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On Generalized Bent and Negabent Functions

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Abstract From the last few years, generalized bent functions gain a lot of attention in research as they have many applications in various fields such as combinatorial design, sequence design theory, cryptography, CDMA communication, etc. A deep and broad study of generalized bent functions with their properties is done in literature. Kumar et al.[11] first gave the concept of generalized bent function. Many researchers studied the properties and characterizations of generalized bent functions. In [2] authors introduced the concept of generalized (q-ary) negabent functions and studied some properties of generalized (q-ary) negabent functions. In this paper, we study the generalized (q-ary) bent functions $f : \mathbb{Z}_q^n \rightarrow \mathbb{Z}_q$, where $\mathbb{Z}_q$ is the ring of integers with mod q, $\mathbb{Z}_q^n$ is the vector space of dimension n over $\mathbb{Z}_q$ and $q \geq 2$ is any positive integer. We discuss several properties of generalized (q-ary) bent functions with respect to their nega-Hadamard transform. We also study the relation between generalized nega-Hadamard transforms and generalized nega-autocorrelations. Furthermore, we prove the necessary and sufficient conditions for the bentness and negabentness of generalized (q-ary) bent function generated by the secondary construction for $\mathbb{Z}_q^n$, where $q = 2^s$.

Keywords Generalized Bent Function, Generalized Walsh-Hadamard Transform, Generalized Nega-Hadamard Transform

1 Introduction

Day-by-day data security becomes very significant for the society. Cryptography is the art of writing secure data. The term cryptography originated from the Greek words *"Kryptos"* which means hidden and *"Graphein"* which means writing. Hence the word cryptography means "secret writing". In [15] Tayal et al. investigated that with the advent of World Wide Web, social networks and commerce applications, organizations across the world produce a huge amount of data daily. Due to this, information security is an excessive issue in terms of transmitting data safely through the web. In this paper the authors provide an overview on the various techniques to enhance network security. Generalized bent functions have vast applications in cryptography, coding theory and sequence design. They are used for signal design with good correlation for wireless communication([13, 12, 18]). The concept of generalized bent function was first introduced by Kumar et al.[11]. Later many researchers studied the generalized bent functions due to their importance in cryptography([3, 4, 7, 10]). Hodžić et al.[14] studied the generalized bent functions and provided the necessary and sufficient conditions for generalized bent functions to represent them as a linear combination of generalized bent and Boolean bent functions. In [8] and [9] Stănică et al. introduced several properties of nega-Hadamard transform of Boolean functions. Chaturvedi and Gangopadhyay [1] introduced some properties of generalized nega-Hadamard transform of generalized Boolean function defined from $\mathbb{Z}_2^n$ to $\mathbb{Z}_q$. Recently, Paul et al.[2] propose a generalization of negabent functions to construct generalized negabent functions over $\mathbb{Z}_q$. They raised an open problem for constructing generalized (q-ary) negabent functions in general. In this paper, we partially solve the open problem raised by Paul et al. in [2]. We study several properties of generalized (q-ary) Boolean functions defined from $\mathbb{Z}_q^n$ to $\mathbb{Z}_q$ with respect to nega-Hadamard transform. Also, we prove the necessary and sufficient conditions for the bentness and negabentness of generalized (q-ary) Boolean function on $(n + 1)$ variables generated by secondary construction from the functions on n variables for $\mathbb{Z}_q^n$, where $q = 2^s$.

1.1 Definitions and Notations

Let $\mathbb{Z}_q$ be the ring of integers with mod q and $\mathbb{Z}_q^n$ be the vector space of dimension n over $\mathbb{Z}_q$. Any function from $\mathbb{Z}_q^n$ to $\mathbb{Z}_q$ ($q \geq 2$, a positive integer) is known as a *generalized Boolean function* on n variables. The set of all generalized Boolean functions is denoted by $\mathcal{GB}_n^q$ and the cardinality of $\mathcal{GB}_n^q$ is q^{q^n}. For $q = 2$, we obtain the classical Boolean function.

Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{Z}_q^n$, then the *Hamming distance*[6] between these two vectors x and y, denoted by $dist(x, y)$, is the number of places where they differ. The *Hamming weight*[6] of a vector $x = (x_1, x_2, ..., x_n)$, denoted by $wt(x)$, is the number of non-zero x_i.

The *Walsh-Hadamard transform* of a generalized Boolean function[5] $f \in \mathcal{GB}_n^q$ at any point $\lambda \in \mathbb{Z}_q^n$ is the complex valued function defined by

$$\mathcal{W}_f(\lambda) = q^{-\frac{n}{2}} \sum_{x \in \mathbb{Z}_q^n} \zeta^{f(x)+<\lambda,x>},$$

where ζ is the q-th primitive root of unity. The function $f \in \mathcal{GB}_n^q$ is said to be q-ary bent function if $|\mathcal{W}_f(\lambda)| = 1$, for all $\lambda \in \mathbb{Z}_q^n$. That is, if Walsh-Hadamard spectrum of f is flat.

The *nega-Hadamard transform* of a generalized Boolean function[2] $f \in \mathcal{GB}_n^q$ at any point $\lambda \in \mathbb{Z}_q^n$ is defined by

$$\mathcal{N}_f(\lambda) = q^{-\frac{n}{2}} \sum_{x \in \mathbb{Z}_q^n} \zeta^{f(x)+<\lambda,x>} \omega^{\sum x_i},$$

where $\zeta = e^{\frac{2\pi i}{q}}$ is the q-th primitive root of unity and ω is the $(2q)$-th primitive root of unity.

The *nega crosscorrelation*[2] of two q-ary functions $f, g \in \mathcal{GB}_n^q$ at any point $\lambda \in \mathbb{Z}_q^n$ is defined by

$$\mathcal{C}_{f,g}(\lambda) = \sum_{x \in \mathbb{Z}_q^n} \zeta^{f(x)+g(x+\lambda)} (-1)^{n_q(x,\lambda)},$$

where $n_q(x, \lambda) = \sum_{i=1}^{n} \lfloor \frac{x_i+\lambda_i}{q} \rfloor = |\{i : x_i + \lambda_i \geq q\}|$.

The identity $\sum_{i=1}^{n} \lfloor \frac{x_i+\lambda_i}{q} \rfloor = |\{i : x_i + \lambda_i \geq q\}|$ holds true as $x_i + \lambda_i < 2q$, for $x_i,\ \lambda_i \in \mathbb{Z}_q^n$. If $f = g$, then the sum $\mathcal{C}_{f,f}(\lambda) = \mathcal{C}_f(\lambda)$ is known as *nega autocorrelation* of $f \in \mathcal{GB}_n^q$ at $\lambda \in \mathbb{Z}_q^n$. This whole approach conflicts with the classical Boolean case when $q = 2$.

1.2 Preliminary results

We recall the following results.

Lemma 1. *[16] Given any positive integer n. If $v \in \mathbb{Z}_2^n$, then*

$$\sum_{x \in \mathbb{Z}_2^n} (-1)^{v \cdot x} = \begin{cases} 2^n, & if\ v = 0 \\ 0, & otherwise. \end{cases}$$

Lemma 2. *[11] Given any positive integer n. If $u \in \mathbb{Z}_q^n$, then*

$$\sum_{x \in \mathbb{Z}_q^n} \zeta^{<u,x>} = \begin{cases} q^n, & if\ u = 0 \\ 0, & otherwise. \end{cases}$$

where $n_q(x, y) = \sum_{i=1}^{n} \lfloor \frac{x_i+y_i}{q} \rfloor = |\{i : x_i + y_i \geq q\}|$.

2 Properties of q-ary Functions in terms of Nega-Hadamard Transform

Here we provide some properties of generalized Boolean functions (q-ary functions) in terms of nega-Hadamard transforms.

Lemma 3. *[2] Let $x, y, z \in \mathbb{Z}_q^n$ and $z = x + y$. Then*

$$\sum z_i = \sum x_i + \sum y_i - qn_q(x, y),$$

where $n_q(x, y) = \sum_{i=1}^{n} \lfloor \frac{x_i+y_i}{q} \rfloor = |\{i : x_i + y_i \geq q\}|$.

Theorem 4. *Let f, g and h are q-ary functions. Then the following statements are true:-*

(i) For any affine function, $k_{u,v}(x) =< u, x > +v$,

$$\mathcal{N}_{f+k_{u,v}}(\lambda) = \zeta^v \mathcal{N}_f(u + \lambda).$$

(ii) If $l_{u,v}(x) = \frac{q}{2} < u, x > +v$ is any affine function then

$$\sum_{\lambda \in \mathbb{Z}_q^n} \mathcal{N}_{l_{u,v}}(\lambda) \zeta^{<-\lambda,x>} = q^{\frac{n}{2}} \zeta^v \sum_{x \in \mathbb{Z}_q^n} (-1)^{u \cdot x} \omega^{\sum x_i}.$$

(iii) If $g(x, y) = f_1(x) + f_2(y)$, for $x, y \in \mathbb{Z}_q^n$, then

$$\mathcal{N}_g(\lambda, \gamma) = \mathcal{N}_{f_1}(\lambda) \mathcal{N}_{f_2}(\gamma).$$

(iv) If $g(x) = f_1(x) + f_2(x)$ on $\mathbb{Z}_q^n$, then for $\gamma \in \mathbb{Z}_q^n$

$$\mathcal{N}_g(\gamma) = q^{-\frac{n}{2}} \sum_{\lambda \in \mathbb{Z}_q^n} \mathcal{N}_{f_1}(\lambda) \mathcal{W}_{f_2}(\gamma - \lambda)$$
$$= q^{-\frac{n}{2}} \sum_{\lambda \in \mathbb{Z}_q^n} \mathcal{W}_{f_1}(\lambda) \mathcal{N}_{f_2}(\gamma - \lambda).$$

Proof. (i) Let $k_{u,v}(x) =< u, x > +v$. Then as defined on generalized nega-Hadamard transform, we have

$$\begin{aligned} \mathcal{N}_{f+k_{u,v}}(\lambda) &= q^{-\frac{n}{2}} \sum_{x \in \mathbb{Z}_q^n} \zeta^{f+k_{u,v}(x)+<\lambda,x>} \omega^{\sum x_i} \\ &= q^{-\frac{n}{2}} \sum_{x \in \mathbb{Z}_q^n} \zeta^{f(x)+<u,x>+v+<\lambda,x>} \omega^{\sum x_i} \\ &= q^{-\frac{n}{2}} \zeta^v \sum_{x \in \mathbb{Z}_q^n} \zeta^{f(x)+<u+\lambda,x>} \omega^{\sum x_i} \\ &= \zeta^v \mathcal{N}_f(u + \lambda). \end{aligned}$$

(ii) Let $l_{u,v}(x) = (\frac{q}{2}) < u, x > +v$. Then we have

$$\begin{aligned} &\sum_{\lambda \in \mathbb{Z}_q^n} \mathcal{N}_{l_{u,v}}(\lambda) \zeta^{<-\lambda,x>} \\ &= q^{-\frac{n}{2}} \sum_{\lambda \in \mathbb{Z}_q^n} \sum_{y \in \mathbb{Z}_q^n} \zeta^{l_{u,v}(y)+<\lambda,y>} \omega^{\sum y_i} \zeta^{<-\lambda,x>} \\ &= q^{-\frac{n}{2}} \sum_{\lambda \in \mathbb{Z}_q^n} \sum_{y \in \mathbb{Z}_q^n} \zeta^{\frac{q}{2}<u,y>+v+<\lambda,y>+<-\lambda,x>} \omega^{\sum y_i} \\ &= q^{-\frac{n}{2}} \zeta^v \sum_{y \in \mathbb{Z}_q^n} \zeta^{\frac{q}{2}<u,y>} \omega^{\sum y_i} \sum_{\lambda \in \mathbb{Z}_q^n} \zeta^{<\lambda,-x+y>} \\ &= q^{\frac{n}{2}} \zeta^v \sum_{x \in \mathbb{Z}_q^n} (-1)^{<u,x>} \omega^{\sum x_i}. \qquad (using\ Lemma\ 2) \end{aligned}$$

(iii) Let $g(x,y) = f_1(x) + f_2(y)$, for all $x, y \in \mathbb{Z}_q^n$. Then we have

$$\begin{aligned}
&\mathcal{N}_g(\lambda,\gamma)\\
&= q^{-n}\sum_{x,y\in\mathbb{Z}_q^n}\zeta^{f_1(x)+f_2(y)+<\lambda,x>+<\gamma,y>}\omega^{\sum x_i+\sum y_i}\\
&= q^{-\frac{n}{2}}\sum_{x\in\mathbb{Z}_q^n}\zeta^{f_1(x)+<\lambda,x>}\omega^{\sum x_i}q^{-\frac{n}{2}}\sum_{y\in\mathbb{Z}_q^n}\zeta^{f_2(y)+<\gamma,y>}\omega^{\sum y_i}\\
&= \mathcal{N}_{f_1(\lambda)}N_{f_2(\gamma)}.
\end{aligned}$$

(iv) Let $g(x) = f_1(x) + f_2(x)$, for $x \in \mathbb{Z}_q^n$. Then we have

$$\mathcal{N}_{f_1}(\lambda) = q^{-\frac{n}{2}}\sum_{x\in\mathbb{Z}_q^n}\zeta^{f_1(x)+<\lambda,x>}\omega^{\sum x_i}$$

and

$$\mathcal{W}_{f_2}(\gamma-\lambda) = q^{-\frac{n}{2}}\sum_{y\in\mathbb{Z}_q^n}\zeta^{f_2(y)+<\gamma-\lambda,y>}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}_q^n}\mathcal{N}_{f_1}(\lambda)\mathcal{W}_{f_2}(\gamma-\lambda)\\
&= q^{-n}\sum_{\lambda,x,y\in\mathbb{Z}_q^n}\zeta^{f_1(x)+f_2(y)+<\lambda,x>+<\gamma-\lambda,y>}\omega^{\sum x_i}\\
&= q^{-n}\sum_{x,y\in\mathbb{Z}_q^n}\zeta^{f_1(x)f_2(y)+<\gamma,y>}\omega^{\sum x_i}\sum_{\lambda\in\mathbb{Z}_q^n}\zeta^{<\lambda,x-y>}\\
&= \sum_{x\in\mathbb{Z}_q^n}\zeta^{f_1(x)+f_2(x)+<\gamma,x>}\omega^{\sum x_i} \qquad (using\ Lemma\ 2)\\
&= \sum_{x\in\mathbb{Z}_q^n}\zeta^{g(x)+<\gamma,x>}\omega^{\sum x_i}\\
&= q^{\frac{n}{2}}\mathcal{N}_g(\gamma).
\end{aligned}$$

Again,

$$\mathcal{W}_{f_1}(\lambda) = q^{-\frac{n}{2}}\sum_{x\in\mathbb{Z}_q^n}\zeta^{f_1(x)+<\lambda,x>}$$

and

$$\mathcal{N}_{f_2}(\gamma-\lambda) = q^{-\frac{n}{2}}\sum_{y\in\mathbb{Z}_q^n}\zeta^{f_2(y)+<\gamma-\lambda,y>}\omega^{\sum y_i}.$$

Therefore, we have

$$\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}_q^n}\mathcal{W}_{f_1}(\lambda)\mathcal{N}_{f_2}(\gamma-\lambda)\\
&= q^{-n}\sum_{\lambda,x,y\in\mathbb{Z}_q^n}\zeta^{f_1(x)+f_2(y)+<\lambda,x>+<\gamma-\lambda,y>}\omega^{\sum y_i}\\
&= q^{-n}\sum_{x,y\in\mathbb{Z}_q^n}\zeta^{f_1(x)f_2(y)+<\gamma,y>}\omega^{\sum y_i}\sum_{\lambda\in\mathbb{Z}_q^n}\zeta^{<\lambda,x-y>}\\
&= \sum_{x\in\mathbb{Z}_q^n}\zeta^{f_1(x)+f_2(x)+<\gamma,x>}\omega^{\sum x_i} \qquad (using\ Lemma\ 2)\\
&= \sum_{x\in\mathbb{Z}_q^n}\zeta^{g(x)+<\gamma,x>}\omega^{\sum x_i}\\
&= q^{\frac{n}{2}}\mathcal{N}_g(\gamma).
\end{aligned}$$

□

Lemma 5. *Let $f \in \mathcal{GB}_n^q$ and $\lambda \in \mathbb{Z}_q^n$. then*

$$|\mathcal{N}_f(\lambda)|^2 = q^{-n}\sum_{z\in\mathbb{Z}_q^n}C_f(z)\omega^{-\sum z_i}\zeta^{<-\gamma,z>}.$$

Proof.

$$\begin{aligned}
&|\mathcal{N}_f(\lambda)|^2\\
&= \mathcal{N}_f(\lambda)\overline{\mathcal{N}_f(\lambda)}\\
&= q^{-n}\sum_{x\in\mathbb{Z}_q^n}\zeta^{f(x)+<\lambda,x>}\omega^{\sum x_i}\sum_{y\in\mathbb{Z}_q^n}\zeta^{-f(y)-<\lambda,y>}\omega^{-\sum y_i}\\
&= q^{-n}\sum_{x,y\in\mathbb{Z}_q^n}\zeta^{f(x)-f(y)+<\lambda,x>-<\lambda,y>}\omega^{\sum x_i-\sum y_i}\\
&= q^{-n}\sum_{x,z\in\mathbb{Z}_q^n}\big(\zeta^{f(x)-f(x+z)+<\lambda,x>-<\lambda,x+z>}\\
&\quad \omega^{-\sum z_i+qn_q(x,z)}\big) \qquad (using\ Lemma\ 3)\\
&= q^{-n}\sum_{x,z\in\mathbb{Z}_q^n}\big(\zeta^{f(x)-f(x+z)}\zeta^{<\lambda,x>-<\lambda,x+z>}\\
&\quad (-1)^{n_q(x,z)}\omega^{-\sum z_i}\big)\\
&= q^{-n}\sum_{z\in\mathbb{Z}_q^n}\zeta^{<-\lambda,z>}\mathcal{C}_f(z)\,\omega^{-\sum z_i}.
\end{aligned}$$

□

Theorem 6. *Let f and g are q-ary functions and $\lambda, \gamma \in \mathbb{Z}_q^n$. Then*

$$\begin{aligned}
&q^n\sum_{\lambda\in\mathbb{Z}_q^n}|\mathcal{N}_f(\lambda)|^2|\mathcal{N}_g(\gamma-\lambda)|^2\\
&\quad= \sum_{a\in\mathbb{Z}_q^n}\mathcal{C}_f(a)\mathcal{C}_g(a)\omega^{-2\sum a_i}\zeta^{<-\gamma,a>}
\end{aligned}$$

Proof. From Lemma 5, we have

$$\begin{aligned}
&\sum_{\lambda\in\mathbb{Z}_q^n}|\mathcal{N}_f(\lambda)|^2|\mathcal{N}_g(\gamma-\lambda)|^2\\
&= q^{-2n}\sum_{\lambda\in\mathbb{Z}_q^n}\sum_{a\in\mathbb{Z}_q^n}\big(\mathcal{C}_f(a)\omega^{-\sum a_i}\zeta^{<-\lambda,a>}\\
&\quad \sum_{b\in\mathbb{Z}_q^n}\mathcal{C}_g(b)\omega^{-\sum b_i}\zeta^{<\lambda-\gamma,b>}\big)\\
&= q^{-2n}\sum_{\lambda,a,b\in\mathbb{Z}_q^n}\mathcal{C}_f(a)\mathcal{C}_g(b)\omega^{-\sum a_i-\sum b_i}\zeta^{<-\lambda,a>+<\lambda-\gamma,b>}\\
&= q^{-2n}\sum_{a,b\in\mathbb{Z}_q^n}\mathcal{C}_f(a)\mathcal{C}_g(b)\omega^{-\sum a_i-\sum b_i}\zeta^{<-\gamma,b>}\sum_{\lambda\in\mathbb{Z}_q^n}\zeta^{<\lambda,-a+b>}\\
&= q^{-n}\sum_{a\in\mathbb{Z}_q^n}\mathcal{C}_f(a)\mathcal{C}_g(a)\omega^{-2\sum a_i}\zeta^{<-\gamma,a>}.
\end{aligned}$$

In particular, if $f = g$ then we have

$$q^n\sum_{\lambda\in\mathbb{Z}_q^n}|\mathcal{N}_f(\lambda)|^2|\mathcal{N}_f(\gamma-\lambda)|^2 = \sum_{a\in\mathbb{Z}_q^n}(\mathcal{C}_f(a))^2\omega^{-2\sum a_i}\zeta^{<-\gamma,a>}.$$

□

3 Secondary construction of generalized bent functions on $(n+1)$ variables from the functions on n variables on $\mathbb{Z}_{2^s}^n$

In this section, we proposed a necessary and sufficient condition for the bentness of a generalized Boolean function.

Theorem 7. *Let $h \in \mathcal{GB}_{n+1}^{2^s}$ be expressed as*

$$h(\mathbf{x}, y) = f(\mathbf{x})(1+y) + g(\mathbf{x})y,$$

for all $\mathbf{x}, y \in \mathbb{Z}_{2^s}^n \times \mathbb{Z}_{2^s}$ and $f, g \in \mathcal{GB}_n^{2^s}$. Then h is generalized bent if and only if the following conditions are satisfied.

(i) $|\sum_{r=0}^{2^s-1} \mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s}$, *for all* $\mathbf{v} \in \mathbb{Z}_{2^s}^n$.

(ii) $\frac{\sum_{r=0}^{2^s-1} \cos(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})}{\sum_{r=0}^{2^s-1} \sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})} = \psi(\mathbf{v})$, *for $a = 1$ to $a = 2^{s-1} - 1$,*
$\frac{\sum_{r=0}^{2^{s-1}-1} \mathcal{W}_{h_{2r}}(\mathbf{v})}{\sum_{r=0}^{2^{s-1}-1} \mathcal{W}_{h_{2r+1}}(\mathbf{v})} = i\rho(\mathbf{v})$, *where $\psi(\mathbf{v}), \rho(\mathbf{v}) \in \mathbb{R}$.*

(iii) $\sum_{r=0}^{2^s-1} |\mathcal{W}_{h_r}(\mathbf{v})|^2 + \sum_{m=0}^{2^s-5}\sum_{n=0}^{k} \mathcal{W}_{h_m}(\mathbf{v})\overline{\mathcal{W}_{h_{4n+m+4}}(\mathbf{v})} + \sum_{m=0}^{2^s-5}\sum_{n=0}^{k} \overline{\mathcal{W}_{h_m}(\mathbf{v})}\mathcal{W}_{h_{4n+m+4}}(\mathbf{v}) = 2^s$, *where k is chosen such that $(4n+m+4) < 2^s$, and*
$\sum_{m=0}^{2^s-3}\sum_{n=0}^{j} \mathcal{W}_{h_m}(\mathbf{v})\overline{\mathcal{W}_{h_{4n+m+2}}(\mathbf{v})} + \sum_{m=0}^{2^s-3}\sum_{n=0}^{j} \overline{\mathcal{W}_{h_m}(\mathbf{v})}\mathcal{W}_{h_{4n+m+2}}(\mathbf{v}) = 0$, *where j is chosen such that $(4n+m+2) < 2^s$.*

(iv) $|\sum_{r=0}^{2^s-1} \cos(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2 + |\sum_{r=0}^{2^s-1} \sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2 = 2^s$, *for $a = 1$ to $a = 2^{s-1} - 1$.*
Here $h_r(\mathbf{v}) = h(\mathbf{v}, r) = f(\mathbf{v})(1+r) + g(\mathbf{v})r$, for all $\mathbf{v} \in \mathbb{Z}_{2^s}^n, r \in \mathbb{Z}_{2^s}$.

Proof. Let

$$h(\mathbf{x}, y) = f(\mathbf{x})(1+y) + g(\mathbf{x})y$$

be a generalized bent function, for all $\mathbf{x}, y \in \mathbb{Z}_{2^s}^n \times \mathbb{Z}_{2^s}$. Then the generalized Walsh-Hadamard transform of h at $(\mathbf{v}, a) \in \mathbb{Z}_{2^s}^n \times \mathbb{Z}_{2^s}$ is given by

$$\begin{aligned}\mathcal{W}_h(\mathbf{v}, a) &= \frac{1}{(2^s)^{\frac{n+1}{2}}} \sum_{(\mathbf{x},y)\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}} \zeta^{h(\mathbf{x},y)+a\cdot y+<\mathbf{v},\mathbf{x}>} \\ &= \frac{1}{(2^s)^{\frac{n+1}{2}}} \sum_{r=0}^{2^s-1}\sum_{\mathbf{x}\in\mathbb{Z}_{2^s}^n} \zeta^{h_r(\mathbf{x})+a\cdot r+<\mathbf{v},\mathbf{x}>} \\ &= \frac{1}{\sqrt{2^s}} \sum_{r=0}^{2^s-1} \zeta^{a\cdot r}\mathcal{W}_{h_r}(\mathbf{v}).\end{aligned}$$

Since h is generalized bent function, it follows that $|\mathcal{W}_h(\mathbf{v}, a)| = 1$, for all $(\mathbf{v}, a) \in \mathbb{Z}_{2^s}^n \times \mathbb{Z}_{2^s}$. This implies that,

$$|\sum_{r=0}^{2^s-1} \zeta^{a\cdot r}\mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s}$$

or

$$|\sum_{r=0}^{2^s-1} \Big(\cos(\frac{ar\pi}{2^{s-1}}) + i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s},$$

where $a \in \mathbb{Z}_{2^s}$.This implies that,

$$|\sum_{r=0}^{2^s-1} \mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s}, \tag{1}$$

$$|\sum_{r=0}^{2^s-1} \Big(\cos(\frac{ar\pi}{2^{s-1}}) + i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s}, \tag{2}$$

where $a = 1$ to $a = 2^{s-1} - 1$,

$$|\sum_{r=0}^{2^s-1} (-1)^r \mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s}, \tag{3}$$

$$|\sum_{r=0}^{2^s-1} \Big(\cos(\frac{ar\pi}{2^{s-1}}) + i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s}, \tag{4}$$

where $a = 2^{s-1} + 1$ to $a = 2^s - 1$.
Since cosine and sine are periodic functions, it follows that equation (4) can also be written as

$$|\sum_{r=0}^{2^s-1} \Big(\cos(\frac{ar\pi}{2^{s-1}}) - i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\mathcal{W}_{h_r}(\mathbf{v})| = \sqrt{2^s}, \tag{5}$$

where $a = 1$ to $a = 2^{s-1} - 1$.
Let $\frac{\sum_{r=0}^{2^{s-1}-1} \mathcal{W}_{h_{2r}}(\mathbf{v})}{\sum_{r=0}^{2^{s-1}-1} \mathcal{W}_{h_{2r+1}}(\mathbf{v})} = \phi(\mathbf{v})$, where $\sum_{r=0}^{2^{s-1}-1} \mathcal{W}_{h_{2r+1}}(\mathbf{v}) \neq 0$.
Solving (1) and (3), we get $\phi(\mathbf{v}) = -\overline{\phi(\mathbf{v})}$, which implies that $\phi(\mathbf{v})$ is purely imaginary, i.e.,

$$\phi(\mathbf{v}) = i\rho(\mathbf{v}), \quad for\ all\ \rho(\mathbf{v}) \in \mathbb{R} \tag{6}$$

Similarly by solving (2) and (5), we get

$$\frac{\sum_{r=0}^{2^s-1} \cos(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})}{\sum_{r=0}^{2^s-1} \sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})} = \psi(\mathbf{v}), \quad \psi \in \mathbb{R}\ and \tag{7}$$

$a = 1$ to $a = 2^{s-1} - 1$.
From (2) and (7), we obtain

$$|\sum_{r=0}^{2^s-1} \sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2 |i + \psi(\mathbf{v})|^2 = 2^s,$$

$$|\sum_{r=0}^{2^s-1} \sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2 (1 + (\psi(\mathbf{v}))^2) = 2^s,$$

$$|\sum_{r=0}^{2^s-1} \cos(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2 + |\sum_{r=0}^{2^s-1} \sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2 = 2^s, \tag{8}$$

where $a = 1$ to $a = 2^{s-1} - 1$.
Similarly from (3) and (6), we have

$$|\sum_{r=0}^{2^{s-1}-1} \mathcal{W}_{h_{2r}}(\mathbf{v})|^2 + |\sum_{r=0}^{2^{s-1}-1} \mathcal{W}_{h_{2r+1}}(\mathbf{v})|^2 = 2^s. \tag{9}$$

Again, from (8), we have

$$|\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2+|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2=2^s, \tag{10}$$

where $a=1$ to $a=2^{s-2}-1$.

$$|\sum_{r=0}^{2^{s-1}-1}(-1)^r\mathcal{W}_{h_{2r}}(\mathbf{v})|^2+|\sum_{r=0}^{2^{s-1}-1}(-1)^r\mathcal{W}_{h_{2r+1}}(\mathbf{v})|^2=2^s \tag{11}$$

$$|\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2+|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2=2^s, \tag{12}$$

where $a=2^{s-2}+1$ to $a=2^{s-1}-1$. Solving (9) and (11), we get

$$\sum_{r=0}^{2^s-1}|\mathcal{W}_{h_r}(\mathbf{v})|^2+\sum_{m=0}^{2^s-5}\sum_{n=0}^{k}\mathcal{W}_{h_m}(\mathbf{v})\overline{\mathcal{W}_{h_{4n+m+4}}(\mathbf{v})}+\sum_{m=0}^{2^s-5}\sum_{n=0}^{k}\overline{\mathcal{W}_{h_m}(\mathbf{v})}\mathcal{W}_{h_{4n+m+4}}(\mathbf{v})=2^s,$$

where k is chosen such that $(4n+m+4)<2^s$ and

$$\sum_{m=0}^{2^s-3}\sum_{n=0}^{j}\mathcal{W}_{h_m}(\mathbf{v})\overline{\mathcal{W}_{h_{4n+m+2}}(\mathbf{v})}+\sum_{m=0}^{2^s-3}\sum_{n=0}^{j}\overline{\mathcal{W}_{h_m}(\mathbf{v})}\mathcal{W}_{h_{4n+m+2}}(\mathbf{v})=0,$$

where j is chosen such that $(4n+m+2)<2^s$.

Conversely, let us assume that the conditions (i), (ii), (iii) and (iv) are true. Then from the condition (i), we have

$$|\mathcal{W}_h(\mathbf{v},0)|=1.$$

Using conditions (ii) and (iv), we get

$$\begin{aligned}
&2^s|\mathcal{W}_h(\mathbf{v},1)|^2\\
&=|\sum_{r=0}^{2^s-1}\Big(\cos(\frac{r\pi}{2^{s-1}})+i\ \sin(\frac{r\pi}{2^{s-1}})\Big)\mathcal{W}_{h_r}(\mathbf{v})|^2\\
&=|\sum_{r=0}^{2^s-1}\sin(\frac{r\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2|\psi(\mathbf{v})+i|^2\\
&=|\sum_{r=0}^{2^s-1}\sin(\frac{r\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2(\psi(\mathbf{v})^2+1)\\
&=|\sum_{r=0}^{2^s-1}\cos(\frac{r\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2+|\sum_{r=0}^{2^s-1}\sin(\frac{r\pi}{2^{s-1}})\mathcal{W}_{h_r}(\mathbf{v})|^2\\
&=2^s.
\end{aligned}$$

This implies that,

$$|\mathcal{W}_h(\mathbf{v},1)|=1.$$

Similarly from conditions (ii), (iii) and (iv), for $a=2$ to $a=2^s-1$, we have $|\mathcal{W}_h(\mathbf{v},a)|=1$.
Therefore, we have

$$|\mathcal{W}_h(\mathbf{v},a)|=1,\ \ for\ all\ (\mathbf{v},a)\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}.$$

This completes the proof. □

4 Secondary construction of generalized negabent functions on $(n+1)$ variables from the functions on n variables on $\mathbb{Z}_{2^s}^n$

In this section, we proposed a necessary and sufficient condition for the negabentness of a generalized Boolean function.

Theorem 8. *Let $h\in\mathcal{GB}_{n+1}^{2^s}$ can be expressed as*

$$h(\mathbf{x},y)=f(\mathbf{x})(1+y)+g(\mathbf{x})y,$$

for all $\mathbf{x},y\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}$ and $f,g\in\mathcal{GB}_n^{2^s}$. Then h is generalized negabent if and only if the following conditions are satisfied.

(i) $|\sum_{r=0}^{2^s-1}\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s}$, *for all* $\mathbf{v}\in\mathbb{Z}_{2^s}^n$.

(ii) $\frac{\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})}{\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})}=\psi(\mathbf{v})$, *for* $a=1$ *to* $a=2^{s-1}-1$,
$\frac{\sum_{r=0}^{2^{s-1}-1}\omega^{2r}\mathcal{N}_{h_{2r}}(\mathbf{v})}{\sum_{r=0}^{2^{s-1}-1}\omega^{2r+1}\mathcal{N}_{h_{2r+1}}(\mathbf{v})}=i\rho(\mathbf{v})$, *where* $\psi(\mathbf{v}),\rho(\mathbf{v})\in\mathbb{R}$.

(iii) $\sum_{r=0}^{2^s-1}|\mathcal{N}_{h_r}(\mathbf{v})|^2+\sum_{m=0}^{2^s-5}\sum_{n=0}^{k}\mathcal{N}_{h_m}(\mathbf{v})\overline{\omega^{4n+m+4}\mathcal{N}_{h_{4n+m+4}}(\mathbf{v})}+\sum_{m=0}^{2^s-5}\sum_{n=0}^{k}\overline{\mathcal{N}_{h_m}(\mathbf{v})}\omega^{4n+m+4}\mathcal{N}_{h_{4n+m+4}}(\mathbf{v})=2^s$, *where k is chosen such that $(4n+m+4)<2^s$ and*

$\sum_{m=0}^{2^s-3}\sum_{n=0}^{j}\mathcal{N}_{h_m}(\mathbf{v})\overline{\omega^{4n+m+2}\mathcal{N}_{h_{4n+m+2}}(\mathbf{v})}+\sum_{m=0}^{2^s-3}\sum_{n=0}^{j}\overline{\mathcal{N}_{h_m}(\mathbf{v})}\omega^{4n+m+2}\mathcal{N}_{h_{4n+m+2}}(\mathbf{v})=0$, *where j is chosen such that $(4n+m+2)<2^s$.*

(iv) $|\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2+|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2=2^s$, *for $a=1$ to $a=2^{s-1}-1$.*
Here $h_r(\mathbf{v})=h(\mathbf{v},r)=f(\mathbf{v})(1+r)+g(\mathbf{v})r$, for all $\mathbf{v}\in\mathbb{Z}_{2^s}^n, r\in\mathbb{Z}_{2^s}$.

Proof. Let

$$h(\mathbf{x},y)=f(\mathbf{x})(1+y)+g(\mathbf{x})y,$$

be a generalized negabent function for all $\mathbf{x},y\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}$. Then the generalized nega-Hadamard transform of h at $(\mathbf{v},a)\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}$ is given by

$$\begin{aligned}
\mathcal{N}_h(\mathbf{v},a)&=\frac{1}{(2^s)^{\frac{n+1}{2}}}\sum_{(\mathbf{x},a)\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}}\zeta^{h(\mathbf{x},y)+a\cdot y+<\mathbf{v},\mathbf{x}>}\omega^{\sum x_i+y}\\
&=\frac{1}{(2^s)^{\frac{n+1}{2}}}\sum_{r=0}^{2^s-1}\sum_{\mathbf{x}\in\mathbb{Z}_{2^s}^n}\zeta^{h_r(\mathbf{x})+a\cdot r+<\mathbf{v},\mathbf{x}>}\omega^{\sum x_i+y}\\
&=\frac{1}{\sqrt{2^s}}\sum_{r=0}^{2^s-1}\zeta^{a\cdot r}\omega^r\mathcal{N}_{h_r}(\mathbf{v}).
\end{aligned}$$

Since h is generalized negabent function, it follows that $|\omega^r\mathcal{N}_h(\mathbf{v},a)|=1$, for all $(\mathbf{v},a)\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}$. This implies that

$$|\sum_{r=0}^{2^s-1}\zeta^{a\cdot r}\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s}$$

or

$$|\sum_{r=0}^{2^s-1}\Big(\cos(\frac{ar\pi}{2^{s-1}})+i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s},$$

where $a\in\mathbb{Z}_{2^s}$. This implies that

$$|\sum_{r=0}^{2^s-1}\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s}, \tag{13}$$

$$|\sum_{r=0}^{2^s-1}\Big(\cos(\frac{ar\pi}{2^{s-1}})+i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s}, \tag{14}$$

where $a=1$ to $a=2^{s-1}-1$,

$$|\sum_{r=0}^{2^s-1}(-1)^r\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s}, \tag{15}$$

$$|\sum_{r=0}^{2^s-1}\Big(\cos(\frac{ar\pi}{2^{s-1}})+i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s}, \tag{16}$$

where $a=2^{q-1}+1$ to $a=2^s-1$.
Since cosine and sine are periodic functions, it follows that equation (16) is written as

$$|\sum_{r=0}^{2^s-1}\Big(\cos(\frac{ar\pi}{2^{s-1}})-i\ \sin(\frac{ar\pi}{2^{s-1}})\Big)\omega^r\mathcal{N}_{h_r}(\mathbf{v})|=\sqrt{2^s}, \tag{17}$$

where $a=1$ to $a=2^{s-1}-1$.
Let $\frac{\sum_{r=0}^{2^{s-1}-1}\omega^{2r}\mathcal{N}_{h_{2r}}(\mathbf{v})}{\sum_{r=0}^{2^{s-1}-1}\omega^{2r+1}\mathcal{N}_{h_{2r+1}}(\mathbf{v})}=\phi(\mathbf{v})$, where $\sum_{r=0}^{2^{s-1}-1}\omega^{2r+1}\mathcal{N}_{h_{2r+1}}(\mathbf{v})\neq 0$.
Solving (13) and (15), we get $\phi(\mathbf{v})=-\overline{\phi(\mathbf{v})}$, which implies that $\phi(\mathbf{v})$ is purely imaginary, i.e.,

$$\phi(\mathbf{v})=i\rho(\mathbf{v}),\ \ for\ all\ \rho(\mathbf{v})\in\mathbb{R}. \tag{18}$$

Solving (14) and (17), we get

$$\frac{\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})}{\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})}=\psi(\mathbf{v}),\ \ \psi\in\mathbb{R}, \tag{19}$$

where $a=1$ to $a=2^{s-1}-1$.
From (14) and (19), we obtain

$$|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2|i+\psi(\mathbf{v})|^2=2^s,$$

$$|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2(1+(\psi(\mathbf{v}))^2)=2^s,$$

$$\begin{aligned}&|\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2\\&+|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2=2^s,\end{aligned} \tag{20}$$

where $a=1$ to $a=2^{s-1}-1$.
Similarly from (15) and (18), we have

$$|\sum_{r=0}^{2^{s-1}-1}\omega^{2r}\mathcal{N}_{h_{2r}}(\mathbf{v})|^2+|\sum_{r=0}^{2^{s-1}-1}\omega^{2r+1}\mathcal{N}_{h_{2r+1}}(\mathbf{v})|^2=2^s. \tag{21}$$

Again, from (20), we have

$$\begin{aligned}&|\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2\\&+|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2=2^s,\end{aligned} \tag{22}$$

where $a=1$ to $a=2^{s-2}-1$,

$$\begin{aligned}&|\sum_{r=0}^{2^{s-1}-1}(-1)^r\omega^{2r}\mathcal{N}_{h_{2r}}(\mathbf{v})|^2\\&+|\sum_{r=0}^{2^{s-1}-1}(-1)^r\omega^{2r+1}\mathcal{N}_{h_{2r+1}}(\mathbf{v})|^2=2^s,\end{aligned} \tag{23}$$

$$\begin{aligned}&|\sum_{r=0}^{2^s-1}\cos(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2\\&+|\sum_{r=0}^{2^s-1}\sin(\frac{ar\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2=2^s,\end{aligned} \tag{24}$$

where $a=2^{s-2}+1$ to $a=2^{s-1}-1$.
Solving (21) and (23), we get

$$\begin{aligned}&\sum_{r=0}^{2^s-1}|\mathcal{N}_{h_r}(\mathbf{v})|^2+\sum_{m=0}^{2^s-5}\sum_{n=0}^{k}\mathcal{N}_{h_m}(\mathbf{v})\overline{\omega^{4n+m+4}\mathcal{N}_{h_{4n+m+4}}(\mathbf{v})}\\&+\sum_{m=0}^{2^s-5}\sum_{n=0}^{k}\overline{\mathcal{N}_{h_m}(\mathbf{v})}\omega^{4n+m+4}\mathcal{N}_{h_{4n+m+4}}(\mathbf{v})=2^s,\end{aligned}$$

where k is chosen such that $(4n+m+4)<2^s$ and

$$\begin{aligned}&\sum_{m=0}^{2^s-3}\sum_{n=0}^{j}\mathcal{N}_{h_m}(\mathbf{v})\overline{\omega^{4n+m+2}\mathcal{N}_{h_{4n+m+2}}(\mathbf{v})}\\&+\sum_{m=0}^{2^s-3}\sum_{n=0}^{j}\overline{\mathcal{N}_{h_m}(\mathbf{v})}\omega^{4n+m+2}\mathcal{N}_{h_{4n+m+2}}(\mathbf{v})=0,\end{aligned}$$

where j is chosen such that $(4n+m+2)<2^s$.

Conversely, let us assume that the conditions (i), (ii), (iii) and (iv) are true. Then from condition (i), we have

$$|\mathcal{N}_h(\mathbf{v},0)|=1.$$

Using conditions (ii) and (iv), we get

$$2^s|\mathcal{N}_h(\mathbf{v},1)|^2$$
$$=|\sum_{r=0}^{2^s-1}\Big(\cos(\frac{r\pi}{2^{s-1}})+i\ \sin(\frac{r\pi}{2^{s-1}})\Big)\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2$$
$$=|\sum_{r=0}^{2^s-1}\sin(\frac{r\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2|\psi(\mathbf{v})+i|^2$$
$$=|\sum_{r=0}^{2^s-1}\sin(\frac{r\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2(\psi(\mathbf{v})^2+1)$$
$$=|\sum_{r=0}^{2^s-1}\cos(\frac{r\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2+|\sum_{r=0}^{2^s-1}\sin(\frac{r\pi}{2^{s-1}})\omega^r\mathcal{N}_{h_r}(\mathbf{v})|^2$$
$$=2^s.$$

This implies that

$$|\mathcal{N}_h(\mathbf{v},1)|=1.$$

Similarly from conditions (ii), (iii) and (iv), for $a=2$ to $a=2^s-1$, we have $|\mathcal{N}_h(\mathbf{v},a)|=1$.
Therefore, we have

$$|\mathcal{N}_h(\mathbf{v},a)|=1,\ \ for\ all\ (\mathbf{v},a)\in\mathbb{Z}_{2^s}^n\times\mathbb{Z}_{2^s}.$$

This completes the proof. □

5 Conclusions

In this paper, we have discussed the properties of generalized (q-ary) Boolean functions with respect to nega- Hadamard transform and their relation with generalized (q-ary) nega autocorrealation. We also determine the necessary and sufficient conditions for bentness and negabentness of generalized Boolean function for secondary construction on $\mathbb{Z}_q^n$, for $q=2^s$. In other words, we obtain general constructions for generalized (q-ary) bent functions and generalized (q-ary) negabent functions. These results may have great significance for their feasible applications in numerous wireless communications.

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