Mathematics and Statistics 10(3): 515-522, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100306

$(\in, \in \vee q_{\tilde{\kappa}})$-Anti-Intuitionistic Fuzzy Soft b-Ideals in BCK/BCI-Algebras

Aiyared Iampan[1,*],**M. Balamurugan**[2],**V. Govindan**[3]

[1]Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand

[2]Department of Mathematics, Sri Vidya Mandir Arts and Science College (Autonomous), India

[3]Department of Mathematics, Dmi St John The Baptist University, Malawi

Received January 3, 2022; Revised March 25, 2022; Accepted April 15, 2022

Cite This Paper in the following Citation Styles

***(a)**: [1] Aiyared Iampan, M. Balamurugan, V. Govindan, "$(\in, \in \vee q_{\tilde{\kappa}})$-Anti-Intuitionistic Fuzzy Soft b-Ideals in BCK/BCI-Algebras," Mathematics and Statistics, Vol.10, No.3, pp. 515-522, 2022. DOI: 10.13189/ms.2022.100306*

***(b)**: Aiyared Iampan, M. Balamurugan, V. Govindan, (2022). $(\in, \in \vee q_{\tilde{\kappa}})$-Anti-Intuitionistic Fuzzy Soft b-Ideals in BCK/BCI-Algebras. Mathematics and Statistics, 10(3), 515-522 DOI: 10.13189/ms.2022.100306*

Abstract Among many algebraic structures, algebras of logic form an essential class of algebras. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6, 7] in 1966 and have been extensively investigated by many researchers. The concept of fuzzy soft sets is introduced in [17] to generalize standard soft sets [21]. The concept of intuitionistic fuzzy soft sets is introduced by Maji et al. [18], which is based on a combination of the intuitionistic fuzzy set [2] and soft set models. The first section will discuss the origins and importance of studies in this article. Section 2 will review the definitions of a BCK/BCI-algebra, a soft set, a fuzzy soft set, and an intuitionistic fuzzy soft set and show the essential properties of BCK/BCI-algebras to be applied in the next section. In Section 3, the concept of an anti-intuitionistic fuzzy soft b-ideal (AIFSBI) is discussed in BCK/BCI-algebras, and essential properties are provided. A set of conditions is provided for an AIFSBI to be an anti-intuitionistic fuzzy soft ideal (AIFSI). The definition of quasi-coincidence of an intuitionistic fuzzy soft point with an intuitionistic fuzzy soft set (IFSS) is considered in a more general form. In Section 4, the concepts of a $(\in, \in \vee q_{\tilde{\kappa}})$-AFSBI and a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$ are introduced, and some characterizations of $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI are discussed using the concept of an AIFSBI with thresholds. Finally, conditions are given for a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI to be a $(\in, \in)$-AIFSBI.

Keywords BCK/BCI-algebra, Intuitionistic Fuzzy Soft Set, Anti-intuitionistic Fuzzy Soft B-ideal and $(\in, \in \vee q_{\tilde{\kappa}})$-anti-intuitionistic Fuzzy Soft B-ideal

1 Introduction

Several authors have investigated the algebraic structure of certain uncertainty theories. Zadeh's [30] theory of "fuzzy sets" is the best solution to deal with uncertainty. Many other hybrid concepts arose as a result of the development of fuzzy sets. The concept of "intuitionistic fuzzy set as generalization of fuzzy sets" was introduced by Atanassov [2]. Imai and Iséki [6] presented the concept of "BCK-algebra". Molodstov [21] has introduced the soft set concept as a new mathematical tool to address uncertainties free of issues that are plagued by traditional methods. Maji et al. [16] examined a number of operations based on soft set theory. The concept of "fuzzy soft sets" is introduced in [17] to generalize standard soft sets [21]. Maji et al. [18, 19, 20] introduced the "intuitionistic fuzzy soft set theory", which is based on a combination of the intuitionistic fuzzy set [2] and soft set models. Muhiuddin et al. [22, 25, 26, 29] introduced various concepts used in BCK/BCI-algebras.

Bhakat et al. [5], which is a viable generalization of the "Rosenfeld fuzzy subgroup", has introduced the definition of "$(\in, \in \vee q)$-fuzzy subgroup". Similar generalizations of existing fuzzy subsystems of other algebraic structures are now naturally explored. Jun et al. [12, 14] defined $(\in, \in \vee q)$-fuzzy subalgebras, and ideals. These ideals are fuzzy set generalizations. The concept of quasi-coincidence of an intuitionistic fuzzy point within an intuitionistic fuzzy set was introduced by Jana et al. [9]. We introduced the notions of $(\in, \in \vee q)$-intuitionistic fuzzy BCI-subalgebras and explored their properties with the help of this new idea. Jana et al. [8] introduced and investigated the concept of "generalized intuitionistic fuzzy ideals of BCK/BCI-algebras based on 3-valued logic".

Jun [10] introduced the concept of "soft BCK/BCI-algebras" by applying soft sets to the theory of BCK/BCI-algebras. With several examples, Balamurugan et al. [3] introduced the concept of "anti-intuitionistic fuzzy soft ideals in BCK/BCI-algebras". Balasubramanian et al. [4] investigated "$(\in, \in \vee q)$-anti intuitionistic fuzzy soft subalgebras of BG-algebras". Muhiuddin et al. [1, 23] pioneered the use of "normal unisoft filters in R_0-algebras". Critical soft points, concave soft sets, and union-soft ideals of ordered semigroups were developed by Jun et al. [15]. Muhiuddin et al. [24, 28] discussed cubic soft set-based subalgebras of BCK/BCI-algebras with applications in BCK/BCI-algebras. Muhiuddin et al. [27] defined the "N-soft p-ideal of BCI-algebras".

In this paper, we present an AIFSBI in BCK/BCI-algebras. It is demonstrated that an AIFSBI is equivalent to an AIFSI. Characterizations of $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI are discussed using the concept of an AIFSBI with thresholds. Finally, it is demonstrated that a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI is a $(\in, \in)$-AIFSBI.

2 Preliminaries

A BCI-algebra $(\widehat{\Theta}, \ominus, 0)$ of type $(2, 0)$ that meets the following conditions for every $\eth, \varkappa, \varpi \in \widehat{\Theta}$:
(C_1) $((\eth \ominus \varkappa) \ominus (\eth \ominus \varpi)) \ominus (\varpi \ominus \varkappa) = 0$,
(C_2) $(\eth \ominus (\eth \ominus \varkappa)) \ominus \varkappa = 0$,
(C_3) $\eth \ominus \eth = 0$,
(C_4) $\eth \ominus \varkappa = 0$ and $\varkappa \ominus \eth = 0 \Rightarrow \eth = \varkappa$. The partial ordering is demarcated as $\eth \leq \varkappa$ if and only if $\eth \ominus \varkappa = 0$.

If a BCI-algebra $\widehat{\Theta}$ satisfies $0 \ominus \eth = 0$ for every $\eth \in \widehat{\Theta}$, then $\widehat{\Theta}$ is a BCK-algebra. $\widehat{\Theta}$ denotes a BCK/BCI-algebra, other than or otherwise stated. Assume $\widehat{\Theta}$ is a BCK/BCI-algebra. Then $\widehat{\Theta}$ meets the following conditions for any $\eth, \varkappa, \varpi \in \widehat{\Theta}$,
(C_5) $\eth \ominus 0 = \eth$,
(C_6) $(\eth \ominus \varkappa) \ominus \varpi = (\eth \ominus \varpi) \ominus \varkappa$,
(C_7) $0 \ominus (\eth \ominus \varkappa) = (0 \ominus \eth) \ominus (0 \ominus \varkappa)$,
(C_8) $\eth \ominus (\eth \ominus (\eth \ominus \varkappa)) = \eth \ominus \varkappa$,
(C_9) $\eth \leq \varkappa$ imply $\eth \ominus \varpi \leq \varkappa \ominus \varpi$ and $\varpi \ominus \varkappa \leq \varpi \ominus \eth$,
(C_{10}) $(\eth \ominus \varpi) \ominus (\varkappa \ominus \varpi) \leq \eth \ominus \varkappa$,
(C_{11}) $(\eth \ominus \varpi) \ominus \varkappa \leq \eth \ominus \varkappa$,
(C_{12}) $0 \ominus (0 \ominus ((\eth \ominus \varpi) \ominus (\varkappa \ominus \varpi))) = (0 \ominus \varkappa) \ominus (0 \ominus \eth)$,
(C_{13}) $0 \ominus (0 \ominus (\eth \ominus \varkappa)) = (0 \ominus \varkappa) \ominus (0 \ominus \eth)$,
where $\eth \leq \varkappa$ if and only if $\eth \ominus \varkappa = 0$.

An ideal of $\widehat{\Theta}$ is a non-empty subset $\widehat{I}$ of $\widehat{\Theta}$ that satisfies (C_{14}) $0 \in \widehat{I}$ and (C_{15}) $\eth \ominus \varkappa \in \widehat{I}$ and $\varkappa \in \widehat{I} \Rightarrow \eth \in \widehat{I}$ for every $\eth, \varkappa \in \widehat{\Theta}$.

A non-empty subset $\widehat{I}$ of $\widehat{\Theta}$ is called a b-ideal of $\widehat{\Theta}$ if it satisfies (C_{14}) and (C_{16}) $(\eth \ominus \varpi) \ominus \varkappa \in \widehat{I}$ and $\varkappa \in \widehat{I} \Rightarrow \eth \in \widehat{I}$ for every $\eth, \varkappa \in \widehat{\Theta}$.

Definition 2.1. [21] A pair $(\widehat{\Lambda}, \widehat{\xi})$ is referred to as a soft set over $\widehat{\Theta} \Leftrightarrow \widehat{\Lambda} : \widehat{\xi} \to \wp(\widehat{\Theta})$.

Definition 2.2. [17] Let $\widehat{\xi} \subseteq \widehat{\Xi}$ be a set of factors, and $\widehat{\Theta}$ be a primary set under consideration. Then $(\widehat{\Lambda}, \widehat{\xi})$ is a fuzzy soft set over $\widehat{\Theta}$, where $\widehat{\Lambda}$ is a mapping $\widehat{\xi}$ into the set all fuzzy set $\widehat{\Lambda}(\widehat{\Theta})$.

Definition 2.3. [18] Let $\widehat{\xi} \subseteq \widehat{\Xi}$ be a set of factors, and $\widehat{\Theta}$ be a primary set under consideration. Then $(\widehat{\Lambda}, \widehat{\xi})$ is called an intuitionistic fuzzy soft set over $\widehat{\Theta}$, where $\widehat{\Lambda}$ is a mapping $\widehat{\xi}$ into the collection of all intuitionistic fuzzy set $\mathbb{I}\widehat{\Lambda}(\widehat{\Theta})$.

3 Anti-intuitionistic fuzzy soft b-ideals

Definition 3.1. [3] An IFSS $(\widehat{\Lambda}, \widehat{\xi})$ is called an AIFSI of $\in \widehat{\Theta}$ if $\widehat{\Lambda}[\breve{\delta}] = \{(\eth, \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth), \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)) \mid \eth \in \widehat{\Theta} \text{ and } \breve{\delta} \in \widehat{\xi}\}$ is an AIFI of $\in \widehat{\Theta}$ satisfies for each $\eth, \varkappa \in \widehat{\Theta}$:
(AIFI1) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)$ and $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)$,
(AIFI2) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$,
(AIFI3) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$.

Definition 3.2. An IFSS $(\widehat{\Lambda}, \widehat{\xi})$ is called an AIFSBI of $\in \widehat{\Theta}$ if $\widehat{\Lambda}[\breve{\delta}] = \{(\eth, \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth), \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)) \mid \eth \in \widehat{\Theta} \text{ and } \breve{\delta} \in \widehat{\xi}\}$ is an AIFBI of $\in \widehat{\Theta}$ satisfies for each $\eth, \varkappa, \varpi \in \widehat{\Theta}$:
(AIFBI1) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)$ and $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)$,
(AIFBI2) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$,
(AIFBI3) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$.

Example 3.3. Consider a BCI-algebra $\widehat{\Theta} = \{0, \eth, \varkappa, \varpi\}$ with Cayley table.

$\ominus$	0	$\eth$	$\varkappa$	ϖ
0	0	$\eth$	$\varkappa$	ϖ
$\eth$	$\eth$	0	ϖ	$\varkappa$
$\varkappa$	$\varkappa$	ϖ	0	$\eth$
ϖ	ϖ	$\varkappa$	$\eth$	0

Let $\widehat{\Lambda}[\breve{\delta}] = (\omega_{\widehat{\Lambda}[\breve{\delta}]}, \varsigma_{\widehat{\Lambda}[\breve{\delta}]})$ be an IFSS in $\widehat{\Theta}$ defined as

$\widehat{\Lambda}$	0	$\eth$	$\varkappa$	ϖ
$\breve{\delta}$	[.15, .65]	[.25, .55]	[.55, .15]	[.55, .15]

Then $\widehat{\Lambda}[\breve{\delta}]$ is an AIFBI of $\widehat{\Theta}$. Thus, $(\widehat{\Lambda}, \widehat{\xi})$ is an AIFSBI of $\widehat{\Theta}$.

Theorem 3.4. *Any AIFSBI is an AIFSI of* $\widehat{\Theta}$.

Proof. Let $(\widehat{\Lambda}, \widehat{\xi})$ be an AIFSBI of $\widehat{\Theta}$. Then

(i) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. If we substitute 0 for ϖ, we have $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus 0) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. Since $\eth \ominus 0 = \eth$ for all $\eth \in \widehat{\Theta}$, we have $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$.

(ii) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. If we substitute 0 for ϖ, we have $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus 0) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. Since $\eth \ominus 0 = \eth$ for all $\eth \in \widehat{\Theta}$, we have $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$.

Hence, $(\widehat{\Lambda}, \widehat{\xi})$ is an AIFSI of $\widehat{\Theta}$. □

Theorem 3.5. *Any AIFSI is an AIFSBI of* $\widehat{\Theta}$.

Proof. Let $(\widehat{\Lambda}, \widehat{\xi})$ be an AIFSI of $\widehat{\Theta}$. Then

(i) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. Since $\eth \ominus 0 = \eth$ for all $\eth \in \widehat{\Theta}$, we have $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus 0) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. If we substitute ϖ for 0, we have $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$.

(ii) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. Since $\eth \ominus 0 = \eth$ for all $\eth \in \widehat{\Theta}$, we have $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus 0) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. If we substitute ϖ for 0, we have $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$.

Therefore, $(\widehat{\Lambda}, \widehat{\xi})$ is an AIFSBI of $\widehat{\Theta}$. □

Theorem 3.6. *Let* $(\widehat{\Lambda}, \widehat{\xi})$ *be an AIFSBI of* $\widehat{\Theta}$. *Then it satisfies for all* $\eth, \varkappa, \varpi \in \widehat{\Theta}$ *and* $\breve{\delta} \in \widehat{\xi}$,

$$(i)\ \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus (\varkappa \ominus \varpi)) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa); \tag{3.1}$$

$$(ii)\ \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus (\varkappa \ominus \varpi)) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa). \tag{3.2}$$

Proof. Let $(\widehat{\Lambda}, \widehat{\xi})$ be an AIFSBI of $\widehat{\Theta}$. Since the inequality $(\eth \ominus \varpi) \ominus \varkappa \leq (\eth \ominus \varkappa)$ holds in $\widehat{\Theta}$. It follows, that $((\eth \ominus \varpi) \ominus \varkappa) \ominus (\eth \ominus \varkappa) = 0$. Since $(\widehat{\Lambda}, \widehat{\xi})$ is an AIFSI of $\widehat{\Theta}$ by Theorem 3.4, we have

$$\begin{aligned}
(i)\ \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) &\leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(((\eth \ominus \varpi) \ominus \varkappa) \ominus (\eth \ominus \varkappa)) \\
&\quad \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \\
\omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) &\leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \\
\omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) &\leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa), \\
(ii)\ \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) &\geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(((\eth \ominus \varpi) \ominus \varkappa) \ominus (\eth \ominus \varkappa)) \\
&\quad \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \\
\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) &\geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \\
\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) &\geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa)
\end{aligned}$$

for all $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. □

Theorem 3.7. *Let* $(\widehat{\Lambda}, \widehat{\xi})$ *be an AIFSI of* $\widehat{\Theta}$ *and if it satisfies for all* $\eth, \varkappa, \varpi \in \widehat{\Theta}$ *and* $\breve{\delta} \in \widehat{\xi}$,

$$(i)\ \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \geq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus (\varkappa \ominus \varpi)), \tag{3.3}$$

$$(ii)\ \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \leq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus (\varkappa \ominus \varpi)). \tag{3.4}$$

Then $(\widehat{\Lambda}, \widehat{\xi})$ *is an AIFSBI of* $\widehat{\Theta}$.

Proof. (i) Let $(\widehat{\Lambda}, \widehat{\xi})$ be an AIFSI of $\widehat{\Theta}$ satisfies (3.3). Then $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. Since $\eth \ominus 0 = \eth$ for all $\eth \in \widehat{\Theta}$, we have $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus 0) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. If we substitute ϖ for 0, we have $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$.

(ii) Let $(\widehat{\Lambda}, \widehat{\xi})$ be an AIFSI of $\widehat{\Theta}$ satisfies (3.4). Then $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. Since $\eth \ominus 0 = \eth$ for all $\eth \in \widehat{\Theta}$, we have $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus 0) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$. If we substitute ϖ for 0, we have $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ for all $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$.

Hence, $(\widehat{\Lambda}, \widehat{\xi})$ is an AIFSBI of $\widehat{\Theta}$. □

Theorem 3.8. *If* $(\widehat{\Lambda}, \widehat{\xi})$ *and* $(\widehat{\Lambda}, \widehat{\xi})^c$ *are both AIFSBI of* $\widehat{\Theta}$, *then* $(\widehat{\Lambda}, \widehat{\xi})$ *is a constant function.*

Proof. Let $(\widehat{\Lambda}, \widehat{\xi})$ and $(\widehat{\Lambda}, \widehat{\xi})^c$ be both AIFSBI of $\widehat{\Theta}$. It is enough to prove that
(i) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) = \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)$.
(ii) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) = \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)$.
(iii) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) = \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$.
(iv) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) = \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus r) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$.

(i) Let $\eth \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. Then

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \tag{3.5}$$

$$\begin{aligned}
\text{and}\quad \omega^c_{\widehat{\Lambda}[\breve{\delta}]}(0) &\leq \omega^c_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \\
\Rightarrow 1 - \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) &\leq 1 - \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)
\end{aligned}$$

$$\Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth). \tag{3.6}$$

From Equations (3.5) and (3.6), we have

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) = \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \tag{3.7}$$

(ii) Let $\eth \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. Then

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \tag{3.8}$$

$$\begin{aligned}
\text{and}\quad \varsigma^c_{\widehat{\Lambda}[\breve{\delta}]}(0) &\geq \varsigma^c_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \\
\Rightarrow 1 - \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) &\geq 1 - \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)
\end{aligned}$$

$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth). \tag{3.9}$$

From Equations (3.8) and (3.9), we have

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) = \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth). \tag{3.10}$$

(iii) Let $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. Then

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \tag{3.11}$$

$$\text{and}\quad \omega^c_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega^c_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega^c_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$$

$$\begin{aligned}
\Rightarrow 1 - \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) &\leq (1 - \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa)) \\
&\quad \vee (1 - \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa))
\end{aligned} \tag{3.12}$$

$$\Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus r) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa). \tag{3.13}$$

From Equations (3.11) and (3.13), we have

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) = \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa). \tag{3.14}$$

(iv) Let $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$. Then

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \tag{3.15}$$

$$\text{and}\quad \varsigma^c_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma^c_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma^c_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$$

$$\begin{aligned}
\Rightarrow 1 - \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) &\geq (1 - \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa)) \\
&\quad \wedge (1 - \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa))
\end{aligned} \tag{3.16}$$

$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa). \tag{3.17}$$

From Equations (3.15) and (3.17), we have

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) = \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa). \tag{3.18}$$

Hence, from (3.7), (3.10), (3.14) and (3.18), we get $(\widehat{\Lambda}, \widehat{\xi})$ is a constant function. $\square$

4 $(\in, \in \vee q_{\tilde{\kappa}})$-anti-intuitionistic fuzzy soft b-ideals

Definition 4.1. A FSS $(\widehat{\Lambda}, \widehat{\xi})$ of $\widehat{\Theta}$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AFSBI of $\widehat{\Theta}$ if $\widehat{\Lambda}[\breve{\delta}] = \{(\eth, \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)) \mid \eth \in \widehat{\Theta} \text{ and } \breve{\delta} \in \widehat{\xi}\}$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AFBI of $\widehat{\Theta}$ for every $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ satisfies:
(i) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \vee \frac{1-\tilde{\kappa}}{2}$,
(ii) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \vee \frac{1-\tilde{\kappa}}{2}$.

Definition 4.2. An IFSS $(\widehat{\Lambda}, \widehat{\xi})$ of $\widehat{\Theta}$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$ if $\widehat{\Lambda}[\breve{\delta}] = \{(\eth, \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth), \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)) \mid \eth \in \widehat{\Theta} \text{ and } \breve{\delta} \in \widehat{\xi}\}$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFBI of $\widehat{\Theta}$ for every $\eth, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ satisfies:
(i) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \vee \frac{1-\tilde{\kappa}}{2}$,
(ii) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \wedge \frac{1-\tilde{\kappa}}{2}$,
(iii) $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \vee \frac{1-\tilde{\kappa}}{2}$,
(iv) $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \wedge \frac{1-\tilde{\kappa}}{2}$.

Theorem 4.3. *An IFSS $(\widehat{\Lambda}, \widehat{\xi})$ of $\widehat{\Theta}$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$ if and only if*
(a) $(\eth, \tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\vartheta})\overline{\in \wedge q_{\tilde{\kappa}}}\omega_{\widehat{\Lambda}[\breve{\delta}]}$,
(b) $(\eth, \tilde{\varrho})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\varrho})\overline{\in \wedge q_{\tilde{\kappa}}}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$,
(c) $((\eth \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (\eth, \tilde{\vartheta} \vee \tilde{\varrho})\overline{\in \wedge q_{\tilde{\kappa}}}\omega_{\widehat{\Lambda}[\breve{\delta}]}$,
(d) $((\eth \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (\eth, \tilde{\vartheta} \wedge \tilde{\varrho})\overline{\in \wedge q_{\tilde{\kappa}}}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$ *for all* $\eth, \varkappa, \varpi \in \widehat{\Theta}$ *and* $\breve{\delta} \in \widehat{\xi}$*, where* $\tilde{\vartheta}, \tilde{\varrho} \in (0, 1]$.

Proof. (i) First, let $(\widehat{\Lambda}, \widehat{\xi})$ be a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$. To prove (a), and (c). Since $(\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$, we have

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \vee \frac{1-\tilde{\kappa}}{2} \tag{4.1}$$

$$\text{and} \quad \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(((\eth \ominus \varpi) \ominus \varkappa)) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \vee \frac{1-\tilde{\kappa}}{2}. \tag{4.2}$$

Let $\eth \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ and $\tilde{\vartheta} \in [0, 1]$ be such that $(\eth, \tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) < \tilde{\vartheta}$. Now,

$$\begin{aligned}
(4.1) \Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \vee \frac{1-\tilde{\kappa}}{2} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta} \vee \frac{1-\tilde{\kappa}}{2} = \begin{cases} \tilde{\vartheta} & \text{if } \tilde{\vartheta} > \frac{1-\tilde{\kappa}}{2} \\ \frac{1-\tilde{\kappa}}{2} & \text{if } \tilde{\vartheta} \leq \frac{1-\tilde{\kappa}}{2} \end{cases} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta} \quad \text{or} \quad \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < \frac{1-\tilde{\kappa}}{2} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta} \quad \text{or} \\
& \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} < \frac{1-\tilde{\kappa}}{2} + \frac{1-\tilde{\kappa}}{2} = 1 - \tilde{\kappa} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < t \quad \text{or} \quad \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} + \tilde{\kappa} < 1 \\
\Rightarrow & (0, \tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \quad \text{or} \quad (0, \tilde{\vartheta})\bar{q}\kappa\omega_{\widehat{\Lambda}[\breve{\delta}]} \\
\Rightarrow & (0, \tilde{\vartheta})\overline{\in \wedge q_{\tilde{\kappa}}}\omega_{\widehat{\Lambda}[\breve{\delta}]}.
\end{aligned}$$

Therefore, $(\eth, \tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\vartheta})\overline{\in \wedge q_{\tilde{\kappa}}}\omega_{\widehat{\Lambda}[\breve{\delta}]}$ which proves (a).

Again, let $\eth, \varkappa, \varpi \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ and $\tilde{\vartheta}, \tilde{\varrho} \in (0, 1]$ be such that $((\eth \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) < \tilde{\vartheta}$ and $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) < \tilde{\varrho}$.

$$\begin{aligned}
(4.2) \Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \vee \frac{1-\tilde{\kappa}}{2} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) < \tilde{\vartheta} \vee \tilde{\varrho} \vee \frac{1-\tilde{\kappa}}{2} \\
& = \begin{cases} \tilde{\vartheta} & \text{if } \tilde{\vartheta} \vee s > \frac{1-\tilde{\kappa}}{2} \\ \frac{1-\tilde{\kappa}}{2} & \text{if } \tilde{\vartheta} \vee \tilde{\varrho} \leq \frac{1-\tilde{\kappa}}{2} \end{cases} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) < \tilde{\vartheta} \vee \tilde{\varrho} \quad \text{or} \quad \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) < \frac{1-\tilde{\kappa}}{2} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) < \tilde{\vartheta} \vee \tilde{\varrho} \quad \text{or} \\
& \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) + \tilde{\vartheta} \vee \tilde{\varrho} < \frac{1-\tilde{\kappa}}{2} + \frac{1-\tilde{\kappa}}{2} = 1 - \tilde{\kappa} \\
\Rightarrow & \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) < \tilde{\vartheta} \vee \tilde{\varrho} \quad \text{or} \quad \omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth) + \tilde{\vartheta} \vee \tilde{\varrho} + \tilde{\kappa} < 1 \\
\Rightarrow & (\eth, \tilde{\vartheta} \vee \tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \quad \text{or} \quad (\eth, \tilde{\vartheta} \vee \tilde{\varrho})\bar{q}\kappa\omega_{\widehat{\Lambda}[\breve{\delta}]} \\
\Rightarrow & (\eth, \tilde{\vartheta} \vee \tilde{\varrho})\overline{\in \wedge q_{\tilde{\kappa}}}\omega_{\widehat{\Lambda}[\breve{\delta}]}.
\end{aligned}$$

Therefore, $((\eth \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (\eth, \tilde{\vartheta} \vee \tilde{\varrho})\overline{\in \wedge q_{\tilde{\kappa}}}\omega_{\widehat{\Lambda}[\breve{\delta}]}$ which proves (c).

Conversely, suppose $(\widehat{\Lambda}, \widehat{\xi})$ satisfies (a) and (c). To prove $(\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$. If possible $(\widehat{\Lambda}, \widehat{\xi})$ is not a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$, then there exists $ß, \wp, \ell \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ such that at least one of $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) > \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) \vee \frac{1-\tilde{\kappa}}{2}$ or $\omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\wp) \vee \frac{1-\tilde{\kappa}}{2}$ holds. Suppose $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) > \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) \vee \frac{1-\tilde{\kappa}}{2}$ holds. Choose a real number $\tilde{\vartheta}$ such that

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\vartheta} > \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) \vee \frac{1-\tilde{\kappa}}{2} \quad (4.3)$$
$$\Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) < \tilde{\vartheta} \Rightarrow (ß, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$$
$$\Rightarrow (0, \tilde{\vartheta}) \overline{\in \wedge q_{\tilde{\kappa}}} \omega_{\widehat{\Lambda}[\breve{\delta}]} \text{ [By condition } (a)]$$
$$\Rightarrow (0, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \text{ or } (0, \tilde{\vartheta}) \bar{q\kappa} \omega_{\widehat{\Lambda}[\breve{\delta}]}$$
$$\Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta} \text{ or } \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} + \tilde{\kappa} \leq 1$$

first part is not true by (4.3).

Therefore, $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} + \tilde{\kappa} \leq 1 \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} \leq 1 - \tilde{\kappa} \Rightarrow 1 - \tilde{\kappa} \geq \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} > \tilde{\vartheta} + \tilde{\vartheta} = 2\tilde{\vartheta}$ [Since $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\vartheta}$ by (4.3)] $\Rightarrow \tilde{\vartheta} \leq 1 - \frac{1-\tilde{\kappa}}{2}$, which contradicts (4.3) again. Hence, we must have

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) \vee \frac{1-\tilde{\kappa}}{2}. \quad (4.4)$$

Again, if $\omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) > \omega_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\wp) \vee \frac{1-\tilde{\kappa}}{2}$ holds for some $ß, \wp, \ell \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$. Then choose a real number $\tilde{\vartheta}$ such that

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) > \tilde{\vartheta} > \omega_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\wp) \vee \frac{1-\tilde{\kappa}}{2} \quad (4.5)$$
$$\Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp < \tilde{\vartheta} \text{ and } \omega_{\gamma[\breve{\delta}]}(\wp) < \tilde{\vartheta}$$
$$\Rightarrow ((ß \ominus \ell) \ominus \wp, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \text{ and } (\wp, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$$
$$\Rightarrow (ß, \tilde{\vartheta} \vee \tilde{\vartheta}) \overline{\in \wedge q_{\tilde{\kappa}}} \omega_{\widehat{\Lambda}[\breve{\delta}]} \text{ [By condition (c)]}$$
$$\Rightarrow (ß, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \text{ or } \Rightarrow (ß, \tilde{\vartheta}) \bar{q\kappa} \omega_{\widehat{\Lambda}[\breve{\delta}]}$$
$$\Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) < \tilde{\vartheta} \text{ or } \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) + \tilde{\vartheta} + \tilde{\kappa} \leq 1$$

first part is not true by (4.5)

$$\Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) + \tilde{\vartheta} \leq 1 - \tilde{\kappa}$$
$$\Rightarrow 1 - \tilde{\kappa} \geq \omega_{\widehat{\Lambda}[\breve{\delta}]}(ß) + \tilde{\vartheta} > \tilde{\vartheta} + \tilde{\vartheta} = 2\tilde{\vartheta}$$
$$\Rightarrow \tilde{\vartheta} \leq \frac{1-\tilde{\kappa}}{2} \quad \text{which contradicts (4.5).}$$

Hence,

$$\omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus r) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \vee \frac{1-\tilde{\kappa}}{2} \quad (4.6)$$

From (4.4) and (4.6), we get

$$\omega_{\widehat{\Lambda}[\breve{\delta}]} \text{ is an } (\in, \in \vee q_{\tilde{\kappa}}) - \text{AFBI of } \widehat{\Theta}. \quad (4.7)$$

(ii) First, let $(\widehat{\Lambda}, \widehat{\xi})$ be a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$. To prove conditions (b), and (d). Since $(\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$, we have

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) \wedge \frac{1-\tilde{\kappa}}{2} \quad (4.8)$$
$$\text{and} \quad \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \wedge \frac{1-\tilde{\kappa}}{2}. \quad (4.9)$$

Let $ð \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ and $\tilde{\varrho} \in [0,1]$ be such that $(ð, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\varrho}$. Now,

$$(4.8) \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) \wedge \frac{1-\tilde{\kappa}}{2}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\varrho} \wedge \frac{1-\tilde{\kappa}}{2} = \begin{cases} \tilde{\varrho} & \text{if } \tilde{\varrho} < \frac{1-\tilde{\kappa}}{2} \\ \frac{1-\tilde{\kappa}}{2} & \text{if } \tilde{\varrho} \geq \frac{1-\tilde{\kappa}}{2} \end{cases}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\varrho} \text{ or } \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \frac{1-\tilde{\kappa}}{2}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\varrho} \text{ or}$$
$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\varrho} > \frac{1-\tilde{\kappa}}{2} + \frac{1-\tilde{\kappa}}{2} = 1 - \tilde{\kappa}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\varrho} \text{ or } \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\varrho} + \tilde{\kappa} > 1$$
$$\Rightarrow (ð, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \text{ or } (0, \tilde{\varrho}) \bar{q\kappa} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$$
$$\Rightarrow (0, \tilde{\varrho}) \overline{\in \wedge q_{\tilde{\kappa}}} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}.$$

Therefore, $(ð, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\varrho}) \overline{\in \wedge q_{\tilde{\kappa}}} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$ which prove (b).

Again, $ð, \varkappa, \varpi \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ and $\tilde{\varrho}, \tilde{\vartheta} \in (0,1]$ such that $((ð \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) > \tilde{\vartheta}$ and $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) > \tilde{\varrho}$.

$$(4.9) \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \wedge \frac{1-\tilde{\kappa}}{2}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\vartheta} \wedge s \wedge \frac{1-\tilde{\kappa}}{2}$$
$$= \begin{cases} \tilde{\vartheta} & \text{if } t \wedge \tilde{\varrho} < \frac{1-\tilde{\kappa}}{2} \\ \frac{1-\tilde{\kappa}}{2} & \text{if } \tilde{\vartheta} \wedge \tilde{\varrho} \geq \frac{1-\tilde{\kappa}}{2} \end{cases}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\vartheta} \wedge \tilde{\varrho} \text{ or } \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(x) > \frac{1-\tilde{\kappa}}{2}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\vartheta} \wedge \tilde{\varrho} \text{ or}$$
$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) + \tilde{\vartheta} \wedge \tilde{\varrho} > \frac{1-\tilde{\kappa}}{2} + \frac{1-\tilde{\kappa}}{2} = 1 - \tilde{\kappa}$$
$$\Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\vartheta} \wedge s \text{ or } \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) + \tilde{\vartheta} \wedge s + \tilde{\kappa} > 1$$
$$\Rightarrow (ð, \tilde{\vartheta} \wedge \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \text{ or } (ð, \tilde{\vartheta} \wedge \tilde{\varrho}) \bar{q\kappa} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$$
$$\Rightarrow (ð, \tilde{\vartheta} \wedge \tilde{\varrho}) \overline{\in \wedge q_{\tilde{\kappa}}} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}.$$

Therefore, $((ð \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (ð, \tilde{\vartheta} \wedge \tilde{\varrho}) \overline{\in \wedge q_{\tilde{\kappa}}} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$ which prove (d).

Conversely, suppose $(\widehat{\Lambda}, \widehat{\xi})$ satisfies (b) and (d). To prove $(\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$. If possible $(\widehat{\Lambda}, \widehat{\xi})$ is not a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of Ω, then there exists $ß, \wp, \ell \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ such that at least one of $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) < \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) \wedge \frac{1-\tilde{\kappa}}{2}$ or $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\wp) \wedge \frac{1-\tilde{\kappa}}{2}$ holds. Suppose $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) <$

$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) \wedge \frac{1-\tilde{\kappa}}{2}$ holds. Choose a real number $\tilde{\vartheta}$ such that

$$\begin{aligned}
&\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta} < \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) \wedge \frac{1-\tilde{\kappa}}{2} \quad (4.10)\\
\Rightarrow &\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) > \tilde{\vartheta}\\
\Rightarrow &(ß, \tilde{\vartheta}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}\\
\Rightarrow &(0, \tilde{\vartheta}) \overline{\in \wedge q_{\tilde{\kappa}}} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \text{ [By condition } (b)]\\
\Rightarrow &(0, \tilde{\vartheta}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \quad \text{or} \quad (0, \tilde{\vartheta}) \bar{q}\kappa \varsigma_{\widehat{\Lambda}[\breve{\delta}]}\\
\Rightarrow &\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\vartheta} \quad \text{or} \quad \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} + \tilde{\kappa} \geq 1\\
&\text{second part is not true by (4.10).}
\end{aligned}$$

Therefore, $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} + \tilde{\kappa} \geq 1 \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} \geq 1 - \tilde{\kappa} \Rightarrow 1 - \tilde{\kappa} \leq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) + \tilde{\vartheta} < \tilde{\vartheta} + \tilde{\vartheta} = 2\tilde{\vartheta}$ [Since $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta}$ by (4.8)] $\Rightarrow \tilde{\vartheta} \leq \frac{1-\tilde{\kappa}}{2}$, which contradicts (4.10) again. Hence, we must have

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) \wedge \frac{1-\tilde{\kappa}}{2}. \quad (4.11)$$

Again, if $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) < \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\wp) \wedge \frac{1-\tilde{\kappa}}{2}$ holds for some $ß, \wp, \ell \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$. Then choose a real number t such that

$$\begin{aligned}
&\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) < \tilde{\vartheta} < \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\wp)\\
&\wedge \frac{1-\tilde{\kappa}}{2} \quad (4.12)\\
\Rightarrow &\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ß \ominus \ell) \ominus \wp > \tilde{\vartheta} \text{ and } \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\wp) > \tilde{\vartheta}\\
\Rightarrow &((ß \ominus \ell) \ominus \wp, \tilde{\vartheta}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \text{ and } (\wp, \tilde{\vartheta}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}\\
\Rightarrow &(ß, \tilde{\vartheta} \wedge \tilde{\vartheta}) \overline{\in \wedge q_{\tilde{\kappa}}} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \text{ [By condition (d)]}\\
\Rightarrow &(ß, \tilde{\vartheta}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \quad \text{or} \quad \Rightarrow (ß, \tilde{\vartheta}) \bar{q}\kappa \varsigma_{\widehat{\Lambda}[\breve{\delta}]}\\
\Rightarrow &\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) > \tilde{\vartheta} \quad \text{or} \quad \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) + \tilde{\vartheta} + \tilde{\kappa} \geq 1\\
&\text{second part is not true by (4.12).}\\
\Rightarrow &\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) + \tilde{\vartheta} \geq 1 - \tilde{\kappa}\\
\Rightarrow &1 - \tilde{\kappa} \leq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ß) + \tilde{\vartheta} < \tilde{\vartheta} + \tilde{\vartheta} = 2\tilde{\vartheta}\\
\Rightarrow &\tilde{\vartheta} \geq \frac{1-\tilde{\kappa}}{2} \quad \text{which contradicts (4.12).}
\end{aligned}$$

Hence,

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) \wedge \frac{1-\tilde{\kappa}}{2} \quad (4.13)$$

From (4.11) and (4.13), we get

$$\varsigma_{\widehat{\Lambda}[\breve{\delta}]} \text{ is an } (\in, \in \vee q_{\tilde{\kappa}}) - \text{AFBI of } \Omega. \quad (4.14)$$

(4.7) and (4.14) $\Rightarrow \widehat{\Lambda}[\breve{\delta}]$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFBI of $\widehat{\Theta}$. Thus, $(\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in \vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$. □

Theorem 4.4. *An intuitionistic fuzzy soft $(\widehat{\Lambda}, \widehat{\xi})$ of $\widehat{\Theta}$ is an AIFSBI of $\widehat{\Theta}$ if and only if $(\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in)$-AIFSBI of $\widehat{\Theta}$.*

Proof. Let $(\widehat{\Lambda}, \widehat{\xi})$ be an AIFSBI of $\widehat{\Theta}$. We prove that $(\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in)$-AIFSBI of $\widehat{\Theta}$. It is enough to show that

(i) $(ð, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$.
(ii) $(ð, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$.
(iii) $((ð \ominus \varpi) \ominus (\varkappa \ominus \varpi), \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (ð, \tilde{\vartheta} \vee \tilde{\varrho}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$.
(iv) $((ð \ominus \varpi) \ominus (\varkappa \ominus \varpi), \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (ð, \tilde{\vartheta} \wedge \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$
for all $ð, \varkappa, \varpi \in \Omega$ and $\breve{\delta} \in \widehat{\xi}$.

(i) Let $ð \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ be such that $(ð, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$ where $\tilde{\vartheta} \in (0, 1)$. Then $\omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) < \tilde{\vartheta}$. Now, $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) < \tilde{\vartheta} \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta} \Rightarrow (0, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$. Therefore, $(ð, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\vartheta}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$.

(ii) Let $ð \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ be such that $(ð, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$ where $\tilde{\varrho} \in (0, 1)$. Then $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\varrho}$. Now, $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\varrho} \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\vartheta} \Rightarrow (0, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$. Therefore, $(ð, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$.

(iii) Let $ð, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ be such that $((ð \ominus \varpi) \ominus (\varkappa \ominus \varpi), \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$, where $\tilde{\vartheta}, \tilde{\varrho} \in (0, 1)$. Then $\omega_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) < \tilde{\vartheta}, \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) < \tilde{\varrho}$. Now, $\omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) \leq \omega_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) = \tilde{\vartheta} \vee \tilde{\varrho} \Rightarrow (ð, \tilde{\vartheta} \vee s) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$. Therefore, $((ð \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (ð, \tilde{\vartheta} \vee \tilde{\varrho}) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]}$.

(iv) Let $ð, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ be such that $((ð \ominus \varpi) \ominus \varkappa), \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$, where $\tilde{\vartheta}, \tilde{\varrho} \in (0, 1)$. Then $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) > \tilde{\vartheta}, \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) > \tilde{\varrho}$. Now, $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) \geq \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) = \tilde{\vartheta} \wedge \tilde{\varrho} \Rightarrow (ð, \tilde{\vartheta} \wedge s) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]}$. Therefore, $((ð \ominus \varpi) \ominus \varkappa, \tilde{\vartheta}), (\varkappa, \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (ð, \tilde{\vartheta} \wedge \tilde{\varrho}) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \widehat{\Lambda}[\breve{\delta}]$ is a $(\in, \in)$-AIFBI of $\widehat{\Theta} \Rightarrow (\widehat{\Lambda}, \widehat{\xi})$ is a $(\in, \in)$-AIFSBI of $\widehat{\Theta}$.

Conversely, let $(\widehat{\Lambda}, \widehat{\xi})$ be a $(\in, \in)$-AIFSBI of $\widehat{\Theta}$.

(i) Let $ð \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ and $\omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) = \tilde{\vartheta}$, where $\tilde{\vartheta} \in [0, 1]$. Then $\omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) < \tilde{\vartheta} + \epsilon$, where ϵ is an arbitrary small positive number. Therefore, $(ð, \tilde{\vartheta} + \epsilon) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\vartheta} + \epsilon) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) < \tilde{\vartheta} + \breve{\delta} \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(0) \leq \tilde{\vartheta} = \omega_{\widehat{\Lambda}[\breve{\delta}]}(ð)$.

(ii) Let $ð \in \widehat{\Theta}, \breve{\delta} \in \widehat{\xi}$ and $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) = \tilde{\varrho}$, where $\tilde{\varrho} \in [0, 1]$. Then $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\vartheta} - \epsilon$, where ϵ is an arbitrary small positive number. Therefore, $(ð, \tilde{\varrho} - \epsilon) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (0, \tilde{\varrho} - \epsilon) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) > \tilde{\varrho} - \breve{\delta} \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0) \geq \tilde{\varrho} = \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð)$.

(iii) Let $ð, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ be such that $\omega_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) = \tilde{\vartheta}, \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) = \tilde{\varrho}$, where $\tilde{\vartheta}, \tilde{\varrho} \in [0, 1]$. Then $\omega_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) < \tilde{\vartheta} + \epsilon, \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) < \tilde{\varrho} + \epsilon$, where ϵ is an arbitrary small positive number. Therefore, $((ð \ominus \varpi) \ominus \varkappa, \tilde{\vartheta} + \epsilon), (\varkappa, \tilde{\varrho} + \epsilon) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (ð, (\tilde{\vartheta} + \epsilon) \vee (\tilde{\varrho} + \epsilon)) \bar{\in} \omega_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) < (\tilde{\vartheta} + \epsilon) \vee (\tilde{\varrho} + \epsilon) \Rightarrow \omega_{\widehat{\Lambda}[\breve{\delta}]}(ð) < \tilde{\vartheta} \vee \tilde{\varrho} = \omega_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) \vee \omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$.

(iv) Let $ð, \varkappa, \varpi \in \widehat{\Theta}$ and $\breve{\delta} \in \widehat{\xi}$ be such that $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) = \tilde{\vartheta}, \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) = \tilde{\varrho}$, where $\tilde{\vartheta}, \tilde{\varrho} \in [0, 1]$. Then $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) > \tilde{\vartheta} - \epsilon, \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa) > \tilde{\varrho} - \epsilon$, where ϵ is an arbitrary small positive number. Therefore, $((ð \ominus \varpi) \ominus \varkappa, \tilde{\vartheta} - \epsilon), (\varkappa, \tilde{\varrho} - \epsilon) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow (ð, (\tilde{\vartheta} - \epsilon) \wedge (\tilde{\varrho} - \epsilon)) \bar{\in} \varsigma_{\widehat{\Lambda}[\breve{\delta}]} \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > (\tilde{\vartheta} - \epsilon) \wedge (\tilde{\varrho} - \epsilon) \Rightarrow \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(ð) > \tilde{\vartheta} \wedge \tilde{\varrho} = \varsigma_{\widehat{\Lambda}[\breve{\delta}]}((ð \ominus \varpi) \ominus \varkappa) \wedge \varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)$ $\Rightarrow \widehat{\Lambda}[\breve{\delta}]$ is an AIFBI of $\widehat{\Theta} \Rightarrow (\widehat{\Lambda}, \widehat{\xi})$ is an AIFSBI of $\widehat{\Theta}$. □

Theorem 4.5. *Let $(\widehat{\Lambda},\widehat{\xi})$ be a $(\in,\in\vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$. Then*
(i) If $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0)>\frac{1-\tilde{\kappa}}{2}$ for some $\eth\in\widehat{\Theta}$, then $(\widehat{\Lambda},\widehat{\xi})$ is a $(\in,\in)$-AFSBI of $\widehat{\Theta}$.
(ii) If $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)\leq\frac{1-\tilde{\kappa}}{2}$ for some $\eth\in\widehat{\Theta}$, then $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0)\leq\frac{1-\tilde{\kappa}}{2}$.
(iii) If $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0)<\frac{1-\tilde{\kappa}}{2}$ for some $\eth\in\widehat{\Theta}$, then $(\widehat{\Lambda},\widehat{\xi})$ is a $(\in,\in)$-AFSBI of $\widehat{\Theta}$.
(iv) If $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)\geq\frac{1-\tilde{\kappa}}{2}$ for some $\eth\in\widehat{\Theta}$, then $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0)\geq\frac{1-\tilde{\kappa}}{2}$.

Proof. (i) Let $(\widehat{\Lambda},\widehat{\xi})$ be a $(\in,\in\vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$ and $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)>\frac{1-\tilde{\kappa}}{2}$ for some $\eth\in\widehat{\Theta}$. Now, $(\eth,\tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}\Rightarrow\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)<\tilde{\vartheta}$. Therefore, $\frac{1-\tilde{\kappa}}{2}<\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)<\tilde{\vartheta}$. Also $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0)>\frac{1-\tilde{\kappa}}{2}\Rightarrow\omega_{\widehat{\Lambda}[\breve{\delta}]}(0)+\tilde{\vartheta}>\frac{1-\tilde{\kappa}}{2}+\frac{1-\tilde{\kappa}}{2}=1-\tilde{\kappa}\Rightarrow\omega_{\widehat{\Lambda}[\breve{\delta}]}(0)+\tilde{\vartheta}+\tilde{\kappa}>1\Rightarrow(0,\tilde{\vartheta})\omega_{\widehat{\Lambda}[\breve{\delta}]}$. Since $\omega_{\widehat{\Lambda}[\breve{\delta}]}$ is a $(\in,\in\vee q_{\tilde{\kappa}})$-AFBI of $\widehat{\Theta}$, we have $(0,\tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}$. Hence, $(\eth,\tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}\Rightarrow(0,\tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}$.

Again, if $((\eth\ominus\varpi)\ominus\varkappa,\tilde{\vartheta}),(\varkappa,\tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}$, then $\omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth\ominus\varpi)\ominus\varkappa)<\tilde{\vartheta}$ and $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)<\tilde{\varrho}$. Therefore, $\frac{1-\tilde{\kappa}}{2}<\omega_{\widehat{\Lambda}[\breve{\delta}]}((\eth\ominus\varpi)\ominus\varkappa)<\tilde{\vartheta}$ and $\frac{1-\tilde{\kappa}}{2}<\omega_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)<\tilde{\varrho}\Rightarrow\tilde{\vartheta}\vee\tilde{\varrho}>\frac{1-\tilde{\kappa}}{2}$. Also $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)>\frac{1-\tilde{\kappa}}{2}$. Thus, $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)+\tilde{\vartheta}\vee\tilde{\varrho}>\frac{1-\tilde{\kappa}}{2}+\frac{1-\tilde{\kappa}}{2}=1-\tilde{\kappa}\Rightarrow\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)+(\tilde{\vartheta}\vee\tilde{\varrho})+\tilde{\kappa}>1\Rightarrow(\eth,\tilde{\vartheta}\vee\tilde{\varrho})q_{\tilde{\kappa}}\omega_{\widehat{\Lambda}[\breve{\delta}]}$. Since $\omega_{\widehat{\Lambda}[\breve{\delta}]}$ is a $(\in,\in\vee q_{\tilde{\kappa}})$-AFBI of $\widehat{\Theta}$, we have $(\eth,\tilde{\vartheta}\vee\tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}$. Hence, $((\eth\ominus\varpi)\ominus\varkappa,\tilde{\vartheta})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]},(\varkappa,\tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}\Rightarrow(\eth,\tilde{\vartheta}\vee\tilde{\varrho})\bar{\in}\omega_{\widehat{\Lambda}[\breve{\delta}]}$. Therefore, $\widehat{\Lambda}[\breve{\delta}]$ is a $(\in,\in)$-AFBI of $\widehat{\Theta}$. Thus, $(\widehat{\Lambda},\widehat{\xi})$ is a $(\in,\in\vee q_{\tilde{\kappa}})$-AFSBI of $\widehat{\Theta}$.

(ii) Let $(\widehat{\Lambda},\widehat{\xi})$ be a $(\in,\in\vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$ and $\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)\leq\frac{1-\tilde{\kappa}}{2}$ for all $\eth\in\widehat{\Theta}$ and $\breve{\delta}\in\widehat{\xi}$. Now, $\omega_{\widehat{\Lambda}[\breve{\delta}]}(0)\leq\omega_{\widehat{\Lambda}[\breve{\delta}]}(\eth)\vee\frac{1-\tilde{\kappa}}{2}=\frac{1-\tilde{\kappa}}{2}\vee\frac{1-\tilde{\kappa}}{2}=\frac{1-\tilde{\kappa}}{2}$.

(iii) Let $(\widehat{\Lambda},\widehat{\xi})$ be a $(\in,\in\vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$ and $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)<\frac{1-\tilde{\kappa}}{2}$ for some $\eth\in\widehat{\Theta}$. Now, $(\eth,\tilde{\vartheta})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}\Rightarrow\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)>\tilde{\vartheta}$. Therefore, $\frac{1-\tilde{\kappa}}{2}>\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)>\tilde{\vartheta}$. Also $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0)<\frac{1-\tilde{\kappa}}{2}\Rightarrow\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0)+\tilde{\vartheta}<\frac{1-\tilde{\kappa}}{2}+\frac{1-\tilde{\kappa}}{2}=1-\tilde{\kappa}\Rightarrow\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0)+\tilde{\vartheta}+\tilde{\kappa}<1\Rightarrow(0,\tilde{\vartheta})\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$. Since $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$ is a $(\in,\in\vee q_{\tilde{\kappa}})$-AFBI of $\widehat{\Theta}$, we have $(0,\tilde{\vartheta})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$. Hence, $(\eth,\tilde{\vartheta})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}\Rightarrow(0,\tilde{\vartheta})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$.

Again, if $((\eth\ominus\varpi)\ominus\varkappa,\tilde{\vartheta}),(\varkappa,\tilde{\varrho})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$, then $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth\ominus\varpi)\ominus\varkappa)>\tilde{\vartheta}$ and $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)>\tilde{\varrho}$. Therefore, $\frac{1-\tilde{\kappa}}{2}>\varsigma_{\widehat{\Lambda}[\breve{\delta}]}((\eth\ominus\varpi)\ominus\varkappa)>\tilde{\vartheta}$ and $\frac{1-\tilde{\kappa}}{2}>\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\varkappa)>\tilde{\varrho}\Rightarrow\tilde{\vartheta}\wedge\tilde{\varrho}<\frac{1-\tilde{\kappa}}{2}$. Also $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)<\frac{1-\tilde{\kappa}}{2}$. Thus, $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)+\tilde{\vartheta}\wedge\tilde{\varrho}<\frac{1-\tilde{\kappa}}{2}+\frac{1-\tilde{\kappa}}{2}=1-\tilde{\kappa}\Rightarrow\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)+(\tilde{\vartheta}\wedge\tilde{\varrho})+\tilde{\kappa}<1\Rightarrow(\eth,\tilde{\vartheta}\wedge\tilde{\varrho})q_{\tilde{\kappa}}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$. Since $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$ is a $(\in,\in\vee q_{\tilde{\kappa}})$-AFBI of $\widehat{\Theta}$, we have $(\eth,\tilde{\vartheta}\wedge\tilde{\varrho})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$. Hence, $((\eth\ominus\varpi)\ominus\varkappa,\tilde{\vartheta})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]},(\varkappa,\tilde{\varrho})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}\Rightarrow(\eth,\tilde{\vartheta}\wedge\tilde{\varrho})\bar{\in}\varsigma_{\widehat{\Lambda}[\breve{\delta}]}$. Therefore, $\widehat{\Lambda}[\breve{\delta}]$ is a $(\in,\in)$-AFBI of $\widehat{\Theta}$. Hence, $(\widehat{\Lambda},\widehat{\xi})$ is a $(\in,\in\vee q_{\tilde{\kappa}})$-AFSBI of $\widehat{\Theta}$.

(iv) Let $(\widehat{\Lambda},\widehat{\xi})$ be a $(\in,\in\vee q_{\tilde{\kappa}})$-AIFSBI of $\widehat{\Theta}$ and $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)\geq\frac{1-\tilde{\kappa}}{2}$ for all $\eth\in\widehat{\Theta}$ and $\breve{\delta}\in\widehat{\xi}$. Now, $\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(0)\geq\varsigma_{\widehat{\Lambda}[\breve{\delta}]}(\eth)\wedge\frac{1-\tilde{\kappa}}{2}=\frac{1-\tilde{\kappa}}{2}\wedge\frac{1-\tilde{\kappa}}{2}=\frac{1-\tilde{\kappa}}{2}$. □

5 Conclusions

In this paper, the notion of AIFSI and AIFSBI was pioneered, and related properties were investigated. Furthermore, characterizations of $(\in,\in\vee q_{\tilde{\kappa}})$-AIFSBI's using the concept of an AIFSBI with thresholds are provided. Finally, certain properties are given for a $(\in,\in\vee q_{\tilde{\kappa}})$-AIFSBI to be a $(\in,\in)$-AIFSBI.

Acknowledgements

For their insightful comments, the authors would like to thank anonymous reviewers.

REFERENCES

[1] A. M. Al-roqi, G. Muhiuddin, S. Aldhafeeri. Normal unisoft filters in R_0-algebras, Cogent Math., Vol.1, No.4, Article ID 1383006, 2017.

[2] K. T. Atanassov. Intuitionistic fuzzy sets, Fuzzy Sets Syst., Vol.20, No.1, 87-96, 1986.

[3] M. Balamurugan, C. Ragavan, G. Balasubramanian. Anti-intuitionistic fuzzy soft ideals in BCK/BCI-algebras, Materials Today: Proceedings, Vol.16, No.2, 532-539, 2019.

[4] G. Balasubramanian, M. Balamurugan, C. R. Ragavan. Generalizations of $(\in,\in\vee q)$-anti intuitionistic fuzzy soft subalgebras of BG-algebras, Int. J. Appl. Eng. Res., Vol.13, No.23, 16376-16393, 2018.

[5] S. K. Bhakat, P. Das. $(\in,\in\vee q)$-fuzzy subgroup, Fuzzy Sets Syst., Vol.80, No.3, 359-368, 1996.

[6] Y. Imai, K. Iséki. On axiom systems of propositional calculi, XIV, Proc. Japon Acad., Vol.42, No.1, 19-22, 1966.

[7] K. Iséki. An algebra related with a propositional calculus, Proc. Japan Acad., Vol.42, no.1, 26-29, 1966.

[8] C. Jana, M. Pal. Generalized intuitionistic fuzzy ideals of BCK/BCI-algebras based on 3-valued logic and its computational study, Fuzzy Inf. Eng., Vol.9, No.4, 455-478, 2017.

[9] C. Jana, T. Senapati, M. Pal. $(\in,\in\vee q)$-intuitionistic fuzzy BCI-subalgebras of BCI-algebra, J. Intell. Fuzzy Syst., Vol.31, 613-621, 2016.

[10] Y. B. Jun. Soft BCK/BCI-algebras, Comput. Math. Appl., Vol.56, No.5, 1408-1413, 2008.

[11] Y. B. Jun. Fuzzy subalgebras with thresholds in BCK/BCI-algebras, Commun. Korean Math. Soc., Vol.22, No.2, 173-181, 2007.

[12] Y. B. Jun. Generalizations of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras, Comput. Math. Appl., Vol.58, No.7, 1383-1390, 2009.

[13] Y. B. Jun, K. J. Lee, C. H. Park. Fuzzy soft set theory applied to BCK/BCI-algebras, Comput. Math. Appl., Vol.59, No.9, 3180-3192, 2010.

[14] Y. B. Jun, K. J. Lee, C. H. Park. New types of ideals in BCK/BCI-algebras, Comput. Math. Appl., Vol.60, No.3, 771-785, 2010.

[15] Y. B. Jun, S. Z. Song, G. Muhiuddin. Concave soft sets, critical soft points, and union-soft ideals of ordered semigroups, Sci. World J., Vol.2014, Article ID 467968, 2014.

[16] P. K. Maji, R. Biswas, A. R. Roy. Soft set theory, Comput. Math. Appl., Vol.45, No.4-5, 555-562, 2003.

[17] P. K. Maji, R. Biswas, A. R. Roy. Fuzzy soft sets, J. Fuzzy Math., Vol.9, No.3, 589-602, 2001.

[18] P. K. Maji, R. Biswas, A. R. Roy. Intuitionistic fuzzy soft sets, J. Fuzzy Math., Vol.9, No.3, 677-692, 2001.

[19] P. K. Maji, R. Biswas, A. R. Roy. On intuitionistic fuzzy soft sets, J. Fuzzy Math., Vol.12, No.3, 669-683, 2004.

[20] P. K. Maji. More on intuitionistic fuzzy soft sets, Lect. Notes Comput. Sci., Vol.5908, 231-240, 2009.

[21] D. Molodstov. Soft set theory-first results, Comput. Math. Appl., Vol.37, No.4-5, 19-31, 1999.

[22] G. Muhiuddin. Hesitant fuzzy filters and hesitant fuzzy G-filters in residuated lattices, J. Comput. Anal. Appl., Vol.21, No.2, 394-404, 2016.

[23] G. Muhiuddin, A. M. Al-roqi. Unisoft filters in R_0-algebras, J. Comput. Anal. Appl., Vol.19, No.1, 133-143, 2015.

[24] G. Muhiuddin, A. M. Al-roqi. Cubic soft sets with applications in BCK/BCI-algebras, Ann. Fuzzy Math. Inform., Vol.8, No.2, 291-304, 2014.

[25] G. Muhiuddin, A. M. Al-roqi, S. Aldhafeeri. Filter theory in MTL-algebras based on uni-soft property, Bull. Iranian Math. Soc., Vol.43, No.7, 2293-2306, 2017.

[26] G. Muhiuddin, S. Aldhafeeri. Subalgebras and ideals in BCK/BCI-algebras based on uni-hesitant fuzzy set theory, Eur. J. Pure Appl. Math., Vol.11, No.2, 417-430, 2018.

[27] G. Muhiuddin, S. Aldhafeeri. N-Soft p-ideal of BCI-algebras, Eur. J. Pure Appl. Math., Vol.12, No.1, 79-87, 2019.

[28] G. Muhiuddin, F. Feng, Y. B. Jun. Subalgebras of BCK/BCI-algebras based on cubic soft sets, Sci. World J., Vol.2014, Article ID 458638, 2014.

[29] G. Muhiuddin, H. S. Kim, S. Z. Song, Y. B. Jun. Hesitant fuzzy translations and extensions of subalgebras and ideals in BCK/BCI-algebras, J. Intell. Fuzzy Syst., Vol.32, No.1, 43-48, 2017.

[30] L. A. Zadeh. Fuzzy sets, Inf. Control, Vol.8, No.3, 338-353, 1965.