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Exact Run Length Computation on EWMA Control Chart for Stationary Moving Average Process with Exogenous Variables

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Abstract The exponentially weighted moving average (EWMA) control chart is a popular tool used to monitor and identify slight unnatural variations in the manufacturing, industrial, and service processes. In general, control charts operate under the assumption of normality observation of the attention quality feature, but it is not easy to maintain this assumption in practice. In such situations, the data of random processes are correlated data, such as stock price in the economic field or air pollution data in the environment field. The characteristics and performance of the control chart are measured by the average run length (ARL). In this article, we present the new explicit formula of ARL for EWMA control chart based on MAX(q,r) process. The proposed explicit formula of ARL for the MAX(q,r) process is proved using the Fredholm integral equation technique. Moreover, ARL values are also assessed using the numerical integral equations method based on Gaussian, midpoint, and trapezoidal rules. Banach's fixed point theorem guarantees the existence and uniqueness of the solution. Furthermore, the accuracy of the proposed explicit formula is assessed in absolute percentage relative error compared with the numerical integral equations method. The results found that the explicit formula's ARL values are similar to those obtained using the numerical integral equation method; the absolute percentage relative errors are less than 0.0001 percent. As a result, the essential conclusion is that the explicit formula outperforms the numerical method in computational time. Consequently, the proposed explicit formula and the numerical integral equation have been the alternative approaches for computing ARL values of the EWMA control chart. They would be applied in various fields, including economics, environment, biology, engineering, and others.

Keywords EWMA, Moving Average, Average Run Length, Explanatory Variable, Exponential White Noise

1. Introduction

Quality control is a critical tool for minimizing the numerous flaws that occur during the production process. Statistical Process Control (SPC) is used to establish better processes. Control charts are an efficient tool for monitoring and regulating production operations in real-time. Walter A. Shewhart devised the control chart in 1924 to assist factories in reducing waste and improving quality. Since then, control charts have been widely used to identify and monitor the effects of processes on their quality in various applications, including industrial production, public health, computer networking and telecommunications, finance and economics, and environmental research, among others. Nowadays, industrial processes often employ Shewhart control charts, cumulative sum (CUSUM) charts and exponentially weighted moving average control charts (EWMA). Robert [1] invented the EWMA control chart to identify a slight change in the mean compatible with the normal distribution. It is now widely understood that a robust control chart is necessary for detecting slight and moderate process changes rapidly. Control chart performance is frequently quantified in the average run length (ARL). The ARL_0 value showed that the process is still in control, but the ARL_1 value represents that process is really out of control and should be kept small. The ARL of the EWMA control chart may be computed using the Monte Carlo Simulations (MC) approach, which is the standard method for validating and comparing it to other techniques. However, the processing is time-consuming, the Martingale approach sees [2-3], Markov chain technique sees [4-5], and integral equation method all contribute to this [6-12].

Additionally, due to the serial correlation of the data, this can affect the performance of control chart processes such as first-order autoregressive (AR(1)) and first-order moving average (MA(1)) [13]. In conclusion, several authors evaluate the ARL when a serial correlation is involved. For example, Lu and Reynolds [14] analyzed the ARL using the integral equation approach when the data were AR(1) and ARMA(1,1) processes on the EWMA control chart. Suriyakat et al. [15] derived the explicit ARL formula for the AR(1) process with exponential white noise using the EWMA control chart. Next, Suriyakat et al. [16] established the ARL solution by developing an explicit formula for the trend exponential AR(1) processes on the EWMA chart. Petcharat et al. [16] published explicit expressions for the ARL of EWMA and CUSUM control charts for the moving average process of order q (MA(q)). Later that year, Busababodin [17] published an explicit formula for determining the ARL of the CUSUM control chart when the data are seasonal AR(p) processes. Petcharat et al. [18] employed an analytical procedure to derive the average run length of a CUSUM chart on a moving average (MA) process of order q with exponential white noise. Then, Petcharat [19] analyzed the average run length by using the explicit formula on the EWMA chart for a seasonal moving average model of order q with exponential white noise. Additionally, Petcharat [20] presented the exact ARL solution for the $SAR(P)_L$ process for the EWMA control chart and compared it to the CUSUM ARL value using the numerical integral method. The EWMA chart is more effective than the CUSUM control chart for detecting small process changes. Furthermore, Suriyakat [21] proposed an analytical solution of ARL for trend exponential Autoregressive order p model based on the EWMA control chart. Sukparungsee and Areepong [22] presented the explicit formulas of ARL for the EWMA chart. Additionally, Suriyakat [23] investigated the sensitivity of the EWMA control chart for detecting mean changes in processes using AR(1), ARMA(1,1), and IMA(1) with exponential white noise and discovered that the EWMA control chart has a high sensitivity for identifying tiny mean changes in processes. Sunthornwat and Areepong [24] produced analytical ARL values and approximate numerical ARL values for the CUSUM control chart for seasonal and non-seasonal moving average processes. Recently, Petcharat [25] proposed the exact solution of average run length for $SAR(P)_L$ with a trend based on the CUSUM control chart. We enhance Petcharat [26] by deriving explicit equations for the average run length (ARL) of the EWMA control chart for moving average processes with exogenous variables (MAX(q,r)) with exponential white noise and comparing them to numerical integration methods.

This article may be structured in the following manner. Section 2 presents the EWMA control chart for the MAX process. Additionally, the explicit formula for computing the ARL of the EWMA control chart when the processes are MAX(q,r) demonstrates and includes numerical integral equation methods. Section 3 contains the results of the simulation. Finally, Section 4 contains conclusions.

2. Methodology

2.1 The EWMA Control Chart for MAX(q,r) Process

The properties of the EWMA control chart for the MAX(q,r) process are discussed in this section. The EWMA control chart is a very effective tool for identifying tiny changes in mean. Assume that Z_t be the sequence of moving average processes with exogenous variables (MAX(q,r)). The MAX(q,r) process with exponential white noise has the following recursive equation:

$$Z_t = \mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \theta_2\varepsilon_{t-2} - \dots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}, \quad (1)$$

where ε_t is exponential white noise : $\varepsilon_t \sim Exp(\alpha)$,

μ is process mean,

θ is moving average coefficient, $-1 \le \theta_i \le 1, \quad i = 1,\dots,q$

X_{it} is exogenous variable,

β_i is a coefficient of X_{it}.

The recursive equation of EWMA statistic based on MAX(q,r) process is defined by

$$Y_t = (1-\lambda)Y_{t-1} + \lambda Z_t, \quad t = 1,2,\dots \quad (2)$$

where Z_t is sequence of MAX(q,r) process,

λ is an exponential smoothing parameter, $0 < \lambda < 1$.

According equation (1) and (2) can be written as

$$Y_t = (1-\lambda)Y_{t-1} + \lambda(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} - \dots - \theta_q\varepsilon_{t-q} + \sum_{i=1}^{r}\beta_i X_{it}), \quad t = 1,2,\dots$$

$$Y_t = (1-\lambda)Y_{t-1} + \lambda\mu + \lambda\varepsilon_t - \lambda\theta_1\varepsilon_{t-1} - ... - \lambda\theta_q\varepsilon_{t-q} + \lambda\sum_{i=1}^{r}\beta_i X_{it},$$

where $t = 1, 2, ...,\ Y_0 = Z_0 = u$

The corresponding stopping time for (2) define as

$$\tau = \inf\{t > 0; Y_t > b\} \quad ,Y_0 = u,\ b > y. \tag{3}$$

where b denote upper control limit, Y_0 is starting value and τ is stopping time.

Let $\mathbb{E}_\omega(.)$ signify the probability that the change-point occurs at point ω where $\omega < \infty$. Thus, the ARL for the MAX(q,r) process with a starting value of $Y_0 = u$ is defined as follows.

$$ARL = j(u) = \mathbb{E}_\infty(\tau_b) < \infty. \tag{4}$$

2.2. The Average Length (ARL) for MAX(q,r) Process Based on EWMA Control Chart

This section presents an explicit formula for the average run length of the EWMA control chart for the MAX(q,r) process with exponential white noise. Let $j(u)$ indicate the average run length of EWMA control chart. Assume that the process is initially in-control $Y_0 = u$. The integral equation denotes as follows:

$$j(u) = 1 + \frac{1}{\lambda}\int_0^b j(z) f\left(\frac{z-(1-\lambda)u}{\lambda} + \begin{pmatrix}\mu + \varepsilon_t + \theta_1\varepsilon_{q-1} + ... \\ +\theta_q\varepsilon_{t-q} - \sum_{i=1}^{r}\beta_i X_{it}\end{pmatrix}\right) d(z), \tag{5}$$

Then

$$j(u) = 1 + \frac{1}{\lambda\alpha}\int_0^b j(z) e^{-\frac{z}{\lambda\alpha}} e^{\begin{pmatrix}\frac{(1-\lambda)u}{\lambda\alpha} + \frac{1}{\alpha}\mu + \varepsilon_t + \theta_1\varepsilon_{t-1} \\ +...+\theta_q\varepsilon_{t-q} - \sum_{i=1}^{r}\beta_i X_{it}\end{pmatrix}} d(z). \tag{6}$$

Suppose $C(u) = e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}$ then equation (6) can be written as

$$j(u) = 1 + \frac{C(u)}{\lambda\alpha}\int_0^b j(z) e^{-\frac{z}{\lambda\alpha}} d(z),\ 0 \le u \le b. \tag{7}$$

In the next step, we use Banach's fix point theorem to verify the existence and uniqueness solution of ARL for EWMA control chart.

Theorem 2.1 (Banach fixed point theorem) *Given (Y, d) be a complete metric space with a contraction mapping $T: Y \to Y$. Therefore, T accepts a unique fixed-point $y^* \in Y$, $T(y^*) = y^*$. Also, y^* can be found as follows: begin with an arbitrary element $y_0 \in Y$ and define a sequence $\{y_n\}$ by $y_n = T(y_{n-1})$, then $y_n \to y^*$.*

Proof

As a result, the right hand side of (7) is continuous, the solution of (7) is also continuous. Consider, the complete metric space $(Y(I), \|\ \|_\infty)$ where $Y(I)$ be space of all continuous functions on compact interval I and the norm $\|j\|_\infty = Sup_{u\in I}|j(u)|$. Noted that the operator T is contraction, if there exists a real constant $0 \le \eta < 1$ such that $\|T(j_1) - T(j_2)\| \le \eta\|j_1 - j_2\|$; $\forall j_1, \forall j_2 \in C(I)$, where $I = [0, b)$ and define the operator T by

$$T(j(u)) = j(u) = 1 + \frac{1}{\lambda\alpha} \times \int_0^b j(z) e^{-\frac{z}{\lambda\alpha}} e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}} d(z). \tag{8}$$

Therefore, the integral equation (8) can be written as $T(j(u)) = j(u)$. According to the Banach's fixed point theorem, if the operator T is a contraction, then fixed point equations $T(j(u)) = j(u)$. have a unique solution. We will prove in the next theorem

Theorem 2.2 *On the metric space $(Y(I), \|\ \|_\infty)$ with the norm $\|j\|_\infty = Sup_{u\in I}|j(u)|$ the operator T is a contraction*

Proof

First, showing T is a contraction $\forall u \in I$, and $j_1, j_2 \in Y(I)$ .The inequality $\|T(j_1) - T(j_2)\| \le \eta\|j_1 - j_2\|$, with $0 \le \eta < 1$. According to (8), then

$$\|T(j_1) - T(j_2)\|_\infty \le \sup_{u\in[0,b)} |j_1(0) - j_2(0)| \int_0^b j(z) e^{-\frac{z}{\lambda\alpha}} d(z) \times \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}$$

$$\le \sup_{u\in[0,b)}\left| \|j_1 - j_2\| \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} (-\lambda\alpha) \right|$$

$$\le \|j_1 - j_2\|_\infty \sup_{u\in[0,b)}\left| \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} (-\lambda\alpha) \right|$$

$$= \|j_1 - j_2\|_\infty \left|1 - e^{-\frac{b}{\lambda\alpha}}\right| \sup_{u\in[0,b)}\left| \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} \right|$$

$$\le \eta\|j_1 - j_2\|_\infty,$$

Where

$$0 \le \eta = \left|1-e^{-\frac{b}{\lambda\alpha}}\right| \sup_{u\in[0,b)} \left|\frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}}\right| < 1.$$

The Triangular inequality can be used and the fact that is

$$|j_1(0)-j_2(0)| \le \sup_{u\in[0,b)} |j_1(u)-j_2(u)| = \|j_1-j_2\|_\infty .$$

The uniqueness of solution is guaranteed via Theorem 2.1 and Theorem 2.2. Next theorem, we use the Fredholm integral equation of second kind to derive the ARL for MAX(q,r) process.

Theorem 2.3 *The explicit solution of the integral equation as follow:*

$$j(u) = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}} e^{-\frac{b}{\lambda\alpha}} - 1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}} + e^{-\frac{b}{\alpha}} - 1},\ u \ge 0.$$

Proof

From equation (6)

$$j(u) = 1 + \frac{C(u)}{\lambda\alpha}\int_0^b j(z) e^{-\frac{z}{\lambda\alpha}} d(z)$$

where $\varepsilon_t \sim exp(\alpha)$, and then Let g be constant as $g = \int_0^b j(z) e^{-\frac{z}{\lambda\alpha}} dz$. The function $j(u)$ can be written as

$$j(u) = 1 + \frac{C(u)}{\lambda\alpha} g \qquad (9)$$

Consider

$$g = \int_0^b L(z) e^{-\frac{z}{\lambda\alpha}} dz \quad = \int_0^b \left(1 + \frac{C(z)}{\lambda\alpha} c\right) e^{-\frac{z}{\lambda\alpha}} dz$$

$$= \int_0^b e^{-\frac{z}{\lambda\alpha}} dz + d\int_0^b \frac{C(z)}{\lambda\alpha} e^{-\frac{z}{\lambda\alpha}} dz$$

$$= -\lambda\alpha\left(e^{-\frac{b}{\lambda\alpha}} - 1\right) + \int_0^b \frac{C(y)}{\lambda\alpha} d e^{-\frac{y}{\lambda\alpha}} dy$$

$$= -\alpha\lambda\left(e^{-\frac{b}{\lambda\alpha}} - 1\right) + \frac{d}{\alpha\lambda} e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+..+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}} \int_0^b e^{-\frac{z}{\lambda\alpha}+\left(\frac{(1-\lambda)z}{\lambda\alpha}\right)} dz$$

$$= \alpha\lambda\left(1 - e^{-\frac{b}{\lambda\alpha}}\right) + \frac{g}{\alpha\lambda}\left[\alpha\left(1 - e^{-\frac{b}{\alpha}}\right)\right] e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}$$

$$= \alpha\lambda\left(1 - e^{-\frac{b}{\lambda\alpha}}\right) + \frac{g}{\lambda}\left[\left(1 - e^{-\frac{b}{\alpha}}\right)\right] \times e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}$$

$$g = \frac{\lambda\alpha\left(1 - e^{-\frac{b}{\lambda\alpha}}\right)}{1 - e^{-\frac{\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}-...-\theta_q\varepsilon_{t-q}+\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}\left(\frac{1 - e^{-\frac{b}{\alpha}}}{\lambda}\right)} \qquad (10)$$

Substitute equation (10) into equation (9), then equation (9) can be rewritten as:

$$j(u) = 1 + \frac{1}{\lambda\alpha} e^{\frac{(1-\lambda)u}{\lambda\alpha}} e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}} \frac{\lambda\alpha\left(1 - e^{-\frac{b}{\lambda\alpha}}\right)}{1 - \left(\frac{1 - e^{-\frac{b}{\alpha}}}{\lambda}\right) e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}}$$

$$= 1 + e^{\frac{(1-\lambda)u}{\lambda\alpha}} e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}} \frac{\lambda\left(1 - e^{-\frac{b}{\lambda\alpha}}\right)}{\lambda - \left(1 - e^{-\frac{b}{\alpha}}\right) e^{-\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}}$$

$$= 1 - \frac{1}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}} \frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}} \lambda\left(e^{-\frac{b}{\lambda\alpha}} - 1\right)}{\lambda + \frac{\left(e^{-\frac{b}{\alpha}} - 1\right)}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}}}$$

$$= 1 - \frac{1}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}} \frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}}\lambda\left(e^{-\frac{b}{\lambda\alpha}}-1\right)}{\frac{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}+\left(e^{-\frac{b}{\alpha}}-1\right)}{e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}}}}$$

$$j(u) = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}} e^{-\frac{b}{\lambda\alpha}} - 1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}} + e^{-\frac{b}{\alpha}} - 1} \quad (11)$$

Assuming $\alpha = \alpha_0$, we verified that the process is in-control state. The explicit formula for ARL_0 of EWMA control chart for the MAX(q,r) process can be rewritten as:

$$ARL_0 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha_0}} e^{-\frac{b}{\lambda\alpha_0}} - 1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha_0}} + e^{-\frac{b}{\alpha_0}} - 1} \quad (12)$$

Contrarily, the process is out-of-control state, the exponential parameter $\alpha = \alpha_1$ where $\alpha_1 = \alpha_0(1+\delta)$ and δ is the shift size. The explicit formula for ARL_1 of EWMA control chart for MAX(q,r) process can be rewritten as:

$$ARL_1 = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha_1}} e^{-\frac{b}{\lambda\alpha_1}} - 1}{\lambda e^{\frac{\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}}{\alpha_1}} + e^{-\frac{b}{\alpha_1}} - 1}, \quad (13)$$

2.3. Numerical Integral Equation (NIE) of Average Run Length for MAX(q,r) on EWMA Control Chart

This section, we present a numerical method to approximate the ARL value of EWMA control chart for MAX(q,r) with an exponential white noise process. Since $F(y) = 1 - e^{-\alpha y}$ and $f(y) = \frac{dF(y)}{du} = \alpha e^{-\alpha y}$.

In this paper, there are three methods, namely midpoint rule, trapezoidal rule and Gaussian's rule. The fundamental approach of numerical integration approximation, differs in approximation as follows:

2.3.1. Midpoint Rule

Consider an integral function $g(y)$ is evaluated over [0,*b*], where [0,*b*] is a finite interval. The value $W(y) = 1$ is chosen and a set of spread equally point is used. The interval [0,*b*] is partitioned into *n* subintervals. $\left\{\left[y_{j-1}, y_j\right], j = 1,2,...,n\right\}$ of equal width $h = \frac{b-0}{n}$ by using spaced point $y_j = y_0 + jh$ for $j = 1,2,...,n$ then $y_0 = 0$ and $y_j = b$. The midpoint is given by $m_j = \frac{1}{2}\left(y_{j-1} + y_j\right) = \left(j - \frac{1}{2}\right)b$. The weight ϖ_j for each midpoint was chosen to be 1. The midpoint rule will exactly integrate a constant function on each subinterval. By combining the rules for *n* subintervals, the composite midpoint rule for subintervals is produced as follows:

$$M(g,h) = h\sum_{j=1}^{n} g\left(\left(j - \frac{1}{2}\right)h\right)$$

The approximation for the integral is given by

$$\int_0^b g(s)ds \approx h\sum_{j=1}^{n} g\left(\left(j - \frac{1}{2}\right)h\right).$$

2.3.2. Trapezoidal Rule

The trapezoidal rule, we give function $W(y) = 1$, the interval of integration [0,*b*] is finite and the set of points are also spread equally. The interval [0,*b*] is partitioned into *n* subintervals $\left\{\left[y_{j-1}, y_j\right], j = 1,2,...,n\right\}$ of equal width $h = \frac{b-0}{n}$ by using spaced point $y_j = y_0 + jh$ for $j = 1,2,...,n$ then $y_0 = 0$ and $y_j = b$, a polynomial of degree 1 will be integrated exactly by the rule. This choice gives equal weight $\frac{1}{2}h$ at the end points of a subinterval. By using the trapezoidal rule, suppose *n* be subinterval on [0,*b*]. By combining the rules for *n* subintervals, the trapezoidal rule for *n* subintervals is produced as follows:

$$T(g,h) = \frac{h}{2}\left(g(0) - g(b)\right) + h\sum_{j=1}^{n} g(s_j), \quad (9)$$

The approximation for the integral is given by

$$\int_0^b g(s)ds \approx \frac{h}{2}\left(g(0) - g(b)\right) + h\sum_{j=1}^{n-1} g(s_j)$$

2.3.3. Gaussian Rule

The integral function $g(y)$ is evaluated over [0,*b*], where [0,*b*] can be infinite interval. The weight function $W(y)$ may be not equal to 1, and the set of point $\{a_j, j = 1,...,n\}$ may not be equally spaced. The approximation of an integral is evaluated by the Gaussian rule as follows:

$$\int_0^b W(y)g(y)dy \approx \sum_{j=1}^{n} \varpi_j g(a_j).$$

Thus a_j be a set of point, $0 \le a_1 \le a_2 \le ... \le a_n \le b$ $\varpi_j = b\ /m \ge\ 0\ ; j=\ 1, 2, .n$ be a set of constant weights.

We give the ARL using NIE method denoted by $j_{NIE}(u)$. The numerical approximation for the integral equation may be approximated as follows:

$$J(a_i) = 1 + \frac{1}{\lambda}\sum_{j=1}^{n}\varpi_j j(a_j) f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right). \text{ i=1,2,…,n} \quad (14)$$

The n linear equations $J(a_1), J(a_2),.., J(a_n)$, as

$$J(a_1) = 1 + \frac{1}{\lambda}\sum_{j=1}^{n}\varpi_j j(a_j) f\left(\frac{a_j-(1-\lambda)a_1}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right)$$
$$J(a_2) = 1 + \frac{1}{\lambda}\sum_{j=1}^{n}\varpi_j j(a_j) f\left(\frac{a_j-(1-\lambda)a_2}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right)$$
$$J(a_n) = 1 + \frac{1}{\lambda}\sum_{j=1}^{n}\varpi_j j(a_j) f\left(\frac{a_j-(1-\lambda)a_n}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right)$$

Or rewritten in matrix form as

$$\boldsymbol{J}_{n\times1} = \mathbf{1}_{n\times1} + \mathbf{R}_{n\times n}\boldsymbol{J}_{n\times1},$$

where $\boldsymbol{J}_{n\times1} = \begin{pmatrix} j(a_1) \\ j(a_2) \\ \vdots \\ j(a_n) \end{pmatrix}$, $\mathbf{1}_{n\times1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$,

$$[R]_{ij} = 1 + \frac{1}{\lambda}\sum_{j=1}^{n}\varpi_j f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right),\ i, j = 1,2,...,n$$
,

and $\mathbf{I}_m = \mathrm{diag}(1,1,...,1)$. If $(\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}$ there exist, then $\boldsymbol{J}_{n\times1} = (\mathbf{1}_n + \mathbf{R}_{n\times n})^{-1}\boldsymbol{J}_{n\times1}$,

Finally, we replace a_i by u in equation (14), then the approximation for $J(u)$ is

$$j_{NIE}(u) = 1 + \frac{1}{\lambda}\sum_{j=1}^{n}\varpi_j j(a_j) f\left(\frac{a_j-(1-\lambda)u}{\lambda}\right.$$
$$\left.+\left(\mu+\varepsilon_t+\theta_1\varepsilon_{t-1}+...+\theta_q\varepsilon_{t-q}-\sum_{i=1}^{r}\beta_i X_{it}\right)\right) \quad (15)$$

where $a_j = \frac{b}{n}\left(j-\frac{1}{2}\right)$ and $\varpi_j = \frac{b}{n}$; $j = 1,2,...,n$.

3. Results

3.1. Numerical Results

In this section, we compare the ARL_0 and ARL_1 obtained from the explicit formulas according to equation (8) for EWMA control chart on MAX(q,r) process with exponential white noise and the ARL evaluated by NIE method using midpoint rule, trapezoidal rule and Gaussian rule on m = 800 subintervals. We set $j(u)$ ARL from explicit formula and $j_{NIE}(u)$ ARL from NIE method. We also compare computational time (CPU time) between 2 methods and absolute percentage relative error (APRE), which defined as

$$(\%)\text{APRE}= \frac{|j(u)-j_{NIE}(u)|}{j(u)}\times 100.$$

In Table 1, the parameters value b for EWMA control chart was selected by setting λ = (0.05, 0.10, 0.20), ARL_0=370 and α_0 = 1 in the case of MAX(2,2) with parameter β_1 = 0.5, β_2 = 0.7, and θ_1 =(0.15, 0.25, 0.30) and θ_2 =(0.10, 0.25,0.30, 0.50), respectively. Note that we set the parameter α_0=1 when process is in-control and the parameter $\alpha_1=\alpha_0(1+\delta)$ when process is out of control where shift size (δ) =0.005, 0.02, 0.04, 0.06, 0.08, 0.1, 0.5, and 1.0. The performance of control chart for detecting change in process is shown in Table 2 to Table 4. In Table 2, the initial ARL_0=370 for MAX(2,1) with parameters $\theta_1 = 0.35$, $\theta_2 = 0.6$ $\beta = 2.0$, λ = 0.05 and b =0.01761086. In Table 3, the initial ARL_0=370 for MAX(2,2) with parameters $\theta_1 = 0.15$, $\theta_2 = 0.25$, $\beta_1 = 0.5$, $\beta_2 = 0.7$, $\lambda = 0.05$ and b = 0.022672201. In Table 4, the initial ARL_0=370 for MAX(3,2) with parameters θ_1 = 0.15, θ_2 = 0.25, θ_3 = 0.45 with β_1 = 0.5, β_2 = 0.7, λ =0.05 and b=0.03579842.

As shown in Table 1, values of ARL_0 from an explicit formula are very close to ARL_0 from NIE method on n =800 subintervals. In the term of APRE were less than 0.0001%. However, the explicit formula takes less than a second to compute, but the NIE method requires 4-5 seconds.

Table 1. Comparison of ARL_0 between using explicit formulas and numerical integration for MAX(2,2) process with parameter $\alpha_0 = 1$ with $\beta_1 = 0.5$, $\beta_2 = 0.7$, for $ARL_0 = 370$

λ	θ_1	θ_2	b	$j(u)$	CPU_{Exp}	$j_{NIE}(u)$	CPU_{NIE}	%APRE
0.05	0.15	0.10	0.019481940	370.5774908	0.014	370.577483	4.765	2.105×10^{-6}
		0.25	0.022672201	370.5144284	0.014	370.514418	4.781	2.807×10^{-6}
		0.30	0.023849150	370.5810878	0.014	370.581076	4.891	3.171×10^{-6}
		0.50	0.029210100	370.4354587	0.014	370.435441	5.250	4.778×10^{-6}
0.10	0.25	0.10	0.043581900	370.3822192	0.014	370.382194	5.250	6.803×10^{-6}
		0.25	0.050820500	370.6270242	0.014	370.626989	4.875	9.487×10^{-6}
		0.30	0.053498300	370.3363528	0.014	370.336328	4.719	6.688×10^{-6}
		0.50	0.065747070	370.2683557	0.014	370.268295	4.953	1.639×10^{-5}
0.20	0.3	0.10	0.093940200	370.2870308	0.014	370.286932	4.875	2.668×10^{-5}
		0.25	0.110015960	370.5272488	0.014	370.527109	4.813	2.105×10^{-5}
		0.30	0.143624000	370.3915332	0.014	370.391282	4.906	2.807×10^{-5}
		0.50	0.143624000	370.3915332	0.014	370.391282	4.828	3.171×10^{-5}

CPU_{Exp} and CPU_{NIE} are computational time (Seconds) from explicit formula and NIE method, respectively.

Table 2. ARL values for MAX(2,1) using numerical integral equation method given $\theta_1 = 0.35$, $\theta_2 = 0.6$ with $\beta = 2.0$, $\lambda = 0.05$, $b = 0.01761086$ for $ARL_0 = 370$

Shift (δ)	Explicit	CPU_{Exp}	NIE					
			Midpoint	CPU_{mid}	Trapezoidal	CPU_{Trap}	Gaussian	CPU_{Gau}
0.000	370.3150	0.014	370.5144	4.781	370.086	4.703	370.315	37.062
0.005	68.8310	0.014	68.8310	4.735	68.789	4.563	68.7890	38.047
0.020	20.6369	0.014	20.6369	4.657	20.6248	4.781	20.6248	38.578
0.040	11.0660	0.014	11.0660	4.735	11.0598	4.906	11.0598	40.781
0.060	7.7585	0.014	7.7585	4.984	7.75435	5.031	7.75435	41.531
0.080	6.0820	0.014	6.0820	4.781	6.07893	5.156	6.07893	41.406
0.100	5.0689	0.014	5.0689	4.985	5.06641	5.031	5.06641	41.719
0.500	1.7953	0.014	1.7953	4.921	1.79483	5.047	1.79483	37.531
1.000	1.3880	0.014	1.3880	4.828	1.38772	5.203	1.38772	42.422

CPU_{Exp}, CPU_{mid}, CPU_{trap} and CPU_{Gau} are computational time (Seconds) from explicit formula and NIE method, respectively

Table 3. ARL values for MAX(2,2) with parameters $\theta_1 = 0.15$, $\theta_2 = 0.25$ with $\beta_1 = 0.5$, $\beta_2 = 0.7$, $\lambda = 0.05$ and $b = 0.022672201$ for $ARL_0 = 370$

Shift (δ)	Explicit	CPU_{Exp}	NIE					
			Midpoint	CPU_{mid}	Trapezoidal	CPU_{Trap}	Gaussian	CPU_{Gau}
0.000	370.5144	0.014	370.5144	4.781	370.486	4.703	370.5144	37.422
0.005	73.8584	0.014	73.8584	4.735	73.8135	4.563	73.8584	37.984
0.020	22.3826	0.014	22.3826	4.657	22.3695	4.781	22.3826	39.515
0.040	12.0144	0.014	12.0145	4.735	12.0078	4.906	12.0145	39.172
0.060	8.4196	0.014	8.4196	4.984	8.4151	5.031	8.4196	39.000
0.080	6.5948	0.014	6.5948	4.781	6.5915	5.156	6.5948	39.828
0.100	5.4910	0.014	5.4910	4.985	5.4882	5.031	5.4910	40.235
0.500	1.9102	0.014	1.9102	4.921	1.9096	5.047	1.9102	39.391
1.000	1.4560	0.014	1.4560	4.828	1.4557	5.125	1.4560	40.594

CPU_{Exp}, CPU_{mid}, CPU_{trap} and CPU_{Gau} are computational time (Seconds) from explicit formula and NIE method, respectively

Table 4. ARL values for MAX(3,2) with parameters $\theta_1 = 0.15$, $\theta_2 = 0.25$, $\theta_3 = 0.45$ with $\beta_1 = 0.5$, $\beta_2 = 0.7$, λ =0.05 and b =0.03579842. for ARL_0= 370

shift (δ)	Explicit	CPU_{Exp}	NIE					
			Midpoint	CPU_{mid}	Trapezoidal	CPU_{Trap}	Gaussian	CPU_{Gau}
0.000	370.5845	0.014	370.5845	4.781	370.3578	5.094	370.5844	37.36
0.005	85.2404	0.014	85.2404	4.907	85.1888	5.172	85.2404	37.563
0.020	26.4916	0.014	26.4916	4.781	26.4761	5.016	26.4916	38.922
0.040	14.2630	0.014	14.2629	4.718	14.2549	5.312	14.2629	38.922
0.060	9.9900	0.014	9.9899	4.829	9.9845	5.719	9.98991	39.86
0.080	7.8138	0.014	7.8138	4.718	7.80973	5.515	7.8138	37.265
0.100	6.4948	0.014	6.4948	4.687	6.49149	5.407	6.49476	37.157
0.500	2.1853	0.014	2.1853	4.735	2.18458	5.437	2.18525	37.578
1.000	1.6209	0.014	1.6209	4.703	1.4557	5.125	1.4560	40.594

CPU_{Exp}, $CPU_{mid,}$ CPU_{trap} and CPU_{Gau} are computational time (Seconds) from explicit formula and NIE method, respectively

Tables 2–4 illustrate that ARL_0 and ARL_1 from the explicit formula are extremely similar to the NIE method on n=800 subintervals. The performance of explicit formula and NIE methods for monitoring various process changes is very equivalent, as seen in Fig. 1 to Fig. 3. However, explicit formula consume significantly less CPU time than the NIE method. Additionally, the numerical method based on the midpoint rule consumes less CPU time than the trapezoidal and Gaussian rules, as seen in Fig. 4 to Fig. 6, respectively.

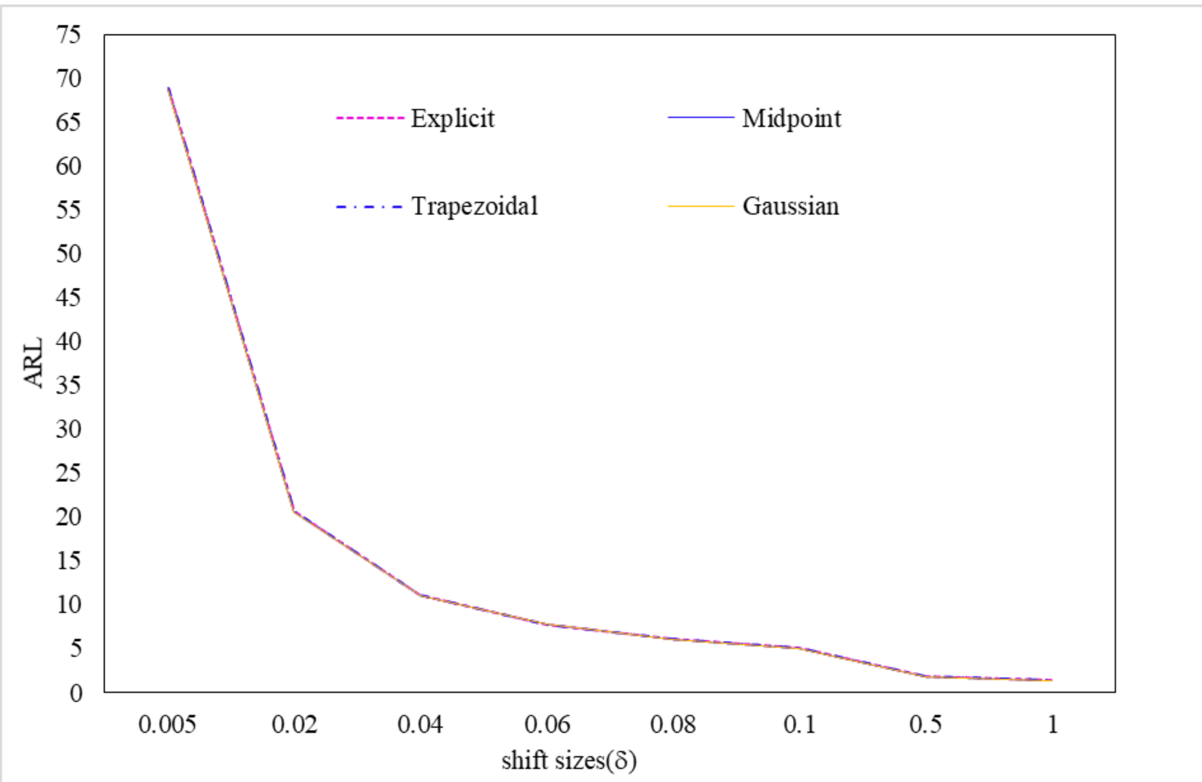

Figure 1. Comparison of ARL_1 of EWMA control chart using explicit formula and NIE methods for the MAX(2,1) with $ARL_0 = 370$

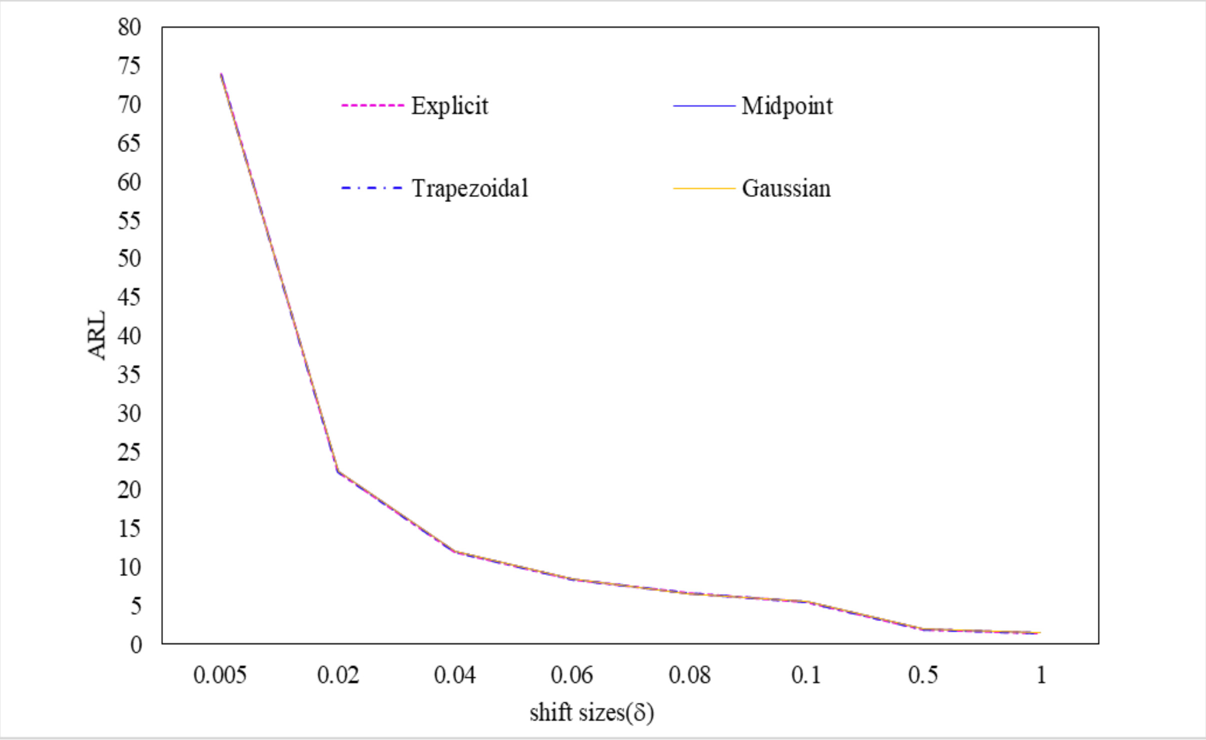

Figure 2. Comparison of ARL_1 of EWMA control chart using explicit formula and NIE methods for the MAX(2,2) with $ARL_0 = 370$

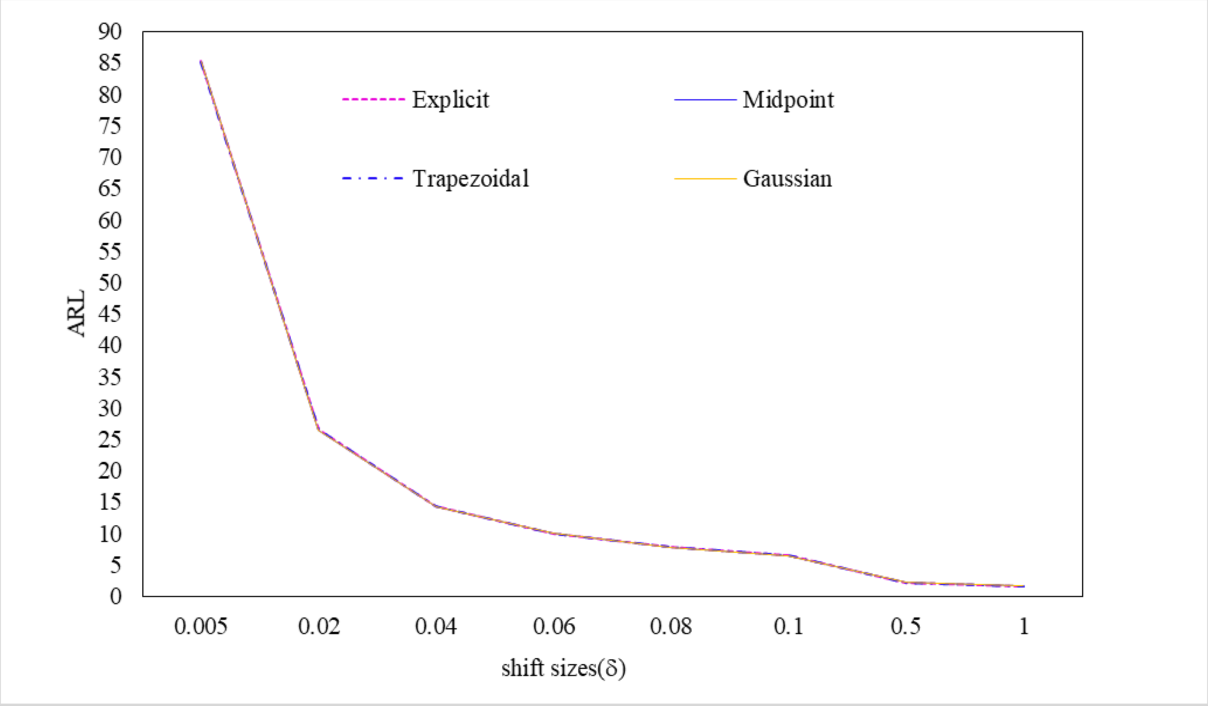

Figure 3. Comparison of ARL_1 of EWMA control chart using explicit formula and NIE methods for the MAX(3,2) with $ARL_0 = 370$

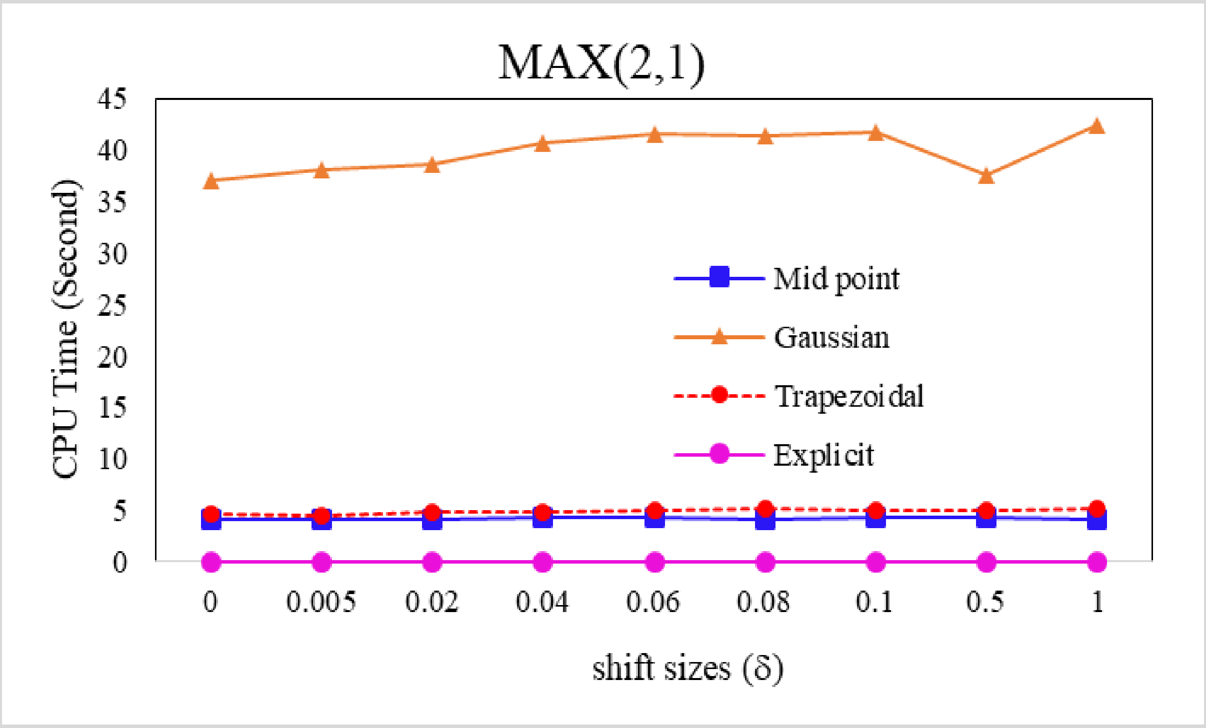

Figure 4. The CPU times for evaluating ARL values of EWMA control chart for MAX(2,1)

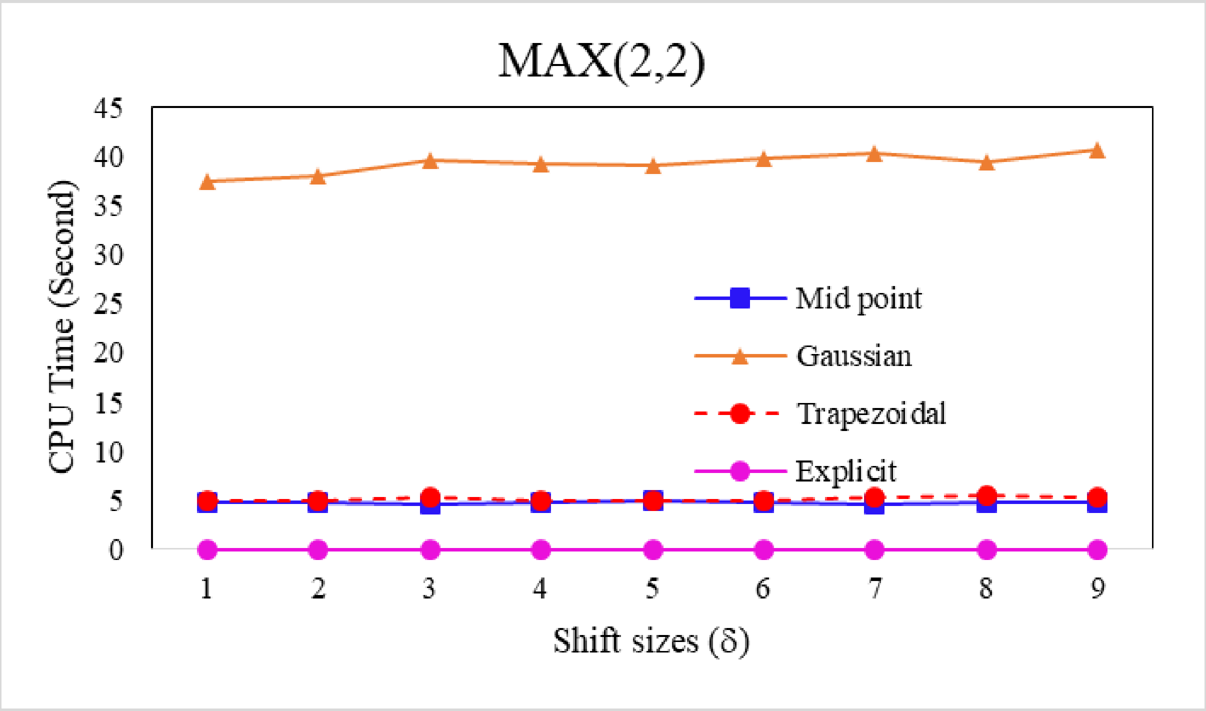

Figure 5. The CPU times for evaluating ARL values of EWMA control chart for MAX(2,2)

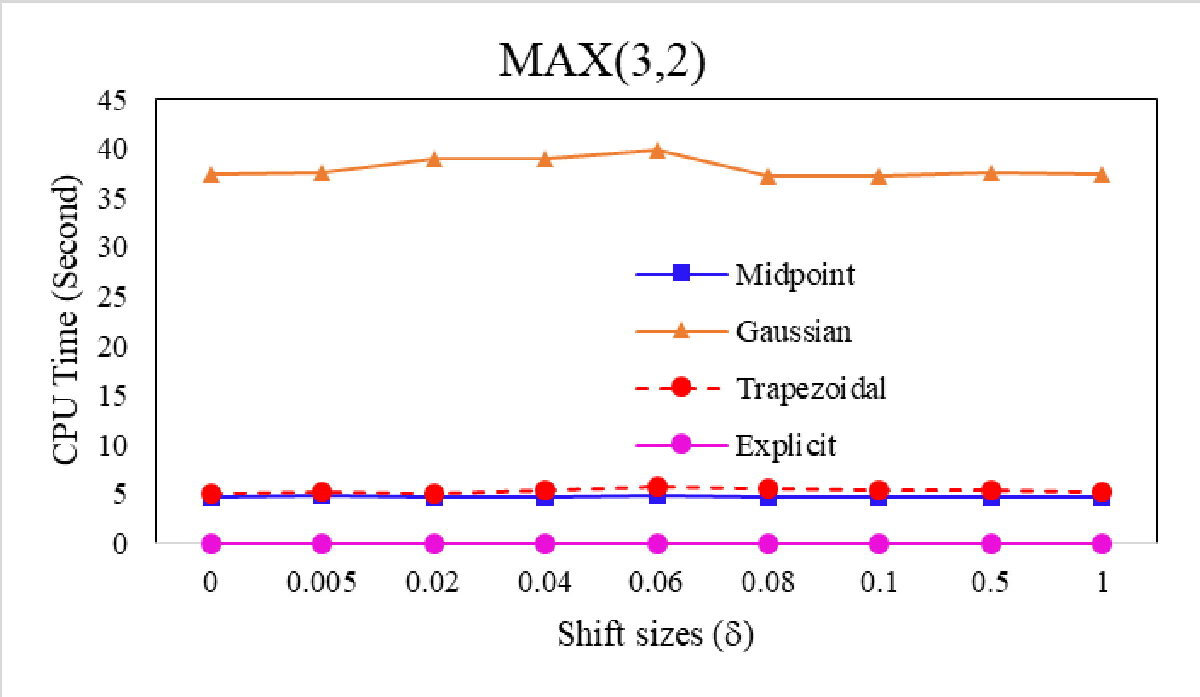

Figure 6. The CPU times for evaluating ARL values of EWMA control chart for MAX(3,2)

4. Conclusions

This research proposed the explicit formula for the average run length of the EWMA control chart for the MAX(q,r) process with exponential white noise. Further, the existence and uniqueness of the explicit ARL have been proven. The results found that the ARL values from the proposed explicit formula are incredibly similar to the numerical method. The proposed explicit formula has the benefit of being simple to calculate and, more significantly, requiring that much less computational time than the NIE method. The computational times for computing explicit formula take less than 1 second as well the numerical integration method takes no more than 11 seconds in the case of MAX(q,r). Additionally, the numerical method based on the midpoint rule consumes less CPU time than the trapezoidal and Gaussian rules

REFERENCES

[1] S. W. Robert. Control chart tests based on geometric moving average, Technometrics, 411-430, 1959.

[2] Y. Areepong, A. Novikov Martingale approach to EWMA control charts for changes in exponential distribution. Journal of Quality Measurement and Analysis. Vol.4, No.1, 197-203, 2008.

[3] S. Supaksree. Martingale approach for closed-form expression of average run length on EWMA chart for detection of a change in number of defects, Far East Journal of Mathematical Sciences, Vol.92, No.2, 141-156, 2014.

[4] D. Brook, D. A. Evans. An approach to the probability distribution of CUSUM run lengths, Biometrika, Vol.59, 539-548, 1972.

[5] J. M. Lucas, and M.S. Saccucci. Exponentially weighted moving average control schemes: properties and enhancements, Technometrics, Vol.32, No.1, 1-29, 1990.

[6] S. Ross. Average Run Lengths for moving average control charts. Probability in the Engineering and Informational Sciences, Vol.13, No.2, 209-220, 1999.

[7] M. S. Srivastava, Y. Wu. Evaluation of optimum weights and average run lengths in EWMA control schemes, Communications in Statistics-Theory and Methods, Vol.26, No.5, 1253-1267, 1997.

[8] Y. Areepong. An Integral Equation Approach for Analysis of Control Charts, PhD Thesis, University of Technology, Australia, 2009.

[9] G. Mititelu, Y. Areepong, S. Sukparungsee, A. A. Novikov. Explicit analytical solutions for the average run length of CUSUM and EWMA charts, East-West J Math. Special 1, 253-265, 2010.

[10] S. Phanyaem. Average run length of cumulative sum control charts for $SARMA(1,1)_L$ models, Thailand Statistician, Vol.15, No.2, 184-195, 2017.

[11] P. Paichit. Exact expression for average run length of control chart of ARX(p) procedure, KKU Science Journal. Vol.45, No.4, 948-958, 2017.

[12] W. Peerajit, Y. Areepong, S. Sukparungsee. Numerical integral equation method for ARL of CUSUM chart for long-memory process with non-seasonal and seasonal ARFIMA models, Thailand Statistician. Vol.16, No.1,

26-37, 2018.

[13] R. A. Johnson, M. Bagshaw. The effect of serial correlation on the performance of CUSUM tests, Technometrics. Vol.16, No.1, 103-112, 1974.

[14] C. W. Lu, M.R. Reynolds. EWMA control charts for monitoring the mean of autocorrelated processes, Journal of Quality Technology. Vol.31, 166-188, 1999.

[15] W. Suriyakat, Y. Areepong, S. Sukparungsee, Mititelu, G. On EWMA procedure for AR(1) observations with exponential white noise, International Journal of Pure and Applied Mathematics, Vol.77, 73–83, 2012.

[16] W. Suriyakat, Y. Areepong, S. Sukparungsee and G. Mititelu, Analytical method of average run length for trend exponential AR(1) processes in EWMA procedure, IAENG International Journal of Applied Mathematics, Vol.42, 250-253, 2012.

[17] K. Petcharat, Y, Areepong, S. Sukparungsee, G. Mititelu. Exact solution of average run length of EWMA chart for MA(q) processes, Far East Journal of Mathematical Sciences. Vol.78, No.2, 291-300, 2013.

[18] P. Busababodin. An analytical expression to CUSUM chart for seasonal AR(p) model, Far East Journal of Mathematical Sciences. Vol.88, No.1, 89-105, 2014.

[19] K. Petcharat, S. Sukparungsee, Y. Areepong. Exact solution of the average run length for the cumulative sum chart for a moving average process of order q, ScienceAsia. Vol. 41, 141-147, 2015.

[20] K. Petcharat. Explicit formula of ARL for $SMA(Q)_L$ with exponential white noise on EWMA chart, International Journal of Applied Physics and Mathematics. Vol.6, No.4, 218-255, 2016.

[21] W.Suriyakat. On sensitivity of control chart for monitoring serially correlated data, Interdisciplinary Research Review, Vol.15, No.3, 44–47, 2020.

[22] S. Sukparungsee, Y. Areepong. An explicit analytical solution of the average run length of an exponentially weighted moving average control chart using an autoregressive mode, Chiang Mai Journal of Science. Vol.44, No.3, 1172-1179, 2017.

[23] W. Suriyakat, Performance of EWMA chart for trend autoregressive model with exponential white noise, Silpakorn University Science and Technology Journal. Vol.11, No.2, 16-22, 2017.

[24] R. Sunthornwat, and Y. Areepong, Average run length on CUSUM control chart for seasonal and non-seasonal moving average processes with exogenous variables, Symmetry.Vol.12, No.1, 2020. Online available from: https://doi.org/ 10.3390/ sym12010173.

[25] K. Petcharat, The Effectiveness of CUSUM control chart for trend stationary seasonal autocorrelated data. Thailand Statistician. Vol.20, No.2, 575-488, 2022.

[26] K. Petcharat, The performance of EWMA control chart for MAX(1,r) process, In Proceedings of the International MultiConference of Engineers and Computer Scientists, 20-22, 111-115, 2021.