for the other parameter β N times.
9. The approximate estimates of SE, loss function are as the following

$$\hat{\xi}_{SE} = \frac{1}{K-M} \sum_{r=M+1}^{K} U(\xi^{(r)}, \beta^{(r)}), \tag{6.23}$$

,where M denoted the number of nburn units and K the loop's number for the MCMC iterations.

7 Simulation studies

Here we introduced the simulation studies to investigate the performances of the MLEs and BEs in the case of SEL, GE and LINEX are carried out. Censoring schemes (C.S) used in the simulation are shown in Table (1). Table (2) shows the results reached in the simulation study.
The PFF censored sample with distribution function F(y) can be shown as a progressive Type-II censored sample from a population with distribution function $1-(1-F(y))^N$, we can generate a PFF censored samples from the continuous random variable using the algorithm that was introduced in [20]. The estimation procedure is carried out in accordance with the next algorithm.

Algorithm (3)

1. Identify the variable values n, k, m.
2. Generate random samples of size m by using Uniform distribution $(0, 1)$, as $(U_1, U_2, ..., U_m)$, $r = 1, 2, ..., m$.
3. Compute the values of the censored schemes, R_r, $r = 1, 2, ..., m$, and such that $\sum_{r=1}^{m} R_r = n - m$.
4. Set $E_{rj} = U_r^{1/(r+\sum_{j=m-r+1}^{m} R_j)}$, and $r = 1, 2, ..., m$.
5. Obtain the censored progressive type-II samples $(U_1^*, U_2^*, ..., U_m^*)$, where $U_r^* = 1 - \prod_{j=m-r+1}^{m} R_j$, $r = 1, 2, ..., m$.
6. Use step (2), to generate random samples $(t_1, t_2, ..., t_m)$, $r = 1, 2, ..., m$. from (3.3) as follows:

$$t_r = \frac{1}{\beta} \ln\left[1 + (-\ln(1 - U(r)))^{\frac{1}{\xi}}\right]$$

, $r = 1, 2, ..., m$.

7. Use the progressively censored data to determine the MLEs of the model parameters by solving the equations of the nonlinear system (4.6)and (4.7).
8. Calculate the BEs of the model parameters relative to SE, GE, and LINEX loss functions by using algorithm (1), where $K = 11000$, $M = 2000$, and K is defined number of mcmc iterations and M is defined nburn.
9. Calculate the CIs bounds with confidence level 95% for the three parameters ξ and β.
10. Calculate 95% credible CIs by using algorithm (2) of the parameters ξ and β.
11. Repeat the steps $((2) - (10))$, 1000 times.
12. Calculate the average values of the MSEs associated with the MLEs and BEs of the parameters.

13. Calculate the average values of the lengths of bootstrap CI, approximate CI, credible CI, and the coverage probability of these CIs of the parameters.

14. Do steps ((1)-(13)) with different values of k, m, and R_i, $r = 1, 2, ..., m$.

Table (1)Censoring schemes used in the simulation.

C.S	[1]	$m = 10, k = 40, R_1 = R_2 = R_3 = 10, R_4 = R_5 = R_6 = R_7 = ... = R_m = 0$
C.S	[2]	$m = 10, k = 40, R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = 5, R_7 = R_8 = ... = R_m = 0$
C.S	[3]	$m = 30, k = 70, R_1 = R_2 = R_3 = R_4 = 10, R_5 = ... = R_m = 0$
C.S	[4]	$m = 30, k = 70, R_1 = 20, R_2 = 10, R_3 = 10, R_4 = ... = R_m = 0$
C.S	[5]	$m = 50, k = 100, R_1 = R_2 = R_3 = R_4 = R_5 = 10, R_6 = ... = R_m = 0$
C.S	[6]	$m = 50, k = 100, R_1 = R_2 = 20, R_3 = 10, R_4 = ... = R_m = 0$
C.S	[7]	$m = 70, k = 120, R_1 = R_2, ... = R_{10} = 5, R_{11} ... = R_m = 0$
C.S	[8]	$m = 70, k = 120, R_1 = R_2 = R_3 = R_4 = R_5 = 10, R_6 = ... = R_m = 0$

Table (2) MSE, CI and HPD

.

$Parameter, C.S[1]$	$[\xi]$	$[\beta]$
MSE_{MlE}	0.01867	0.02467
MSE_{SE}	0.01184	0.00966
$95\%CI$ Lower bound	0.89005	0.61647
$95\%CI$ upper bound	1.83142	1.32441
$95\%CI$ Length	0.70794	0.94138
$bootstrapCI$ lower bound	0.94308	0.66775
$bootstrapCI$ upper bound	1.54467	1.12997
$bootstrapCI$ Length	0.6016	0.46222
$Parameter, C.S[2]$	$[\xi]$	$[\beta]$
MSE_{MlE}	0.02291	0.02176
MSE_{SE}	0.02239	0.02124
$95\%CI$ Lower bound	0.8923	0.58738
$95\%CI$ upper bound	1.86729	1.41589
$95\%CI$ Length	0.82851	0.97499
$bootstrapCI$ lower bound	0.97225	0.66632
$bootstrapCI$ upper bound	1.54459	1.12773
$bootstrapCI$ Length	0.57234	0.46142
$Parameter, C.S[3]$	$[\xi]$	$[\beta]$
MSE_{MlE}	0.01979	0.03151
MSE_{SE}	0.01051	0.02993
$95\%CI$ Lower bound	1.05311	0.7473
$95\%CI$ upper bound	1.70581	1.10924
$95\%CI$ Length	0.36194	0.6527
$bootstrapCI$ lower bound	1.08085	0.69685
$bootstrapCI$ upper bound	1.54193	0.94737
$bootstrapCI$ Length	0.46108	0.25052
$Parameter, C.S[4]$	$[\xi]$	$[\beta]$
MSE_{MlE}	0.01662	0.04125
MSE_{SE}	0.01293	0.03737
$95\%CI$ Lower bound	1.01902	.71945
$95\%CI$ upper bound	1.66737	1.07597
$95\%CI$ Length	0.35652	0.64835
$bootstrapCI$ lower bound	1.04409	0.69904
$bootstrapCI$ upper bound	1.5310	0.93915
$bootstrapCI$ Length	0.49901	0.24011

$Parameter, C.S[5]$	[ξ]	[β]
MSE_{MlE}	0.02028	0.03627
MSE_{SE}	0.01857	0.05120
$95\%CI$ Lower bound	1.14366	0.77957
$95\%CI$ upper bound	1.68177	1.04643
$95\%CI$ Length	0.26686	0.53811
$bootstrapCI$ lower bound	1.13563	0.71804
$bootstrapCI$ upper bound	1.54411	0.89594
$bootstrapCI$ Length	0.40848	0.17790
$Parameter, C.S[6]$	[ξ]	[β]
MSE_{MlE}	0.02037	0.03620
MSE_{SE}	0.01165	0.04481
$95\%CI$ Lower bound	1.12137	0.78121
$95\%CI$ upper bound	1.66911	1.04984
$95\%CI$ Length	0.26862	0.54774
$bootstrapCI$ lower bound	1.13203	0.71243
$bootstrapCI$ upper bound	1.54231	0.90123
$bootstrapCI$ Length	0.41028	0.18881
$Parameter, C.S[7]$	[ξ]	[β]
MSE_{MlE}	0.01692	0.04013
MSE_{SE}	0.01602	0.05599
$95\%CI$ Lower bound	1.18897	0.79006
$95\%CI$ upper bound	1.64837	1.01062
$95\%CI$ Length	0.22056	0.45941
$bootstrapCI$ lower bound	1.16962	0.73223
$bootstrapCI$ upper bound	1.54192	0.87537
$bootstrapCI$ Length	0.37230	0.14314
$Parameter, C.S[8]$	[ξ]	[β]
MSE_{MlE}	0.01741	0.03852
MSE_{SE}	0.01595	0.05220
$95\%CI$ Lower bound	1.17823	0.79422
$95\%CI$ upper bound	1.65555	1.01492
$95\%CI$ Length	0.22070	0.47733
$bootstrapCI$ lower bound	1.16572	0.73467
$bootstrapCI$ upper bound	1.54373	0.87387
$bootstrapCI$ Length	0.37801	0.13920

8 Conclusion

Modified Kies distribution is employed here in the case of PFFC to make the point and interval estimation based on MLE, symmetric Bayesian estimation. From the simulation study section, the Bayesian estimation can be considered to be an approach that is better than classical estimation. Both HPD interval length and the confidence Interval length decrease as k, m are increased. In addition, as the difference between k, m decreases, the MSE error of the parameter decreases. All the outcomes reached are :

1. For $n = 2$ and all the censoring schemes the n, m increase the MSE of the parameters of Modified Kies estimates decrease.
2. At $n = 2$ as k, m increase the Bootstrap and confidence interval lengths of the parameter also decrease.
3. The MSE of $\hat{\theta}_{SE}$, is always smaller than MSE of $\hat{\alpha}_{MLE}$.
4. When m is large, the MSE of $\hat{\xi}_{SE}$, its MSE is very small.
5. When m is large and using BE, the MSE of the parameters tends to zero.
6. When m is large, the $CI, bootstrap\ CI$, lengths also decrease.

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