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Half-Space Model Problem for Navier-Lamé Equations with Surface Tension

Sri Maryani[1,*], **Bambang H Guswanto**[1], **Hendra Gunawan**[2]

[1]Department of Mathematics, Jenderal Soedirman University, Indonesia
[2]Department of Mathematics, Bandung Institute of Technology, Indonesia

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Abstract Recently, we have seen the phenomena in use of partial differential equations (PDEs) especially in fluid dynamic area. The classical approach of the analysis of PDEs were dominated in early nineteenth century. As we know that for PDEs the fundamental theoretical question is whether the model problem consists of equation and its associated side condition is well-posed. There are many ways to investigate that the model problems are well-posed. Because of that reason, in this paper we consider the $\mathcal{R}$-boundedness of the solution operator families for Navier-Lamé equation by taking into account the surface tension in a bounded domain of N- dimensional Euclidean space ($N \geq 2$) as one way to study the well-posedess. We investigate the $\mathcal{R}$- boundedness in half-space domain case. The $\mathcal{R}$-boundedness implies not only the generation of analytic semigroup but also the maximal L_p-L_q regularity for the initial boundary value problem by using Weis's operator valued Fourier multiplier theorem for time dependent problem. it was known that the maximal L_p-L_q regularity class is the powerful tool to prove the well-posesness of the model problem. This result can be used for further research for example to analyze the boundedness of the solution operators of the model problem in bent-half space or general domain case.

Keywords $\mathcal{R}$-sectoriality, Navier-Lamé equation, Surface Tension, Half-space

1 Introduction

Let $\mathbf{u}$ and Ω be a velocity field and a bounded domain in N-dimensional space $\mathbb{R}^N$ ($N \geq 2$), respectively. The formula of Navier-Lamé equation in bounded domain with surface tension is written in the following:

$$\begin{cases} \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} = \mathbf{f} & \text{in } \mathbb{R}^N_+, \\ (\alpha\mathbf{D}(\mathbf{u}) - (\beta-\alpha)\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n} - \sigma(\Delta'_\Gamma\eta)\mathbf{n} = \mathbf{g} & \text{on } \mathbb{R}^N_0, \\ \lambda\eta + \mathbf{a}'\cdot\nabla'\eta - \mathbf{u}\cdot\mathbf{n} = d & \text{on } \mathbb{R}^N_0. \end{cases} \tag{1}$$

where $\mathbf{a}' = (a_1, \ldots, a_{N-1}) \in \mathbb{R}^{N-1}$ and $\mathbf{a}'\cdot\nabla'\eta = \sum_{j=1}^{N-1} a_j\partial_j\eta$. Assume that

$$|\mathbf{a}'| \leq a_0 \tag{2}$$

for some constant $a_0 > 0$. Let $\mathbb{R}^N_+$ and $\mathbb{R}^N_0$ be a half-space and its boundary, respectively. Namely,

$$\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\},$$
$$\mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\},$$

and $\mathbf{n} = (0, \ldots, 0, -1)$ be the unit outer normal to $\mathbb{R}^N_0$. $\mathbf{D}(\mathbf{u})$, $\mathbf{u} = (u_1, \ldots, u_N)$, the doubled deformation tensor whose (i, j) components are $D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i$ ($\partial_i = \partial/\partial x_i$), $\mathbf{I}$ the $N \times N$ identity matrix, α, β are positive constants (α and β are the first and second viscosity coefficients, respectively) such that $\beta - \alpha > 0$.

Meanwhile, Δ_{Γ_t} is the Laplace-Beltrami operator on Δ_{Γ_t}. Let $\mathbb{R}^N_+$ and $\mathbb{R}^N_0$ be a half-space and its boundary, respectively. Namely,

$$\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N > 0\},$$
$$\mathbb{R}^N_0 = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N \mid x_N = 0\}$$

Let $\mathbf{n} = (0, \ldots, 0, -1)$ be the unit outer normal to $\mathbb{R}^N_0$. We consider the following problem:

$$\begin{cases} \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} = \mathbf{f} & \text{in } \mathbb{R}^N_+, \\ (\alpha\mathbf{D}(\mathbf{u}) - (\beta - \alpha)\mathrm{div}\,\mathbf{u})\mathbf{n} - \sigma(\Delta_\Gamma\eta)\mathbf{n} = \mathbf{g} & \text{on } \mathbb{R}^N_0, \\ \lambda\eta - \mathbf{n}\cdot\mathbf{u} = d & \text{on } \mathbb{R}^N_0, \end{cases} \tag{3}$$

where α is uniformly continuous function with respect to $x \in \mathbb{R}^N_+$, which satisfy the assumptions:

$$\rho_*/2 \le \alpha(x) \le 2\rho_*. \tag{4}$$

The aim of this paper is to derive a systematic way proving the existence and the $\mathcal{R}$-boundedness solution operator of the resolvent problem for the equation system of Navier-Lamé (3) with surface tension in half-space. By using the Weis operator valued Fourier multiplier theorem [19], the existence of the $\mathcal{R}$-boundedness solution operator of the problem (1) implies not only the generation of analytic semigroup but also the maximal L_p-L_q regularity. The Navier-Lamé (NL) equation is the fundamental equation of motion in classical linear elastodynamics [7]. Sakhr [13] investigated the Navier-Lamé equation by using Buchwald representation in cylindrical coordinates. The $\mathcal{R}$-sectoriality was introduced by Clément and Prüß[5]. In 2009, Cao [2] investigated the Navier-Stokes and the wave-type extension-Lamé equations by using Fourier expansion. And also investigated the flag partial differential equations by using Xu's method.

In this paper, we investigate the derivation of the $\mathcal{R}$-sectoriality for the model problem in the whole space and half-space by applying Fourier transform to the model problems. In the other side, Denk, Hieber and Prüß[4] proved the $\mathcal{R}$-sectoriality for BVP of the elliptic equation which holds the Lopatinski-Shapiro condition.

Recently, there are many researchers who concern to study $\mathcal{R}$-boundedness case. In 2014, Murata [8] investigated the $\mathcal{R}$-boundedness of the Stokes operator with slip boundary condition. Another researcher who investigated the $\mathcal{R}$-sectoriality is Maryani [10, 11]. She studied the maximal L_p-L_q regularity class in a bounded domain and some unbounded domains which satisfy some uniformity and global well-posedness in the bounded domain case, respectively using the result of $\mathcal{R}$-bundedness of the solution operator of the model problem of the Oldroyd-b model. The main purpose of this paper is to investigate the $\mathcal{R}$-boundedness of the solution operator families for the Navier-Lamé equation with surface tension in half-space problem. A further result in favour of focusing on the main problem is finding the characteristic of η and creating the Laplace- Beltrami operator on Γ. This kind of investigation becomes considerable benefit in studying fluid mechanics.

Several mathematical analysis approach of fluid motion with surface tension have been undertaken in recent years. In 2013, Shibata [15] investigated the generalized resolvent estimates of the Stokes equations with first order boundary condition in a general domain. Later year, Shibata and Shimizu [18] studied a local in time solvability of free surface problems for the Navier-Stokes equations with surface tension. According to those phenomena, it is such an interesting subject to analyze fluid flow of the non-Newtonian compressible type especially model of the Navier-Lamé equations.

The main aim of this study is to prove the existence of the $\mathcal{R}$-bounded solution operator families for Navier-Lamé equations with surface tension in a bounded domain for the resolvent problem (1) in half-space for $\sigma > 0$ and $a = 0$ case. This topic becomes important reference for someone who is concerned with not only local well-posedness but also global well-posedness of Oldroyd-B model fluid flow. And then, applying the definition of $\mathcal{R}$-sectoriality and Weis' operator valued Fourier multiplier theorem in [19], automatically we obtain the generation of analytic semigroup and the maximal L_p-L_q regularity for the equation (3). In 2017, Maryani and Saito [12] investigated $\mathcal{R}$-boundedness of solution operator of two phase problem for Stokes equations.

To state our main results, at this stage we introduce our notation used throughout the paper.

Notation $\mathbb{N}$ denotes the sets of natural numbers and we set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{C}$ and $\mathbb{R}$ denote the sets of complex numbers, and real numbers, respectively. For the sets of all $N \times N$ symmetric and anti-symmetric matrices, we denote $Sym(\mathbb{R}^N)$ and $ASym(\mathbb{R}^N)$, respectively. Let $q' = q/(q-1)$, where q' is the dual exponent of q with $1 < q < \infty$, and satisfies $1/q + 1/q' = 1$. For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{N}^N_0$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial^k_x = \partial^{\kappa_1}_1 \cdots \partial^{\kappa_N}_N$ with $x = (x_1, \ldots, x_N)$. For scalar function f and N-vector of functions $\mathbf{g}$, we set

$$\begin{aligned}\nabla f &= (\partial_1 f, \dots, \partial_N f),\\ \nabla \mathbf{g} &= (\partial_i g_j \mid i, j = 1, \dots, N),\\ \nabla^2 f &= \{\partial_i \partial_j f \mid i, j = 1, \dots, N\},\\ \nabla^2 \mathbf{g} &= \{\partial_i \partial_j g_k \mid i, j, k = 1, \dots, N\}.\end{aligned}$$

$\mathcal{L}(X, Y)$ denotes the set of all bounded linear operators from X into Y, for Banach spaces X and Y and $\mathrm{Hol}\,(\mathrm{U}, \mathcal{L}(\mathrm{X}, \mathrm{Y}))$ the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions defined on a domain U in $\mathbb{C}$. $L_q(D)$, $W^m_q(D)$, $B^s_{p,q}(D)$ and $H^s_q(D)$ denote the usual Lebesgue space, Sobolev space, Besov space and Bessel potential space, respectively, for any domain D in $\mathbb{R}^N$ and $1 \leq p, q \leq \infty$. Whilst, $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W^m_q(D)}$, $\|\cdot\|_{B^s_{q,p}(D)}$ and $\|\cdot\|_{H^s_q(D)}$ denote their respective norms. For $\theta \in (0, 1)$, $H^\theta_p(\mathbb{R}, X)$ denotes the standard X-valued Bessel potential space defined by

$$\begin{aligned}H^\theta_p(\mathbb{R}, X) &= \{f \in L_p(\mathbb{R}, X) \mid \|f\|_{H^\theta_p(\mathbb{R},X)} < \infty\},\\ \|f\|_{H^\theta_p(\mathbb{R},X)} < \infty\} &= \left(\int_{\mathbb{R}} \|\mathcal{F}^{-1}[(1+\tau^2)^{\theta/2}\mathcal{F}[f](\tau)](t)\|^p_X \,dt\right)^{1/p}.\end{aligned}$$

We set $W^0_q(D) = L_q(D)$ and $W^s_q(D) = B^s_{q,q}(D)$. $C^\infty(D)$ denotes the set all C^∞ functions defined on D. $L_p((a, b), X)$ and $W^m_p((a, b), X)$ denote the usual Lebesgue space and Sobolev space of X-valued function defined on an interval (a, b), while $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W^m_p((a,b),X)}$ denote their respective norms.Moreover, we set

$$\|e^{\eta t} f\|_{L_p((a,b),X)} = \left(\int_a^b (e^{\eta t}\|f(t)\|_X)^p dt\right)^{1/p} \quad \text{for } 1 \leq p < \infty.$$

The d-product space of X is defined by $X^d = \{f = (f, \dots, f_d) \mid f_i \in X\,(i = 1, \dots, d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. We set

$$\begin{aligned}&W^{m,\ell}_q(D) = \{(f, \mathbf{g}, \mathbf{H}) \mid f \in W^m_q(D),\\ &\mathbf{g} \in W^\ell_q(D)^N,\ \ \mathbf{H} \in W^m_q(D)^{N\times N}\},\\ &\|(f, \mathbf{g}, \mathbf{H})\|_{W^{m,\ell}_q(\Omega)} = \|(f, \mathbf{H})\|_{W^m_q(\Omega)} + \|\mathbf{g}\|_{W^\ell_q(\Omega)},\\ &L_{p,\gamma_1}(\mathbb{R}, X) = \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R}, X) \mid e^{-\gamma_1 t} f(t) \in L_p(\mathbb{R}, X)\},\\ &L_{p,\gamma_1,0}(\mathbb{R}, X) = \{f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0\ (t < 0)\},\\ &W^m_{p,\gamma_1}(\mathbb{R}, X) = \{f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid e^{-\gamma_1 t}\partial^j_t f(t) \in L_p(\mathbb{R}, X)\\ &(j = 1, \dots, m)\},\\ &W^m_{p,\gamma_1,0}(\mathbb{R}, X) = W^m_{p,\gamma_1} \cap L_{p,\gamma_1,0}(\mathbb{R}, X).\end{aligned}$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}^{-1}_\xi = \mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by

$$\begin{aligned}\mathcal{F}_x[f](\xi) &= \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)dx\\ \mathcal{F}^{-1}_\xi[g](x) &= \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)d\xi.\end{aligned}$$

We also write $\hat{f}(\xi) = \mathcal{F}_x[f](\xi)$. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform, respectively, which are defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^\infty e^{-\lambda t} f(t)dt, \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{\lambda t} g(\tau)d\tau,$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. Given $s \in \mathbb{R}$ and X-valued function $f(t)$, we set

$$\Lambda^s_\gamma f(t) = \mathcal{L}^{-1}_\lambda[\lambda^s \mathcal{L}[f](\lambda)](t).$$

We introduce the Bessel potential space of X-valued functions of order s as follows:

$$\begin{aligned}H^s_{p,\gamma_1}(\mathbb{R}, X) &= \{f \in L_p(\mathbb{R}, X) \mid e^{-\gamma t}\Lambda^s_\gamma[f](t) \in L_p(\mathbb{R}, X)\\ &\qquad \text{for any } \gamma \geq \gamma_1\},\\ H^s_{p,\gamma_1,0}(\mathbb{R}, X) &= \{f \in H^s_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0\ (t < 0)\}.\end{aligned}$$

For $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, we set $\mathbf{x} \cdot \mathbf{y} =< \mathbf{x}, \mathbf{y} >= \sum_{j=1}^{n} x_j y_j$. For scalar functions f, g and N-vectors of functions $\mathbf{k}$, $\mathbf{g}$ we set $(k, g)_D = \int_D kg\, dx$, $(\mathbf{k}, \mathbf{g})_D = \int_D \mathbf{k} \cdot \mathbf{g}\, dx$, $(k, g)_\Gamma = \int_\Gamma kg\, d\sigma$, $(\mathbf{k}, \mathbf{g})_\Gamma = \int_\Gamma \mathbf{k} \cdot \mathbf{g} d\sigma$, where σ is the surface element of Γ. For $N \times N$ matrices of functions $\mathbf{F} = (F_{ij})$ and $\mathbf{G} = (G_{ij})$, we set $(\mathbf{F}, \mathbf{G})_D = \int_D \mathbf{F} : \mathbf{G}\, dx$ and $(\mathbf{F}, \mathbf{G})_\Gamma = \int_\Gamma \mathbf{F} : \mathbf{G}\, d\sigma$, where $\mathbf{F} : \mathbf{G} \equiv \sum_{i,j=1}^{N} F_{ij} G_{ij}$ and $|\mathbf{F}| \equiv \left(\sum_{i,j=1}^{n} F_{ij} F_{ij} \right)^{1/2}$. Moreover, $\mathbf{x} \cdot \mathbf{F}$ means vectors with components $\sum_{i=1}^{n} a_i F_{ij}$. Let $C_0^\infty(G)$ be the set of all C^∞ functions whose supports are compact and contained in G. The letter C denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a, b, \ldots$. The values of constants C and $C_{a,b,\ldots}$ denote a positive constant which may be different even in a single chain of inequalities. We use small boldface letters, e.g. $\mathbf{u}$ to denote vector-valued functions and capital boldface letters, e.g. $\mathbf{H}$ to denote matrix-valued functions, respectively. But, we also use the Greek letters, e.g. $\rho, \theta, \tau, \omega$, such as to denote mass densities, and elastic tensors in case the confusion may occur, although they are $N \times N$ matrices.

Research methodology of this paper is literature review. In this article, we consider the $\mathcal{R}$-Boundedness of the operator solution of the Navier-Lamé equation with surface tension in half-space case. The procedures of how to prove the purpose of the article are explained in the following. First of all, we define half-space and its boundary, then by using the partial Fourier transform and inverse partial Fourier transform of resolvent problem of (1) in whole and half-space, we get new solution formula of velocity and also density of Navier-Lamé equations. In the end, we use Weis's operator valued Fourier multiplier for time dependent problem.

2 Result and Discussion

2.1 Main Theorem

Before stating our main result, firstly, we introduce the definition of $\mathcal{R}$-boundedness and the operator valued Fourier multiplier theorem due to Weis [19]. The following theorem is obtained by Weis [19].

Theorem 2.1. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{(\tau \frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R}\backslash\{0\}\}) \leq \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (5) is extended to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \leq C\kappa$$

for some positive constant C depending on p, X and Y.

Definition 2.2. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, we have the inequality:

$$\left\{ \int_0^1 \| \sum_{j=1}^n r_j(u) T_j x_j \|_Y^p\, du \right\}^{1/p} \leq C \left\{ \int_0^1 \| \sum_{j=1}^n r_j(u) x_j \|_X^p\, du \right\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bounded of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X valued C^∞ functions having compact support and the Schwartz space of rapidly decreasing X valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R}\backslash\{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M \phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)). \tag{5}$$

Remark 2.3. For the definition of UMD space, we refer to a book due to Amann [1]. For $1 < q < \infty$, Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

We quote a proposition [4], which tell us that $\mathcal{R}$-bounds behave like norms.

Lemma 2.4. *Let X, Y and Z be Banach space and $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families*

1. *If X and Y be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then $\mathcal{T} + \mathcal{S} = \{T + S | T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S})$$

2. *If X, Y and Z be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X,Y)$ and $\mathcal{L}(Y,Z)$, respectively. Then $\mathcal{ST} = \{ST | T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X,Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{TS}) \leq \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S})$$

Definition 2.5. Let V be a domain in $\mathbb{C}$, let $\Xi = V \times (\mathbb{R}^{N-1} \setminus \{0\})$, and let $m : \Xi \to \mathbb{C}; (\lambda, \xi') \mapsto m(\lambda, \xi')$ be $\mathcal{C}^1$ with respect to τ, where $\lambda = \gamma + i\tau \in V$, and C^∞ with respect to $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$.

1. $m(\lambda, \xi')$ is called a multiplier of order s with type 1 on Ξ, if the estimates:

$$|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|},$$
$$|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|}$$

hold for any multi-index $\kappa \in \mathbb{N}_0^N$ and $(\lambda, \xi') \in \Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and V.

2. $m(\lambda, \xi')$ is called a multiplier of order s with type 2 on Ξ, if the estimates:

$$|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|},$$
$$|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|}$$

hold for any multi-index $\kappa \in \mathbb{N}_0^N$ and $(\lambda, \xi') \in \Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and V.

Let $\mathbf{M}_{s,i}(V)$ be the set of all multipliers of order s with type i on Ξ for $i = 1, 2$. For $m \in \mathbf{M}_{s,i}(V)$, we set $M(m, V) = \max_{|\kappa'| \leq N} C_{\kappa'}$.

Let $\mathcal{F}_{\xi'}^{-1}$ be the inverse partial Fourier transform defined by

$$\mathcal{F}_{\xi'}^{-1}[f(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{i\xi'\cdot\xi'} f(\xi', x_N)\, d\xi'.$$

Then, we have the following two lemmas which have proved essentially by Shibata and Shimizu [17, Lemma 5.4 and Lemma 5.6].

Lemma 2.6. *Let $\epsilon \in (0, \pi/2)$, $q \in (1, \infty)$ and $\lambda_0 > 0$. Given $m \in \mathbf{M}_{-2,1}(\Sigma_{\epsilon,\lambda_0})$, we define an operator $L(\lambda)$ by*

$$[L(\lambda)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi')\lambda^{1/2} e^{-B(x_n+y_N)}\hat{g}(\xi', y_N)]$$
$$(x')\, dy_N.$$

Then, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N), W_q^{2-j}(\mathbb{R}_+^N)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}\partial_x^\alpha L(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\})$$
$$\leq r_b(\lambda_0) \quad (\ell = 0, 1), (j = 0, 1, 2).$$

where τ denotes the imaginary part of λ, and $r_b(\lambda_0)$ is a constant depending on $M(m, \Sigma_{\epsilon,\lambda_0})$, ϵ, λ_0, N, and q.

Lemma 2.7. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ dan $\lambda_0 > 0$. Let $m(\lambda, \xi')$ be a function defined on $\Sigma_{\epsilon,\lambda_0}$ and $m \in \mathbf{M}_{-2,2}(\Sigma_{\epsilon,\lambda_0})$ such that for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ there exists a constant $C_{\kappa'}$ such that*

$$|\partial_{\xi'}^{\kappa'}\{(\tau\frac{\partial}{\partial\tau})^\ell m(\lambda, \xi')\}| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|^{-2-|\kappa'|})$$
$$(\ell = 0, 1) \tag{6}$$

for any $(\lambda, \xi') \in \Sigma_{\epsilon,\lambda_0}$,. Let $\Psi_j(\lambda)$ $(j = 1, \ldots, 4)$ be operators defined by

$$\Psi_1(\lambda)f = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi') B e^{-B(x_N+y_N)} \mathcal{F}_{x'}[f](\xi', y_N)]$$
$$(x')\, dy_N,$$
$$\Psi_2(\lambda)f = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi') B\, M(x_N+y_N) \mathcal{F}_{x'}[f](\xi', y_N)]$$
$$(x')\, dy_N,$$
$$\Psi_3(\lambda)f = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi') AB\, M(x_N+y_N) \mathcal{F}_{x'}[f](\xi', y_N)]$$
$$(x')\, dy_N,$$
$$\Psi_4(\lambda)f = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi') B^2 M(x_N+y_N) \mathcal{F}_{x'}[f](\xi', y_N)]$$
$$(x')\, dy_N.$$

Then, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+),L_q(\mathbb{R}^N_+)^{\tilde{N}})}(\{(\tau\frac{d}{d\tau})^\ell(G_\lambda\Psi_i(\lambda)) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \le C$$
$$(\ell=0,1,\ i=1,2,3,4)$$

with some constant C. Here and hereafter, $C_{\kappa'}$ denotes a generic constant depending on κ', ϵ, λ_0.

The proof of the Lemma can be seen in [6], [3] and [8].

Lemma 2.8. *Let $1<q<\infty$ and let Λ be a set in $\mathbb{C}$. Let $m=M(\lambda,\xi)$ be a function defined on $\Lambda\times(\mathbb{R}^N\setminus\{0\})$ which is infinitely differentiable with respect to $\xi\in\mathbb{R}^N\setminus\{0\}$ for each $\lambda\in\Lambda$. Assume that for any multi-index $\alpha\in\mathbb{N}_0^N$ there exists a constant C_α depending on α and Λ such that*

$$|\partial_\xi^\alpha m(\lambda,\xi)|\le C_\alpha|\xi|^{-|\alpha|} \tag{7}$$

for any $(\lambda,\xi)\in\Lambda\times(\mathbb{R}^N\setminus\{0\})$. Let K_λ be an operator defined by

$$K_\lambda f=\mathcal{F}^{-1}[m(\lambda,\xi)\mathcal{F}[f](\xi)]. \tag{8}$$

Then, the family of operators $\{K_\lambda \mid \lambda\in\Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda \mid \lambda\in\Lambda\})\le C_{q,N}\max_{|\alpha|\le N+1}C_\alpha \tag{9}$$

for some $C_{q,N}$ depending only on q and N.

The following theorem is the main theorem of this article.

Theorem 2.9. *Let $1<q<\infty,\ 0<\epsilon<\pi/2$ and $N<r<\infty$. Assume that $r\ge\max(q,q')$ and $\lambda\in\Sigma_{\epsilon,\lambda_0}$. Set*

$$\begin{aligned}Z_q(\mathbb{R}^N_+)=&\{(\mathbf{f},\mathbf{g},d)\mid f\in L_q(\mathbb{R}^N_+),\mathbf{g}\in W_q^1(\mathbb{R}^N_+)^N,\\ &d\in W_q^{2-1/q}(\mathbb{R}^N_0)\},\\ \mathcal{Z}_q(\mathbb{R}^N_+)=&\{(\mathbf{F}_1,\mathbf{F}_2,\mathbf{F}_3,F_4)|\mathbf{F}_1\in L_q(\mathbb{R}^N_+)^N,\mathbf{F}_2\in L_q(\mathbb{R}^N_+)^N,\\ &\mathbf{F}_3\in L_q(\mathbb{R}^N_+)^{N^2},F_4\in W_q^{2-1/q}(\mathbb{R}^N_0)\}.\end{aligned}$$

Then, there exists a $\lambda_0\ge1$ and an operator family $R(\lambda)$ and $R_1(\lambda)$ with

$$\begin{aligned}R(\lambda)&\in\mathrm{Hol}(\Sigma_{\epsilon,\lambda_0},\mathcal{L}(\mathcal{Z}_q(\mathbb{R}^N_+),\mathrm{W}_q^2(\mathbb{R}^N_+)))\\ R_1(\lambda)&\in\mathrm{Hol}(\Sigma_{\epsilon,\lambda_0},\mathcal{L}(\mathcal{Z}_q(\mathbb{R}^N_+),\mathrm{W}_q^{3-1/q}(\mathbb{R}^N_0)))\end{aligned} \tag{10}$$

such that for any $(\mathbf{f},\mathbf{g},d)\in Z_q(\mathbb{R}^N_+)$ and $\lambda\in\Sigma_{\epsilon,\lambda_0}$, $\mathbf{u}=R(\lambda)(\mathbf{f},\lambda^{1/2}\mathbf{g},\nabla\mathbf{g},d)$ and $\eta=R_1(\lambda)(\mathbf{f},\lambda^{1/2}\mathbf{g},\nabla\mathbf{g},d)$ are unique solutions to problem (3)*. Moreover, there exists a constant r_b such that*

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(\mathbb{R}^N_+),W_q^{2-j}(\mathbb{R}^N_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}R(\lambda))\mid\lambda\in\Sigma_{\epsilon,\lambda_0}\})\le r_b\\ &(\ell=0,1,\ j=0,1,2),\\ &\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(\mathbb{R}^N_+),W_q^{3-k}(\mathbb{R}^N_+))}(\{(\tau\partial\tau)^\ell(\lambda^kR_1(\lambda))\mid\lambda\in\Sigma_{\epsilon,\lambda_0}\})\le r_b\\ &(\ell=0,1,\ k=0,1),\end{aligned} \tag{11}$$

with $\lambda=\gamma+i\tau$.

Remark 2.10. The $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and F_4 are variables corresponding to $\mathbf{f}$, $\lambda^{1/2}\mathbf{g}$, $\nabla\mathbf{g}$ and d, respectively.

The resolvent parameter λ in problem (3) varies in $\Sigma_{\epsilon,\lambda_0}$ with

$$\begin{aligned}\Sigma_{\epsilon,\lambda_0}=&\{\lambda\in\mathbb{C}\mid|\arg\lambda|\le\pi-\epsilon,\ \ |\lambda|\ge\lambda_0\}\\ &(\epsilon\in(0,\pi/2),\lambda_0>0).\end{aligned} \tag{12}$$

The following section discusses the $\mathcal{R}$-boundedness of the solution operator in the whole space problem.

2.2 On the $\mathcal{R}$-boundedness of the solution operator in $\mathbb{R}^N$

In this section, we consider the $\mathcal{R}$-boundedness of the solution operator of the Navier-Lamé equation:

$$\lambda \mathbf{u} - \alpha\Delta \mathbf{u} - \beta\nabla \mathrm{div}\, \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \tag{13}$$

where α and β are positive constants. Applying div to (13), we have

$$(\lambda - (\alpha+\beta)\Delta)\mathrm{div}\, \mathbf{u} = \mathrm{div}\, \mathbf{f} \tag{14}$$

Substituting (14) to (13) we have the formula of $\mathbf{u}$, that is

$$\mathbf{u} =(\lambda - \alpha\Delta)^{-1}\mathbf{f} + \beta\nabla[(\lambda-\alpha\Delta)^{-1}(\lambda - (\alpha+\beta)\Delta)^{-1}\mathrm{div}\, \mathbf{f}] \tag{15}$$

By the Fourier transform and the inverse Fourier transform for $\mathbf{f} = (f_1, \dots, f_N)$ we have $\mathcal{S}_0(\lambda)\mathbf{f} = (u_1, \dots, u_N)$ then we can write equation (15) to be

$$\begin{aligned}\mathcal{S}_0(\lambda)\mathbf{g} =&\mathcal{F}_\xi^{-1}\left[\frac{\mathcal{F}[\mathbf{f}](\xi)}{\lambda+\alpha|\xi|^2}\right]\\ &+\beta\mathcal{F}_\xi^{-1}\left[\frac{\xi\xi\cdot\mathcal{F}[\mathbf{f}](\xi)}{(\lambda+\alpha|\xi|^2)(\lambda+(\alpha+\beta)|\xi|^2)}\right].\end{aligned} \tag{16}$$

Related to the spectrum, we know the following lemma which is proved by Shibata and Tanaka [14].

Lemma 2.11. *Let $0 < \epsilon < \frac{\pi}{2}$, $\Sigma_{\epsilon,\lambda_0}$ as defined in* (12) *Then we have the following assertion*

1. *For any $\lambda \in \Sigma_\epsilon$ and $\xi \in \mathbb{R}^N$ we have*

$$|\alpha^{-1}\lambda + |\xi|^2| \geq \sin(\frac{\epsilon}{2})(\alpha^{-1}|\lambda| + |\xi|^2) \tag{17}$$

2. *For any $\lambda_0 > 0$ we have*

$$|arg(\alpha^{-1}\lambda)| \leq \pi - \epsilon$$

The following theorem is the main result of this section.

Theorem 2.12. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and we assume that $\alpha > 0$, $\alpha+\beta > 0$. Let $\mathcal{S}_0(\lambda)$ be the operator defined in 16. Then, $\mathcal{S}_0(\lambda) \in Hol(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(L_q(\mathbb{R}^N)^N, W_q^2(\mathbb{R}^N)^N))$. For any $\mathbf{f} \in L_q(\mathbb{R}^N)^N$ and $\lambda \in \Sigma_{\epsilon,\lambda_0}$, $\mathbf{u} = \mathcal{S}_0(\lambda)\mathbf{f}$ is a unique solution to the problem* (13) *and we have*

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N)^N, L_q(\mathbb{R}^N)^{\tilde{N}})}(\{(\tau\frac{d}{d\tau})^\ell(G_\lambda\mathcal{S}_0(\lambda)) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \leq C\\ &(\ell = 0,1)\end{aligned} \tag{18}$$

for $\lambda = \gamma + i\tau$ and some constant C depends solely on ϵ, λ_0, γ, q and N, $G_\lambda\mathbf{u} = (\lambda\mathbf{u}, \gamma\mathbf{u}, \lambda^{1/2}\nabla\mathbf{u}, \nabla^2\mathbf{u})$.

2.3 On the $\mathcal{R}$-boundedness solution operator in $\mathbb{R}^N_+$; $\sigma > 0,\ a = 0$

In this section we consider the following generalized resolvent problem of the equation (3) which can be written in the following:

$$\begin{cases} \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} = \mathbf{f} & \text{in } \mathbb{R}^N_+,\\ (\alpha\mathbf{D}(\mathbf{u}) - (\beta-\alpha)\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n} - \sigma(\Delta_\Gamma\eta)\mathbf{n} = \mathbf{g} & \text{on } \mathbb{R}^N_0,\\ \lambda\eta - \mathbf{u}\cdot\mathbf{n} = d & \text{on } \mathbb{R}^N_0. \end{cases} \tag{19}$$

where $\mathbf{n} = (0, \dots, 0, -1) \in \mathbb{R}^N$ and $\Delta'\eta = \sum_{j=1}^{N-1}\partial^2\eta/\partial x_j^2$.

Furthermore, we consider the following equation system:

$$\begin{cases} \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} = 0 & \text{in } \Omega,\\ (\alpha\mathbf{D}(\mathbf{u}) - (\beta-\alpha)\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n} - \sigma(\Delta_\Gamma\eta)\mathbf{n} = 0 & \text{on } \Gamma,\\ \lambda\eta + \mathbf{a}'\cdot\nabla'\eta - \mathbf{u}\cdot\mathbf{n} = d & \text{on } \mathbb{R}^N_0. \end{cases} \tag{20}$$

Then, we shall prove the following theorem

Theorem 2.13. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $\lambda_1 > 0$ and operator families $\mathcal{U}(\lambda)$ and $\mathcal{V}(\lambda)$ with*

$$\mathcal{U}(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Z}_{\mathrm{q}}(\mathbb{R}^{\mathrm{N}}_+), \mathrm{W}^2_{\mathrm{q}}(\mathbb{R}^{\mathrm{N}}_+)))$$
$$\mathcal{V}(\lambda) \in \mathrm{Hol}(\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Z}_{\mathrm{q}}(\mathbb{R}^{\mathrm{N}}_+), \mathrm{W}^3_{\mathrm{q}}(\mathbb{R}^{\mathrm{N}}_+)))$$

such that for any $d \in W^2_q(\mathbb{R}^N_+)^N$, $\mathbf{u} = \mathcal{U}(\lambda)d$ and $\eta = \mathcal{V}(\lambda)d$ are unique solutions of equation (20)*. Moreover, the following estimate holds:*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(\mathbb{R}^N_+), W^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}\mathcal{U}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b(\lambda_1) \quad (\ell = 0,1,\ j = 0,1,2),$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Z}_q(\mathbb{R}^N_+), W^{3-k}_q(\mathbb{R}^N_+))}(\{(\tau\partial\tau)^\ell(\lambda^{k}\mathcal{V}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le r_b(\lambda_1) \quad (\ell = 0,1,\ k = 0,1).$$

We have Theorem 2.9 immediately with help of the Theorem 2.13.

First of all, applying the partial Fourier transform to equation (20), we have for $x_N > 0$ for first and second equation in the following

$$\begin{cases} \alpha(\alpha^{-1}\lambda + |\xi'|^2)\hat{u}_j - \alpha\partial_N^2\hat{u}_N - \beta i\xi_j(i\xi'\cdot\hat{u}' + \partial_N\hat{u}_N) = 0, \\ \alpha(\alpha^{-1}\lambda + |\xi'|^2)\hat{u}_N - \alpha\partial_N^2\hat{u}_N - \beta\partial_N(i\xi'\cdot\hat{u}' + \partial_N\hat{u}_N) = 0, \\ \alpha(\partial_N\hat{u}_j + i\xi_j\hat{u}_N)\mid_{x_N=0} = 0, \\ 2\alpha\partial_N\hat{u}_N + (\beta-\alpha)(i\xi'\cdot\hat{u}' + \partial_N\hat{u}_N)\mid_{x_N=0} = -\sigma|\xi'|^2\hat{\eta} \\ \lambda\hat{\eta} + \hat{u}_N \mid_{x_N=0} = \hat{d} \end{cases} \tag{21}$$

with $i\xi'\cdot\hat{u}' = \sum_{k=1}^{N-1} i\xi_k\hat{u}_k$, $\xi' = (\xi_1,\ldots,\xi_{N-1})$ and $\hat{f} = \hat{f}(\xi', x_N) = \int_{\mathbb{R}^{N-1}} e^{-ix'\cdot\xi'} f(x', x_N)dx'$. Here and hereafter, j runs from 1 to $N-1$. Since $(\lambda - \alpha\Delta)(\lambda - (\alpha+\beta)\Delta)\hat{\mathbf{u}} = 0$ as was seen in (14), we have $(\partial_N^2 - A^2)(\partial_N^2 - B^2)\hat{\mathbf{u}} = 0$ with

$$A = \sqrt{(\alpha+\beta)^{-1}\lambda + |\xi'|^2}, \quad B = \sqrt{\alpha^{-1}\lambda + |\xi'|^2}.$$

We look for a solution $\hat{\mathbf{u}} = (\hat{u}_1,\ldots,\hat{u}_N)$ of the form

$$\hat{u}_\ell = (P_\ell + Q_\ell)e^{-Bx_N} - P_\ell e^{-Ax_N} \tag{22}$$

for $\ell = 1,\ldots,N$

First of all, by substituting (22) into (21) and equating the coefficients of e^{-Ax_N} and e^{-Bx_N}, we have

$$\begin{cases} \alpha(B^2 - A^2)P_j - \beta i\xi_j(i\xi'\cdot P' - AP_N) = 0, \\ \alpha(B^2 - A^2)P_N + \beta A(i\xi'\cdot P' - AP_N) = 0, \\ i\xi'\cdot P' + i\xi'\cdot Q' - B(P_N + Q_N) = 0, \\ \alpha((B-A)P_j + BQ_j - i\xi_j Q_N) = 0 \\ (\alpha+\beta)(B(P_N+Q_N) - AP_N) - \beta i\xi'\cdot Q' = \sigma|\xi'|^2\hat{\eta} \end{cases} \tag{23}$$

with $i\xi'\cdot R' = \sum_{k=1}^{N-1} i\xi_k R_k$ for $R = P$ and Q. We consider $i\xi'\cdot Q'$ and Q_N as two unknowns to solve the linear equations (23). Then by the second and the third equation in (23), we have

$$i\xi'\cdot P' = \frac{|\xi'|^2}{AB - |\xi'|^2}(i\xi'\cdot Q' - BQ_N),$$
$$P_N = \frac{A}{AB - |\xi'|^2}(i\xi'\cdot Q' - BQ_N) \tag{24}$$

Since $i\xi'\cdot\hat{k}'(0) = \alpha((B-A)i\xi'\cdot P' + Bi\xi'\cdot Q') + \alpha|\xi'|^2 Q_N)$ as follows from the fourth equation of (23), combining this formula with the last equation in (23) and (24) and setting

$$L_{11} = \frac{\alpha A(B^2 - |\xi'|^2)}{AB - |\xi'|^2}$$
$$L_{12} = \frac{\alpha|\xi'|^2(2AB - |\xi'|^2 - B^2)}{AB - |\xi'|^2}$$
$$L_{21} = \frac{2\alpha A(B-A) - (\beta-\alpha)(A^2 - |\xi|^2)}{AB - |\xi'|^2}$$
$$L_{22} = \frac{(\alpha+\beta)B(A^2 - |\xi|^2)}{AB - |\xi'|^2} \tag{25}$$

we have a linear system:

$$L \begin{bmatrix} i\xi' Q' \\ Q_N \end{bmatrix} = \begin{bmatrix} 0 \\ \sigma|\xi'|^2\eta \end{bmatrix} \tag{26}$$

with Lopatinski matrix

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}. \tag{27}$$

The analysis of the Lopatinski determinant can be seen in Götz and Shibata [3].

If $\det L \neq 0$ at $(\lambda, \xi') \in \Sigma_{\epsilon,\lambda_0}$, then it follows from (26) that

$$\begin{aligned} i\xi' \cdot P' &= \frac{|\xi'|^2}{(\det L)(AB - |\xi'|^2)} M, \\ P_N &= \frac{A}{(\det L)(AB - |\xi'|^2)} M \end{aligned} \tag{28}$$

with $M = -(L_{12+BL_{11}})\sigma|\xi'|^2\eta$. By (28), we have

$$i\xi' \cdot P' - AP_N = \frac{(|\xi'|^2 - A^2)}{(\det L)(AB - |\xi'|^2)} M, \tag{29}$$

so that by (23) we have

$$\begin{cases} P_j = -\dfrac{\beta i\xi_j(|\xi'|^2 - A^2)}{\alpha(B^2 - A^2)\det L(AB - |\xi'|^2)}(L_{12+BL_{11}})\sigma|\xi'|^2\hat{\eta} \\ P_N = \dfrac{\beta A(|\xi'|^2 - A^2)}{\alpha(B^2 - A^2)(\det L)(AB - |\xi'|^2)}(L_{12+BL_{11}})\sigma|\xi'|^2\hat{\eta} \\ Q_j = \dfrac{i\xi_j}{B\det L}\Bigg[\dfrac{\beta(|\xi'|^2 - A^2)}{\alpha(A+B)(AB - |\xi'|^2)}(L_{12} + BL_{11}) \\ \quad +L_{11}\Bigg]\sigma|\xi'|^2\hat{\eta} \\ Q_N = \dfrac{L_{11}}{\det L}\sigma|\xi'|^2\hat{\eta} \end{cases} \tag{30}$$

Thus, combining (23) and (30) and setting $\omega = \beta/\alpha$, we have

$$\begin{aligned} \hat{u}_j(\xi', x_N) = &-\frac{\omega(i\xi_j)(L_{12} + BL_{11})}{B(B+A)\det L}\frac{|\xi'|^2 - A^2}{AB - |\xi'|^2} \\ &(BM(x_N) - e^{-Bx_N})\sigma|\xi'|^2\hat{\eta} \\ &+ \frac{(i\xi_j)L_{11}}{B\det L}e^{-Bx_N}\sigma|\xi'|^2\hat{\eta} \end{aligned}$$

and,

$$\begin{aligned} \hat{u}_N(\xi', x_N) = &\frac{\omega A(L_{12} + BL_{11})}{(B+A)\det L}\frac{|\xi'|^2 - A^2}{AB - |\xi'|^2}M(x_N)\sigma|\xi'|^2\hat{\eta} \\ &+ \frac{L_{11}}{\det L}e^{-Bx_N}\sigma|\xi'|^2\hat{\eta}. \end{aligned} \tag{31}$$

with $M(x_N) = \dfrac{e^{-Bx_N} - e^{-Ax_N}}{B - A}$.

Inserting the formula of $\hat{u}_N(\xi', x_N)|_{x_N=0}$ into the last equation of (21), we have

$$\lambda\hat{\eta} + \frac{L_{11}}{\det L}\sigma|\xi'|^2\hat{\eta} = \hat{d}$$

which implies that

$$\hat{\eta} = \frac{\det L}{G}\,\hat{d} \tag{32}$$

with

$$G = (\lambda\det L + L_{11}\sigma|\xi'|^2). \tag{33}$$

Lemma 2.14. *Let* $0 < \epsilon < \pi/2$ *and* G *be the function defined in* (33). *Then, there exist* $\lambda_1 > 0$ *and* $C > 0$ *such that the estimate:*

$$|G| \geq C(|\lambda| + |\xi'|)(|\lambda|^{1/2} + |\xi'|)^3 \tag{34}$$

holds for $(\lambda, \xi') \in \Sigma_{\epsilon,\lambda_1} \times (\mathbb{R}^{N-1} \setminus \{0\})$.

Proof. Firstly, by using Lemma 5.1 in [3] and technique of the proof of the Lemma 2.14 which can be seen in Shibata [16] we can proof the Lemma 2.14. □

Thus, by substituting the solution formula (33), the equation (31) can be written in the following

$$\begin{aligned}\hat{u}_j(\xi', x_N) =& -\frac{\omega(i\xi_j)(L_{12} + BL_{11})}{B(B+A)}\frac{|\xi'|^2 - A^2}{AB - |\xi'|^2}(BM(x_N)\\ & - e^{-Bx_N})\sigma|\xi'|^2\frac{\hat{d}}{G}\\ & + \frac{(i\xi_j)L_{11}}{B}e^{-Bx_N}\sigma|\xi'|^2\frac{\hat{d}}{G}\end{aligned}$$

and,

$$\begin{aligned}\hat{u}_N(\xi', x_N) =& \frac{\omega A(L_{12} + BL_{11})}{(B+A)\det L}\frac{|\xi'|^2 - A^2}{AB - |\xi'|^2}M(x_N)\sigma|\xi'|^2\frac{\hat{d}}{G}\\ & + \frac{L_{11}}{G}e^{-Bx_N}\sigma|\xi'|^2\hat{d}.\end{aligned} \tag{35}$$

By using the Volevich trick

$$\begin{aligned}p(\xi', x_N)q(\xi', 0) =& -\int_0^\infty \frac{\partial}{\partial y_N}(p(\xi', x_N + y_N)q(\xi', y_N))dy_N\\ =& -\int_0^\infty \frac{\partial p}{\partial y_N}(\xi', x_N + y_N)q(\xi', y_N))dy_N\\ & -\int_0^\infty p(\xi', x_N + y_N)\frac{\partial q}{\partial y_N}(\xi', y_N))dy_N\end{aligned}$$

and the identities $1 = \dfrac{\lambda^{\frac{1}{2}}}{\alpha B^2}\lambda^{\frac{1}{2}} - \displaystyle\sum_{k=1}^{N-1}\frac{i\xi_k}{B^2}i\xi_k$ and $\partial_N M(x_N) = -e^{-Bx_N} - AM(x_N)$.

In view of equation (35) The solution formula for $u_j = \mathcal{U}_j(\lambda)d$ and $u_N = \mathcal{U}_N(\lambda)d$ can be written as follow

$$\begin{aligned}
\mathcal{U}_j(x) = &\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(i\xi_j)(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2}\frac{\sigma B}{G} \\
&AM(x_N+y_N)\mathcal{F}[\Delta' d](\xi',y_N)\Big](x')\,dy_N \\
&+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(i\xi_j)(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2}\frac{\sigma B}{G} \\
&e^{-B(x_N+y_N)}\mathcal{F}[\Delta' d](\xi',y_N)\Big](x')\,dy_N \\
&+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2} \\
&\frac{\sigma|\xi'|^2BM(x_N+y_N)}{G}\mathcal{F}[\partial_j\partial_N d](\xi',y_N)\Big](x')\,dy_N \\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(i\xi_j)(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2}\frac{\sigma B}{G} \\
&e^{-B(x_N+y_N)}\mathcal{F}[\Delta' d](\xi',y_N)\Big](x')\,dy_N \\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2}\frac{\sigma|\xi'|^2}{G} \\
&e^{-B(x_N+y_N)}\mathcal{F}[\partial_j\partial_N d](\xi',y_N)\Big](x')\,dy_N \\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{(i\xi_j)L_{11}}{B}\frac{\sigma B}{G} \\
&e^{-B(x_N+y_N)}\mathcal{F}_{x'}[\Delta' d](\xi',y_N)\Big](x')dy_N \\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}[\frac{L_{11}}{B^2}\frac{\sigma B|\xi'|^2}{G} \\
&e^{-B(x_N+y_N)}\mathcal{F}_{x'}[\partial_j\partial_N d](\xi',y_N)](x')dy_N
\end{aligned} \tag{36}$$

$$\begin{aligned}
\mathcal{U}_N(x) = &-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2}\frac{\sigma B}{G} \\
&AM(x_N+y_N)\mathcal{F}[\Delta' d](\xi',y_N)\Big](x')\,dy_N \\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2}\frac{\sigma B}{G} \\
&e^{-B(x_N+y_N)}\mathcal{F}[\Delta' d](\xi',y_N)\Big](x')\,dy_N \\
&+\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{\omega(L_{12}+BL_{11})}{B(B+A)}\frac{|\xi'|^2-A^2}{AB-|\xi'|^2} \\
&\frac{\sigma|\xi'|^2BM(x_N+y_N)}{G}\mathcal{F}[\partial_N d](\xi',y_N)\Big](x')\,dy_N \\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}\Big[\frac{L_{11}}{B}\frac{\sigma B}{G}e^{-B(x_N+y_N)}\mathcal{F}_{x'}[\Delta' d](\xi',y_N)\Big] \\
&(x')dy_N \\
&-\int_0^\infty \mathcal{F}^{-1}_{\xi'}[\frac{L_{11}}{B^2}\frac{\sigma B|\xi'|^2}{G}e^{-B(x_N+y_N)} \\
&\mathcal{F}_{x'}[\partial_N d](\xi',y_N)](x')dy_N
\end{aligned} \tag{37}$$

where we have used $\mathcal{F}[\Delta' d](\xi',y_N) = -|\xi'|^2\hat{d}(\xi',y_N)$. We have $\mathcal{U}_j(\lambda)d = u_j,\ j=1\ldots,N-1$ and $\mathcal{U}_N(\lambda)d = u_N$. By Lemma

2.14 and Lemma 2.15, we have

$$\mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}_+^N),W_q^{2-\kappa}(\mathbb{R}_+^N))}(\{(\tau\partial\tau)^\ell(\lambda^{k/2}\mathcal{U}_j(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b(\lambda_1) \quad (\ell = 0,1,\ k = 0,1,2),$$

where $r_b(\lambda_1)$ is a constant depending on m_0, m_1, m_2 and λ_1. Analogously, we have

$$\mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}_+^N),W_q^{2-\kappa}(\mathbb{R}_+^N))}(\{(\tau\partial\tau)^\ell(\lambda^{k/2}\mathcal{U}_N(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b(\lambda_1) \quad (\ell = 0,1,\ k = 0,1,2).$$

Furthermore, we construct the formula of η. Let $\phi(x_N)$ be a function in C_0^∞ such that $\phi(x_N) = 1$ for $|x_N| \leq 1$ and $\phi(x_N) = 0$ for $|x_N| \geq 2$. We define η by

$$\eta(x) = \phi(x_N)\mathcal{F}_{\xi'}^{-1}\left[e^{-Ax_N}\frac{\det L}{G}\hat{d}(\xi',0)\right](x').$$

By the Volevich trick, we have

$$\begin{aligned}
\eta(x) =& -\phi(x_N)\int_0^\infty \partial_N\mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{\det L}{G}\hat{d}(\xi',y_N)\phi(y_N)\right] \\
&(x')\,dy_N \\
=&\phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{A\det L}{G}\hat{d}(\xi',y_N)\phi(y_N)\right] \\
&(x')\,dy_N \\
&-\phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{\det L}{G}\partial_N(\hat{d}(\xi',y_N)\phi(y_N))\right] \\
&(x')\,dy_N \\
=&\phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{A\det L}{G(1+|\xi'|^2)}\right. \\
&\left.\mathcal{F}'[(1-\Delta')d](\xi',y_N)\phi(y_N)\right](x')\,dy_N \\
&-\phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{\det L}{G(1+|\xi'|^2)}\right. \\
&\left.\left(\partial_N(\hat{d}(\xi',y_N)\phi(y_N)) - \sum_{k-1}^{N-1} i\xi_k\partial_N(\mathcal{F}'[\partial_k d](\xi',y_N)\phi(y_N))\right)\right] \\
&(x')\,dy_N
\end{aligned}$$

Let $\mathcal{V}(\lambda)d\mid_{x_N=0}= \eta$ and recall the definition of η in (32).

By the Volevich trick, we have

$$\begin{aligned}
&\mathcal{V}(\lambda)d \\
&= -\phi(x_N)\int_0^\infty \partial_N\mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{\det L}{G}\hat{d}(\xi',y_N)\phi(y_N)\right](x')\,dy_N \\
&= \phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{A\det L}{G}\hat{d}(\xi',y_N)\phi(y_N)\right](x')\,dy_N \\
&- \phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{\det L}{G}\partial_N(\hat{d}(\xi',y_N)\phi(y_N))\right](x')\,dy_N \\
&= \phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{A\det L}{G(1+|\xi'|^2)}\right. \\
&\left.\mathcal{F}'[(1-\Delta')d](\xi',y_N)\phi(y_N)\right](x')\,dy_N \\
&- \phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\left[e^{-A(x_N+y_N)}\frac{\det L}{G(1+|\xi'|^2)}\right. \\
&\left.\left(\partial_N(\hat{d}(\xi',y_N)\phi(y_N)) - \sum_{k-1}^{N-1} i\xi_k\partial_N(\mathcal{F}'[\partial_k d](\xi',y_N)\phi(y_N))\right)\right](x')\,dy_N
\end{aligned}$$

Let $\mathcal{V}(\lambda)d = \phi(x_N)\{\mathcal{V}^1(\lambda)d + \mathcal{V}^2(\lambda)d\}$ with

$$\begin{aligned}\mathcal{V}^1(\lambda)d &= \int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[e^{-A(x_N+y_N)}\frac{A\det L}{G(1+|\xi'|^2)}\\ &\quad \mathcal{F}'[(1-\Delta')d](\xi',y_N)\phi(y_N)\Big](x')\,dy_N\\ \mathcal{V}^2(\lambda)d &= -\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[e^{-A(x_N+y_N)}\frac{\det L}{G(1+|\xi'|^2)}\\ &\quad \Big(\partial_N(\hat{d}(\xi',y_N)\phi(y_N))\\ &\quad -\sum_{k-1}^{N-1} i\xi_k\partial_N(\mathcal{F}'[\partial_k d](\xi',y_N)\phi(y_N))\Big)\Big]\\ &\quad (x')\,dy_N\end{aligned}$$

To treat η, we use the following lemma which had been proved by Shibata [9].

Lemma 2.15. *Let Σ be a domain in $\mathbb{C}$ and let $1 < q < \infty$. Let ϕ and ψ be two $C_0^\infty((-2,2))$ functions. Given $m_0 \in \mathbf{M}_{0,2}(\Sigma)$, we define an operator $L_6(\lambda)$ and $L_7(\lambda)$ acting on $g \in L_q(\mathbb{R}^N_+)$ by*

$$\begin{aligned}[L_6(\lambda)g](x) &= \phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[e^{-A(x_N+y_N)}m_0(\lambda,\xi')\\ &\quad \hat{g}(\xi',y_N)\psi(y_N)\Big]dy_N,\\ [L_7(\lambda)g](x) &= \phi(x_N)\int_0^\infty \mathcal{F}_{\xi'}^{-1}\Big[Ae^{-A(x_N+y_N)}m_0(\lambda,\xi')\\ &\quad \hat{g}(\xi',y_N)\psi(y_N)\Big]dy_N.\end{aligned}$$

Then,

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+))}(\{(\tau\partial_\tau)^\ell L_k(\lambda) \mid \lambda \in \Sigma\}) \le r_b$$

for some constants $k = 6,7$, $\ell = 0,1$ and r_b depending on $\Sigma_{\epsilon,\lambda_0}$

Proof. The lemma 2.15 of the model has been proved by Shibata [16]. Moreover, for $(j,\alpha',k) \in \mathbb{N}_0 \times \mathbb{N}_0^{N-1} \times \mathbb{N}_0$ with $j+|\alpha'+k| \le 3$ and $j = 0,1$, we write

$$\begin{aligned}\lambda^j\partial_{x'}^{\alpha'}\partial_N^k\mathcal{V}(\lambda)d &= \sum_{n=0}^{k}\binom{n}{k}(\partial_N^{k-n}\phi(x_N))\\ &\quad [\lambda^j\partial_{x'}^{\alpha'}\partial_N^k\mathcal{V}^1(\lambda)d\\ &\quad + \lambda^j\partial_{x'}^{\alpha'}\partial_N^k\mathcal{V}^2(\lambda)d]\end{aligned}$$

and then

$$
\begin{aligned}
&\lambda^j \partial_{x'}^{\alpha'} \partial_N^k \mathcal{V}^1(\lambda) d \\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[A e^{-A(x_N+y_N)} \frac{\lambda^j (i\xi')^{\alpha'} (-|\xi'|)^n \det L}{\tilde{G}(1+|\xi'|^2)} \right. \\
&\left. \mathcal{F}'[(1-\Delta')d](\xi', y_N)\phi(y_N) \right] \\
&\lambda^j \mathcal{V}^2(\lambda) d \\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[e^{-A(x_N+y_N)} \frac{\lambda^j \det L}{\tilde{G}} \partial_N(\hat{d}(\xi', y_N)\phi(y_N)) \right](x') dy_N \\
&\lambda^j \partial_{x'}^{\alpha'} \partial_N^k \mathcal{V}^2(\lambda) d \\
&= \int_0^\infty \mathcal{F}_{\xi'}^{-1} \left[e^{-A(x_N+y_N)} \frac{\lambda^j (i\xi')^{\alpha'} \det L}{\tilde{G}(1+|\xi'|^2)} \right. \\
&\left. \left(\partial_N(\hat{d}(\xi', y_N)\phi(y_N)) - \sum_{k-1}^{N-1} \frac{i\xi_k}{|\xi'|} \partial_N(\mathcal{F}'[\partial_k d](\xi', y_N)\phi(y_N)) \right) \right] \\
&(x') dy_N
\end{aligned} \tag{38}
$$

for $|\alpha'| + n \geq 1$, and we use the formula

$$
1 = \frac{1+|\xi'|^2}{1+|\xi'|^2} = \frac{1}{1+|\xi'|^2} - \sum_{j=1}^{N-1} \frac{|\xi'|}{1+|\xi'|^2} \frac{i\xi_j}{|\xi'|} i\xi_j
$$

for the third equation of (38).

We can see that for the multipliers in the equation (38) hold Lemma 2.15, then we have

$$
\begin{aligned}
&\mathcal{R}_{\mathcal{L}(W_q^2(\mathbb{R}_+^N), W_q^{3-k}(\mathbb{R}_+^N))}(\{(\tau \frac{d}{d\tau})(\lambda^k \mathcal{V}(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_*}\}) \leq r_b \\
&(k = 0, 1).
\end{aligned}
$$

This completes the proof of Theorem 2.13. □

Proof. Furthermore, we prove Theorem 2.9. Let $(\mathbf{f}, \mathbf{g}, d) \in Z_q(\mathbb{R}_+^N)$ and $(\mathbf{u}, \eta)$ be solutions of the equation (3). Setting $\mathcal{U}(\lambda) = (\mathcal{U}_1(\lambda), \ldots, \mathcal{U}_N(\lambda))$, by Theorem 2.13 we see that $\mathbf{u} = \mathcal{U}(\lambda)d$ and $\eta = \mathcal{V}(\lambda)d$ are unique solutions of equation (3), then we can see that given $\epsilon \in (0, \pi/2)$, there exists $\lambda > 0$ and operator families R and R_1 satisfying (10) such that $\mathbf{u} = R(\lambda)(\mathbf{f}, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, d)$ and $\eta = \mathcal{V}(\lambda)(\mathbf{g}, \lambda^{1/2}\mathbf{k}, \nabla\mathbf{k}, d)$ are unique solutions of equation (3). Moreover, the estimate (11) holds. This completes the proof of Theorem 2.9. In fact, in view of Definition $\mathcal{R}$-boundedness solution operator, for any $n \in \mathbb{N}$, we take $\{\lambda_j\}_{j=1}^n \subset \Sigma$, $\{g_j\}_{j=1}^n \subset L_q(\mathbb{R}_+^N)$ and $r_j(u)$ $(j = 1, \ldots, n)$ as Rademacher functions. By the Fubini-Tonelli theorem, we have

$$
\begin{aligned}
&\int_0^1 \| \sum_{j=1}^n r_j(u) L_6(\lambda_j) g_j \|_{L_q(\mathbb{R}_+^N)}^q \, du \\
&= \int_0^t \int_0^\infty \int_{\mathbb{R}^{N-1}} | \sum_{j=1}^n r_j(u) L_6(\lambda_j) g_j |^q \, dy' \, dx_N \, du \\
&= \int_0^\infty \left(\int_0^1 \| \sum_{j=1}^n r_j(u) L_6(\lambda_j) g_j \|_{L_q(\mathbb{R}^{N-1})}^q \, du \right) dx_N.
\end{aligned}
$$

For any $x_N \geq 0$, by Minkowski's integral inequality, Lemma 2.15 and Hölder's inequality, we have

$$\Big(\int_0^1 \|\sum_{j=1}^n r_j(u) L_6(\lambda_j) g_j\|^q_{L_q(\mathbb{R}^{N-1})}\,du\Big)^{1/q}$$
$$= |\phi(x_N)|\Big(\int_0^1 \|\int_0^\infty \mathcal{F}^{-1}_{\xi'}[\sum_{j=1}^n r_j(u) e^{-A(x_N+y_N)}$$
$$m_0(\lambda_j,\xi')\hat{g}_j(\xi',y_N)](y')\psi(y_N)dy_N\|^q_{L_q(\mathbb{R}^{N-1})}\,du\Big)^{1/q}$$
$$\leq |\phi(x_N)|\Big(\int_0^1 \Big(\int_0^\infty \|\mathcal{F}^{-1}_{\xi'}[\sum_{j=1}^n r_j(u) e^{-A(x_N+y_N)}$$
$$m_0(\lambda_j,\xi')\hat{g}_j(\xi',y_N)](y')\psi(y_N)dy_N\|_{L_q(\mathbb{R}^{N-1})}\,dy_N\Big)^q du\Big)^{1/q}$$
$$\leq |\phi(x_N)|\int_0^\infty \Big(\int_0^1 \|\mathcal{F}^{-1}_{\xi'}[\sum_{j=1}^n r_j(u) e^{-A(x_N+y_N)}$$
$$m_0(\lambda_j,\xi')\hat{g}_j(\xi',y_N)](y')\|^q_{L_q(\mathbb{R}^{N-1})}\,du\Big)^{1/q}|\psi(y_N)|\,dy_N$$
$$\leq |\phi(x_N)$$
$$|\int_0^\infty \Big(\int_0^1 \|\mathcal{F}^{-1}_{\xi'}[\sum_{j=1}^n r_j(u)\hat{g}_j(\cdot,y_N)]\|^q_{L_q(\mathbb{R}^{N-1})}\,du\Big)^{1/q}$$
$$|\psi(y_N)|\,dy_N$$
$$\leq |\phi(x_N)$$
$$|\int_0^\infty \Big(\int_0^1 \|\mathcal{F}^{-1}_{\xi'}[\sum_{j=1}^n r_j(u)\hat{g}_j(\cdot,y_N)]\|^q_{L_q(\mathbb{R}^{N-1})}\,du\,dy_N\Big)^{1/q}$$
$$\Big(\int_0^\infty |\psi(y_N)|^{q'}\,dy_N\Big)^{1/q'}$$
$$\leq |\phi(x_N)|\int_0^\infty \Big(\int_0^1 \|\mathcal{F}^{-1}_{\xi'}[\sum_{j=1}^n r_j(u)\hat{g}_j(\cdot,y_N)]\|^q_{L_q(\mathbb{R}^N_+)}\,du\Big)^{1/q}$$
$$\Big(\int_0^\infty |\psi(y_N)|^{q'}\,dy_N\Big)^{1/q'}.$$

In fact since,

$$|\partial^{\alpha'}_{\xi'}(e^{-A(x_N+y_N)}m_0(\lambda,\xi'))| \leq C_{\alpha'}|\xi'|^{-|\alpha'|}$$

for any $x_N \geq 0$, $y_N \geq 0$, $(\lambda,\xi') \in \Sigma \times (\mathbb{R}^{N-1}\setminus\{0\})$, and $\alpha' \in \mathbb{N}^{N-1}$, by Lemma 2.8 we have

$$\int_0^1 \|\sum_{j=1}^n r_j(u)$$
$$\mathcal{F}^{-1}_{\xi'}\Big[e^{-A(x_N+y_N)}m_0(\lambda_j,\xi')\hat{g}_j(\xi',y_N)\Big](y')\|^q_{L_q(\mathbb{R}^{N-1})}\,du$$
$$\leq C\int_0^1 \|\sum_{j=1}^n r_j(u) g_j(\cdot,y_N)\|^q_{L_q(\mathbb{R}^{N-1})}\,du.$$

Putting these inequalities together and using Hölder's inequality gives

$$\int_0^1 \| \sum_{j=1}^n r_j(u) L_q(\lambda_j) g_j \|_{L_q(\mathbb{R}^N_+)}^q \, du$$
$$\leq \int_0^\infty |\phi(x_N)|^q \int_0^1 \| \sum_{j=1}^n r_j(u) g_j \|_{L_q(\mathbb{R}^N_+)}^q \, du \, dx_N$$
$$\left(\int_0^\infty |\psi(y_N)|^{q'} \, dy_N \right)^{q/q'},$$

and so, we have

$$\| \sum_{j=1}^n r_j(u) L_q(\lambda_j) g_j \|_{L_q((0,1), L_q(\mathbb{R}^N_+))}$$
$$\leq C \|\phi\|_{L_q(\mathbb{R})} \|\psi\|_{L_{q'}(\mathbb{R})} \| \sum_{j=1}^n r_j g_j \|_{L_q((0,1), L_q(\mathbb{R}^N_+))}.$$

This shows Lemma 2.15. □

By using Lemma 2.6 and 2.15, we can show Theorem 2.13. These complete the proof of Theorem 2.9.

3 Conclusions

Partial Differential Equation (PDE) can describe the phenomena in our daily life. The aim of PDE problem is well-posedness properties of the model problem. One property of well-posedness is regularity of the solution of the model problem. The $\mathcal{R}$-boundedness of the solution operator families of model problem is one of the methods to get the regularity. Therefore, the $\mathcal{R}$-boundedness of Navier-Lamé equation with surface tension can be used to investigate well-posedness properties of model problem.

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