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A Bounded Maximal Function Operator and Its Acting on $\mathfrak{L}^p(\mathbb{S}^2)$ Functions

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Abstract With a novel generation operator known as Spherical Scaling Wavelet Projection Operator, this study proposes new strategies for achieving the scaling wavelet expansion's convergence of $\mathfrak{L}^p(\mathbb{S}^2)$ functions with respect to $1 < p < \infty$ almost everywhere under generic hypotheses. Hypotheses of results are based on three types of conditions: Space's function f, Kind of Wavelet functions (spherical) and Wavelet Conditions. The results showed that in the case of $f(\omega) \in \mathfrak{L}^p(\mathbb{S}^2, \omega \in \mathbb{S}^2)$ and under the assumption that scaling wavelet function ϕ of a given multiresolution analysis is spherical wavelet with 0-regularity, the convergence of $\mathfrak{L}^p(\mathbb{S}^2)$ expansions almost everywhere will be achieved under a new kind of partial sums operator. We can examine some properties of spherical scaling wavelet functions like rapidity of decreasing and boundedness. After estimating the bounds of spherical scaling wavelet expansions, we examined the limited (bounds) of this operator. The results are established on the almost everywhere wavelet expansions convergence of $\mathfrak{L}^p(\mathbb{S}^2)$ space functions. Several techniques were followed to achieve this convergence, such as the bounded condition of the Spherical Hardy-Littlewood maximal operator is achieved using the maximal inequality and Riesz basis functions conditions. The general wavelet expansions' convergence was demonstrated using the spherical scaling wavelet function and several of its fundamental features. In fact, the partial sums in these expansions are dominated in their magnitude by the maximal function operator, which may be applied to establish convergence. The convergence here may be obtained by assuming minimal regularity for a spherical scaling wavelet function $\phi^k_{j,l}$. The focus of this research is on recent advances in convergence theory issues with respect to spherical wavelet expansions' partial sums operators. The employment of scaling wavelet basis functions defined on $\mathbb{S}^2$ is regarded to be a key in solving convergence problems that occur inside spaces dimension $\mathbb{S}^2$.

Keywords Spherical Scaling Spherical Multiresolution Analysis, Wavelet Expansions, Convergence, Rapidly Decreasing, Maximal Function

1 Introduction

Certain notable challenges in the wavelet theory domain occur if one attempts to rebuild the spherical surface functions $\mathfrak{L}^p(\mathbb{S}^2)$ employing wavelet basis functions. We utilized the solutions by establishing a Spherical Scaling Wavelet Projection Operator through scaling wavelet basis functions described on $\mathbb{S}^2$. This paves an idea to study the convergence of Spherical Wavelet Projection Operators, which relies upon spherical scaling wavelet expansions.
The concept of multiresolution analysis was established by Meyer [8] to construct the orthogonal wavelets. Moreover, Ajmi [10] produced a biorthogonal multiresolution analysis on triangular domains to structure biorthogonal wavelets through a specific geometry of the triangle. Furthermore, the multiresolution analysis concept was developed by Kostadinova et al. [16] to investigate the pointwise behavior with respect to Schwartz distributions in multiple variables.

The convergence problems of multiresolution expansions were examined by various works. For example, Kelly et al. [5, 6] demonstrated the convergence of multiresolution expansions of $\mathfrak{L}^p(\mathbb{R}^n)$ functions with respect to $1 \leq p < \infty$ at every f's Lebesgue point. Moreover, Tao [11] highlighted that if one regards certain minimal regularity with respect to ψ, the point wise classical wavelet operators convergence can be extended to the whole Lebesgue set with respect to f. Furthermore, Raghad et al. [18] expanded the dimension of a wavelet function for four-dimensional wavelet function to improve the results of Tao, in which the results were examined on four-dimensional wavelet projections operators. Also, the convergence of non-convolution integral operators in Lebesque space is studied in [19]. Apart from that, [20] used hyperbolic, trigonometric, exponential and complex to generate abundant exact solutions and to examine a nonlinear class of evolution fractional order equations. Moreover, Zhao et al. [12] employed the approximation approach to converge the wavelet expansions in a multiresolution analysis of $\mathfrak{L}^2(\mathbb{R})$ function to the mean value of its both sides limits at a generalized ongoing point. Here, by applying the biorthogonal B-spline wavelets, Junjian [13] studied the multiresolution expansions convergence property that is non-divergence-free and divergence-free wavelets expansion via the categorization of vector-valued Besov spaces function. Hence, by employing the multiresolution expansions, the Schwartz distributions pointwise behavior in some variables was investigated in [16]. The categorizations of the quasiasymptotic distributions' behavior with regards to the finite points are given, including addressing the associations with respect to $\alpha-$density points of measures. In addition, Singh [7] investigated the point wise wavelet's convergence expansions of multiresolution analysis of $\mathfrak{L}^2(\mathbb{R})$ functions using prolate spheroidal wavelet. Many researchers dealt with pointwise convergence topics in which wavelet expansions are defined on real lines or space. On the other hand, Raghad et al. [17] employed a spherical multiresolution analysis on $\mathbb{S}^2$ surface by discussing the pointwise convergence of SOHO wavelet expansions.

To the best of the author's knowledge, this study is the first one to generalize the findings of Raghad et al. [17] by discussing the pointwise convergence of scaling wavelet expansions under Scaling Wavelet Projections Operators defined on $\mathbb{S}^2$ surfaces. We utilize $\mathfrak{L}^p$ functions and expand them on the unit sphere, referring to the functions represented by wavelet basis functions defined on $\mathbb{S}^2$. In addition, Schröder and Sweldens [4] provided samples of functions defined on $\mathbb{S}^2$ and showed their represention using spherical biorthogonal wavelets efficiently. Subsequently, Ali et al. [1] discussed the construction of wavelets on the manifold. [9] gave us other approaches to construct the identity operator on the sphere as a sum of smooth orthogonal projections with respect to the sphere $\mathbb{S}^d$.

2 MAIN TOOLS TO APPLY THE CONVERGENCE

It is crucial to define a new multi-resolution analysis, named spherical multiresolution analysis using a 0-regular spherical scaling wavelet basis defined on unit sphere $\mathbb{S}^2$, to extend $\mathfrak{L}^p(\mathbb{S}^2, dw)$ function $\mathfrak{f}(w)$ with the property

$$\left(\int_0^{2\pi}\int_0^{\pi} |\mathfrak{f}(\theta,\varphi)|^p \sin\theta d\theta d\varphi\right)^{1/p} < \infty,$$

where $1 < p < \infty$ and $w = (\theta, \varphi)$ express through spherical polar coordinates $\theta \in [0,\pi]$, while $\varphi \in [0, 2\pi]$, while $dw = \sin\theta d\theta d\varphi$ is the standard Lebesgue measure with respect to the unit sphere $\mathbb{S}^2$.

In this work, $\mathfrak{L}^p(\mathbb{S}^2)$ space is analyzed into a set of nested sub-spaces Π_j, for $j = \{j = 0, ..., J\}$, $\{j \in \mathbb{N}\}$ by employing the spherical multi-resolution analysis process ($MSSA$) and Spherical Scaling Wavelet Projection Operator defined on $\mathfrak{L}^p(\mathbb{S}^2)$ space.

Definition 2.0.1. *A multi-spherical spaces analysis* $MSSA = \left\{\Pi_j, \Pi_j \subset \mathfrak{L}^2(\mathbb{S}^2)\right\}$ *is employed to extend* $\mathfrak{L}^p(\mathbb{S}^2, dw)$ *function* $\mathfrak{f}(w)$ *with respect to Spherical Scaling Wavelet Projection Operator. Here,* $\mathfrak{f}(w) \in \mathfrak{L}^p\left(\mathbb{S}^2, dw\right), w \in \mathbb{S}^2$ *as well as* $dw = \sin\theta d\theta d\varphi$. *Thus, for each* $j = \{j = 0, ..., J\}$, $\{j \in \mathbb{N}\}$, *the Spherical Scaling Wavelet projections on spherical spaces* $\{\Pi_j\}$ *is given by:*

$$P_{\Pi_J}\mathfrak{f} = \sum_{j=0}^{J}\sum_{l=1}^{4^j} \lambda_{j,l}\phi_{j,l}, \tag{1}$$

where $\lambda_{j,l} = \left\langle \mathfrak{f}(w), \tilde{\phi}_{j,\tilde{l}} \right\rangle$ *denotes the scaling wavelet expansion coefficient and* $\langle \mathfrak{f}, \mathfrak{g}\rangle = \int_{\mathbb{S}^2} \mathfrak{f}\mathfrak{g}dw$ *is the inner product with respect to the spherical domain. For more consideration of the scaling wavelet expansion, we obtain*

$$\sum_{l=1}^{4} \phi_{j+1,l} = \phi_{j,k}, \tag{2}$$

where $j = \{j = 0, ..., J\}$, $\{j \in \mathbb{Z}^+\}$, $\{k, \tilde{k} \in K(j), \tilde{K}(j)\}$ *and* $\{l, \tilde{l} \in K(j+1), \tilde{K}(j+1))\}$ *respectively as well as* $K(j)$, $\tilde{K}(j)$, $K(j+1)$, $\tilde{K}(j+1)$ *are representing as general index sets.*

Definition 2.0.2. *A multi-spherical spaces analysis* $MSSA = \{\Pi_j, \Pi_j \subset \mathfrak{L}^2(\mathbb{S}^2)\}$ *represents the function of space as a sequence of nested subspaces* $\{\Pi_j\}_{j=0,...,J}$ *of* $\mathfrak{L}^2(\mathbb{S}^2)$ *by implementing scaling wavelet functions, satisfying the given criteria:*

a) $\Pi_{j+1} \subset \Pi_j$.

b) $\overline{\bigcup_{j=0}^{\infty} \Pi_j} = \mathfrak{L}^2\left(\mathbb{S}^2\right)$, *(MSSA is dense in* $\mathfrak{L}^2\left(\mathbb{S}^2\right)$*).*

c) There exists a scaling function $\phi(w) \in \Pi_0$, *in which the sequence* $\{\phi_{j,k}, j \geq 0, k \in K(j)\}$ *acts as a Riesz basis for* Π_j. *For further details, the reader can read [14].*

The convergence in this work relies upon redefining two types of properties. This includes the rapidly decreasing and the bounded scaling wavelet function with 0-integral property by considering the spherical sense for scaling wavelet functions in these properties.

Definition 2.0.3. *A multi-spherical spaces analysis $MSSA = \{\Pi_j, \Pi_j \subset \mathfrak{L}^2(\mathbb{S}^2)\}$ is common to be r-regular provided that the scaling wavelet function $\phi_{j,k}(w) \in C^r$ space, for which $\phi_{j,k}(w)$ is a r-differentiable function, $w = (\theta, \varphi)$ as well as $r = 0, 1, ..., \infty$. Here*

$$\left|\frac{\partial^2 \phi^k_{j,l}(w)}{\partial\theta\partial\varphi}\right| \le \frac{C_\alpha}{4^j(1+|w|)^\alpha}, \tag{3}$$

in which $\alpha \in \mathbb{N}$, $w \in \mathbb{S}^2$, C_α is a constant.
Besides, the spherical scaling wavelet function's derivatives $\frac{\partial^2 \phi^k_{j,l}(w)}{\partial\theta\partial\varphi}$ are also limited by

$$B_\alpha \int_{\mathbb{S}^2} 4^j(1+|w|)^\alpha \left|\frac{\partial^2 \phi^k_{j,l}(w)}{\partial\theta\partial\varphi}\right| dw < \infty, \tag{4}$$

in which $j = \{j = 0, ..., J\}$, $\{j \in \mathbb{N}\}$ with $k, l \in K(j), K(j+1)$ respectively and B_α is a constant.

The other property which is used in the main result is defined here:

Definition 2.0.4. *For every $\mathfrak{f} \in \Pi_j$, there are countable infinite sequences $\{\alpha_{j,k}\}_{j,k\in\mathbb{Z}} \in \mathfrak{L}^2(\mathbb{R})$ and countable sequences of Riesz basis functions $\{\phi_{j,k}(x) : j, k \in \mathbb{Z}\}$ with respect to the multi-space analysis (MSA) of $\mathfrak{L}^2(\mathbb{R})$. In such cases, we have*

$$\mathfrak{f}(x) = \sum_{j,k\in\mathbb{Z}} \alpha_{j,k}\phi_{j,k}(x),$$

the following statements hold true

$$\begin{gathered}\sum_{j,k} |\alpha_{j,k}|^2 < \infty, \\ \imath\sum_{j,k} |\alpha_{j,k}|^2 \le \|\sum_{j,k\in\mathbb{Z}} \alpha_{j,k}\phi_{j,k}(x)\|_2^2 \le \ell\sum_{j,k} |\alpha_{j,k}|^2,\end{gathered} \tag{5}$$

where $0 < \imath \le \ell < \infty$ are constants independent on $\mathfrak{f}$.

For more knowledge on norm definition, we have
$\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$, when $x = [x_1, ..., x_n] \in \mathbb{R}^n$.
Hence, this function accepts the norm's mandatory properties and refers to be the Euclidean norm defined on $\mathbb{R}^n$. For more details, one can read [2].

The almost everywhere convergence is connected between the action of $P_j(\mathfrak{f}(w))$ with respect to the functions $\mathfrak{f}(w)$ and the partial sums of the functions by the scaling wavelets basis functions on $(\mathbb{S}^2)$. Thus, the question arises here is on what may be understood about the almost everywhere convergence of spherical scaling wavelet expansions? Do the expansions converge for their $\mathfrak{L}^p(\mathbb{S}^2)$ functions almost everywhere? The answer to the question is outlined as follows.

3 Main Result with Proof

The primary findings of this research are written as follows:

Theorem 3.0.1. *If $\mathfrak{f}(w) \in \mathfrak{L}^p(\mathbb{S}^2, dw)$ with respect to the assumption that the scaling wavelet function ϕ of a given expansion denotes a 0-regular function, it then yields:*

$$\lim_{J\to\infty} P_{\Pi_J}\mathfrak{f}(w) = \lim_{J\to\infty}\left(\sum_{j=0}^{J}\sum_{l=1}^{4^J} \lambda_{j,l}\phi_{j,l}\right) = \mathfrak{f}(w). \tag{6}$$

Proof. The proof of Formula 6 is divided into several theoretical steps. Our strategy to attain convergence almost everywhere is by studying the bounds on a summation of the scaling wavelet expansions specified on $(\mathbb{S}^2)$. The summation is regulated in its magnitude via the action of the maximal function operator, as explained in the following definition:

Definition 3.0.2. *Let $B(x, r)$ represent the open ball defined by $\mathfrak{L}^p-$norm in $\mathbb{R}^n$ and $|B|$ is its Lebesgue measure, such that, $B(x, r) = \{y \in \mathbb{R}^n : \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p} < r\}$ for any center point $x \in \mathbb{R}^n$ and radius $r > 0$.*

For every $\mathfrak{f}(x) \in L^1_{loc}(\mathbb{R}^n)$ be a locally integrable function in $\mathbb{R}^n$, the Hardy-Littlewood maximal operator M of $\mathfrak{f}(x)$ is defined by:

$$M\mathfrak{f}(x) = \sup_{r>0} |A_r(\mathfrak{f})(x)|,$$

in which A_r denotes a maximal function having

$$A_r(\mathfrak{f})(x) = \frac{1}{|B(x, r)|}\int_{B(x,r)} \mathfrak{f}(y)dy,$$

while

$$|B(x, r)| = w_n r^n.$$

Here, w_n denotes the unit ball volume $B(x, r)$ in $\mathbb{R}^n$ (which is the Lebesgue measure of the unit ball). Typically, we begin with a fixed ball $B(0, 1)$ positioned at the origin as well as describe the maximal function employing all the families of balls achieved through dilations and translations of $B(0, 1)$. For further information, reader can refer to ([3],pp 85-104).

The proof of the main facts is premised on the given theorems.

Theorem 3.0.3. *Let $M(\mathfrak{f})(x)$ be denoted as the supremum of function $\mathfrak{f}$ averages taken across all surfaces of spheres centered at x. Therefore, $\mathfrak{f} \to M(\mathfrak{f})(x)$ is bounded on $\mathfrak{L}^p(\mathbb{R}^n)$, when $p > \dfrac{n}{n-1}$ and $n \ge 3$.*

To prove the aforementioned theorem, see Stein[15].

Theorem 3.0.4. *Let $n \ge 3$, therefore the maximal inequality*

$$\|, M(\mathfrak{f})\|_p \le A_p\|\mathfrak{f}\|_p, \ \ \mathfrak{f} \in S$$

holds whenever $\frac{n}{n-1} < p < \infty$ and S represents a sphere. Provided that $p < \dfrac{n}{n-1}$, the maximal inequality is invalid.

To prove the aforementioned theorem, reader may refer to Stein[15].
From Theorems 3.0.3 and 3.0.4, it is enough to prove the following conditions to verify the convergence of the spherical scaling wavelet expansions.

Theorem 3.0.5. *Suppose $\phi^k_{j,l}$ represents 0-regular spherical scaling basis function with respect to any spherical wavelet function, thus $\phi^k_{j,l}$ is bounded.*

Proof. First, we confirm that $\phi^k_{j,l}$ is bounded. This is due to the fact that ϕ is 0-regular spherical basis function. Moreover, from the formula below

$$\left|\frac{\partial^2\phi^k_{j,l}(w)}{\partial\theta\partial\varphi}\right| \leq \frac{C_\alpha}{4^j(1+|w|)^\alpha}, \tag{7}$$

we have

$$\left|\phi^k_{j,l}(w)\right| \leq \frac{C_N}{4^j(1+|w|)^N}.$$

Here, C_N denotes a constant. We consider $\Upsilon(j,k)$ to be the integer part of the quantity $(1+|w|)$ when j belongs to $\mathbb{N}$ and k belongs to general index set $K(j)$.
Provided that

$$4^j(1+|w|)^N \geq 4^j\Upsilon(j,k)^N,$$

therefore, $\left|\phi^k_{j,l}(w)\right|$ has bounded by the quantity $C_N 4^{-j}\Upsilon(j,k)^{-N}$. □

Theorem 3.0.6. *Assume $\mathfrak{f}(w) \in \mathfrak{L}^p\left(\mathbb{S}^2, dw\right)$. Hence, for nearly each $w \in \mathbb{S}^2$, one possesses under assumption that the scaling function ϕ with respect to a given scaling wavelet expansion refers to a 0-regular spherical wavelet,*

$$|\lambda_{j,k}| \leq \left(2^{-(N+q)}\pi^{1/q}2^{-2j}\|\mathfrak{f}(w)\|_{\mathfrak{L}_p}\right), \tag{8}$$

for nearly each $w=(\theta,\varphi)\in\mathbb{S}^2$, $dw=\sin\theta d\theta d\varphi$ denotes the standard Lebesgue measure on unit sphere $\mathbb{S}^2$, $\{j=0...J\}$, $\{j\in\mathbb{N}\}$ and $k,l\in K(j), K(j+1)$.

Proof. Based on the definition of $\lambda_{j,k}$ coefficient , one gets

$$\lambda_{j,k} = \int_{S^2}\mathfrak{f}(w)\phi_{j,k}dw.$$

After implementing the technique of Hölder's inequality, therefore one obtains

$$|\lambda_{j,k}| = \left(\int_{S^2}|\mathfrak{f}(w)|^p dw\right)^{1/p}\left(\int_{S^2}|\phi_{j,k}(w)|^q dw\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

$$\leq \|\mathfrak{f}(w)\|_{\mathfrak{L}_p}\left(\int_{S^2}|\phi_{j,k}(w)|^q dw\right)^{1/q}.$$

Since ϕ is a rapidly decreasing function, we now have

$$\lambda_{j,k} \leq \|\mathfrak{f}(w)\|_{\mathfrak{L}_p}\left(\int_{S^2}\left(\frac{C_N}{4^j(1+w)^N}\right)^q dw\right)^{1/q}.$$

This work applies on sphere $\mathbb{S}^2$. By using polar coordinates, one obtains $w=(r,\theta,\varphi)$ provided that one converts w into Cartesian coordinates. Therefore, $(r,\theta,\varphi)\equiv(x,y,z)$ in which $x=r\sin\theta\cos\varphi$, $y=r\sin\theta\sin\varphi$ as well as $z=r\cos\theta$.
Therefore,

$$|w| = \sqrt{x^2+y^2+z^2} = \sqrt{r^2(\sin^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi+\cos^2\theta)}$$
$$= r\sqrt{\sin^2\theta(\cos^2\varphi+\sin^2\varphi)+\cos^2\theta},$$

through making use of the equation $(\cos^2\varphi+\sin^2\varphi)=1$. We now obtain

$$|w|\equiv r.$$

Therefore,

$$|\lambda_{j,l}| \leq \|\mathfrak{f}(w)\|_{\mathfrak{L}_p}4^{-j}C_N\left(\int_{\mathbb{S}^2}\frac{1}{2^{Nq}}dw\right)^{1/q}$$
$$\leq 4^{-j}2^{-N}C_N\|\mathfrak{f}(w)\|_{\mathfrak{L}_p}\left(\int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\varphi\right)^{1/q}$$
$$\leq 2^{(-N+\frac{2}{q})}\pi^{1/q}C_N 2^{-2j}\|\mathfrak{f}(w)\|_{\mathfrak{L}_p}.$$

Therefore,

$$\lambda_{j,l} = O\left(2^{-(N+\frac{2}{q})}\pi^{1/q}2^{-2j}\|\mathfrak{f}(w)\|_{\mathfrak{L}_p}\right).$$

□

Theorem 3.0.7. *Consider $\mathfrak{f}(w)\in\mathfrak{L}^p\left(\mathbb{S}^2,dw\right)$ and under assuming a 0-regular spherical scaling wavelet ϕ of a given expansion, then we have:*

$$\sup_{j\leq J}|P_{\Pi_J}\mathfrak{f}(w)| \leq \wp\mu(\mathfrak{f})(w), \tag{9}$$

where $0\leq j\leq J\in\mathbb{N}$ and $\wp$ is a constant.

Proof. To prove Formula 9, the following steps are applied. From Definition 2.0.1, the following fact is obtained:

$$\|P_{\Pi_J}\mathfrak{f}\| \leq \wp\sup_J|P_{\Pi_J}\mathfrak{f}|,$$

where $j\leq J$, as well as

$$\left\|\sum_{j<J}\sum_{l=1}^{4^j}\lambda_{j,l}\phi_{j,l}\right\| \leq \wp\sup_J\left|\sum_{l=1}^{4^j}\lambda_{j,l}\phi_{j,l}\right|.$$

Now, we redefine Definition 3.0.2 on spherical space function $\mathfrak{f}(w)\in\mathfrak{L}^p(\mathbb{S}^2)$, for $1<p<\infty$.

Definition 3.0.8. *For any $\mathfrak{f}(w)\in\mathfrak{L}^p(\mathbb{S}^2)$, the supremum of $\mathfrak{f}(w)$ averages that have covered all surfaces of the sphere $(\mathbb{S}^2)$ and are centred at $w=(\theta,\varphi)$ can be denoted as spherical maximal function operator $\mu(\mathfrak{f})(w)$ and defined by:*

$$\mu(\mathfrak{f})(w) = \sup_{r>0}\frac{1}{|\mathbb{S}^2(w,r)|}\int_{\mathbb{S}^2(w,r)}|\mathfrak{f}(w)|dw, \tag{10}$$

in which dw denotes the Lebesgue measure with respect to the unit sphere for any $w \in \mathbb{S}^2$, $\left|\mathbb{S}^2(w,r)\right| = W^3 r^3$ represents the unit sphere volume, while W^3 denotes the surface measure of unit sphere at $r = 1$.

The condition is proven as follows:

$$\sup_{j\leq J} |P_{\Pi_J}\mathfrak{f}(w)| \leq \wp\mu(\mathfrak{f})(w), \quad (11)$$

in which the following steps are applied.

$$\|P_{\Pi_J}\mathfrak{f}\| \leq \wp \sup_{j\leq \mathrm{J}} |P_{\Pi_J}\mathfrak{f}|,$$

for each $j \leq J$ and $\wp$ is a constant. From Equation 1,
one has the following fact:

$$\|\sum_{j\leq J}\sum_{l=1}^{4^j} \lambda_{j,l}\phi_{j,l}\| \leq \wp \sup_{j\leq \mathrm{J}} \left|\sum_{l=1}^{4^j} \lambda_{j,l}\phi_{j,l}\right|.$$

After simplifying the term $\left|\sum_{l=1}^{4^j} \lambda_{j,l}\phi_{j,l}\right|$, we get

$$\left|\sum_{l=1}^{4^j} \lambda_{j,l}\phi_{j,l}\right| \leq \sum_{l=1}^{4^j} |\lambda_{j,l}|\,|\phi_{j,l}|.$$

From Theorems 3.0.5, 3.0.6 and 3.0.7, we employ the equations to prove the boundedness of the terms $|\lambda_{j,k}|$ and $|\phi_{j,k}|$, respectively. Hence,

$$\sup_{j\leq \mathrm{J}} |P_{\Pi_J}\mathfrak{f}(w)| \leq C_N \Upsilon(j,k)^{-N} \sup_{j\leq \mathrm{J}} \left(2^{(-N+2/q)} 2^{-4j} \pi^{1/q}\right) \left(\left(\int_{\mathbb{S}^2} |\mathfrak{f}(w)|^p dw\right)^{1/p}\right),$$

and

$$\sup_{j\leq \mathrm{J}} |P_{\Pi_J}\mathfrak{f}(w)| \leq \wp\mu(\mathfrak{f})(w),$$

where $\wp$ is a constant.

□

From Theorems 3.0.6 and 3.0.7, the main result of this work is proven. □

4 Conclusions

The convergence of a new kind of scaling wavelet expansions of $\mathfrak{L}^p(\mathbb{S}^2)$ functions is established in this work. Depending on the rapidly decreasing property for the spherical scaling wavelet function, convergence appears almost everywhere. The structure of spherical multi-resolution analysis ($MSSA$) with spherical scaling basis functions is used to form a new kind of wavelet projection operators called Spherical Scaling Wavelet Projection Operators. The bounds of the partial sums in these expansions are achieved using the maximal function operator, which makes the convergence of operators appear almost everywhere.

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