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On Some Properties of Fabulous Fraction Tree

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Abstract Among several properties that real numbers possess, this paper deals with the exciting formation of positive rational numbers constructed in the form of a Tree, in which every number has two branches to the left and right from the root number. This tree possesses all positive rational numbers. Hence it consists of infinite numbers. We call this tree "Fraction Tree". We will formally introduce the Fraction Tree and discuss several fascinating properties including proving the one-one correspondence between natural numbers and the entries of the Fraction Tree. In this paper, we shall provide the connection between the entries of the fraction tree and Fibonacci numbers through some specified paths. We have also provided ideas relating the terms of the Fraction Tree with that of continued fractions. Five interesting theorems related to the entries of the Fraction Tree are proved in this paper. The simple rule that is used to construct the Fraction Tree enables us to prove many mathematical properties in this paper. In this sense, one can witness the simplicity and beauty of making deep mathematics through simple and elegant formulations. The Fraction Tree discussed in this paper which is technically called Stern-Brocot Tree has profound applications in Science as diverse as in clock manufacturing in the early days. In particular, Brocot used the entries of the Fraction Tree to decide the gear ratios of mechanical clocks used several decades ago. A simple construction rule provides us with a mathematical structure that is worthy of so many properties and applications. This is the real beauty and charm of mathematics.

Keywords Fraction Tree, Levels of the Tree, Binary Expansions, One – One Correspondence, Fibonacci Sequence, Continued Fractions

1. Introduction

The concept of Fraction Tree was first discovered independently by Moritz Stern in 1858 and AchilleBrocot in 1861. While Stern was a German Number Theorist, Brocot was a French clockmaker. So, the Fraction Tree which we discuss in this paper is known as "Stern-Brocot Tree" in mathematics literature. The Fraction Tree possesses tremendous beauty in the realm of mathematics and was equipped with several unexpected applications both in mathematics and with other branches of science. We will discuss several aspects of this fascinating Fraction Tree and discuss a few of its applications. For knowing more about Fraction Trees see [1–4].

2. Definition

Let $a, b > 0$ be integers. Then $\frac{a}{b}$ will be a Fraction (either proper fraction or mixed fraction). In fact, such a number will be a positive rational number.

Starting with $\frac{a}{b}$ we generate two new numbers $\frac{a}{a+b}$ and $\frac{a+b}{b}$ called left child and right child of $\frac{a}{b}$ respectively. (2.1)

We can view Figure 1, to understand this creation rule more precisely.

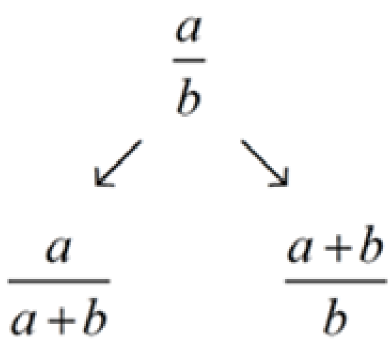

Figure 1. Left Child and Right Child

Given this construction rule, we can consider $\frac{a}{b}$ as parents of both left and right child. Further we also note that the left child $\frac{a}{a+b}$ is always less than 1 whereas, the right child $\frac{a+b}{b}$ is always greater than 1. (2.2)

Beginning with the number $\frac{1}{1}$ called the Apex of the Tree, we now create the Fraction Tree using the construction rule in Figure 1.

We note that every number in the Fraction Tree is a fraction (either proper or mixed) and each fraction has precisely two children namely a left child and a right child as shown in Figure 2. We also notice that Figure 2 represents a Tree structure which has infinite numbers where the Apex number is considered as $\frac{1}{1}$. This number is also called root of the tree. Thus beginning with $\frac{1}{1}$ every fraction in the tree is associated with precisely two numbers namely its left and right child given by the production rule in Figure 1. For example, from Figure 2, the fraction $\frac{1}{3}$ has left child $\frac{1}{4}$ and right child $\frac{4}{3}$. Similarly $\frac{5}{3}$ has left child $\frac{5}{8}$ and right child $\frac{8}{3}$.

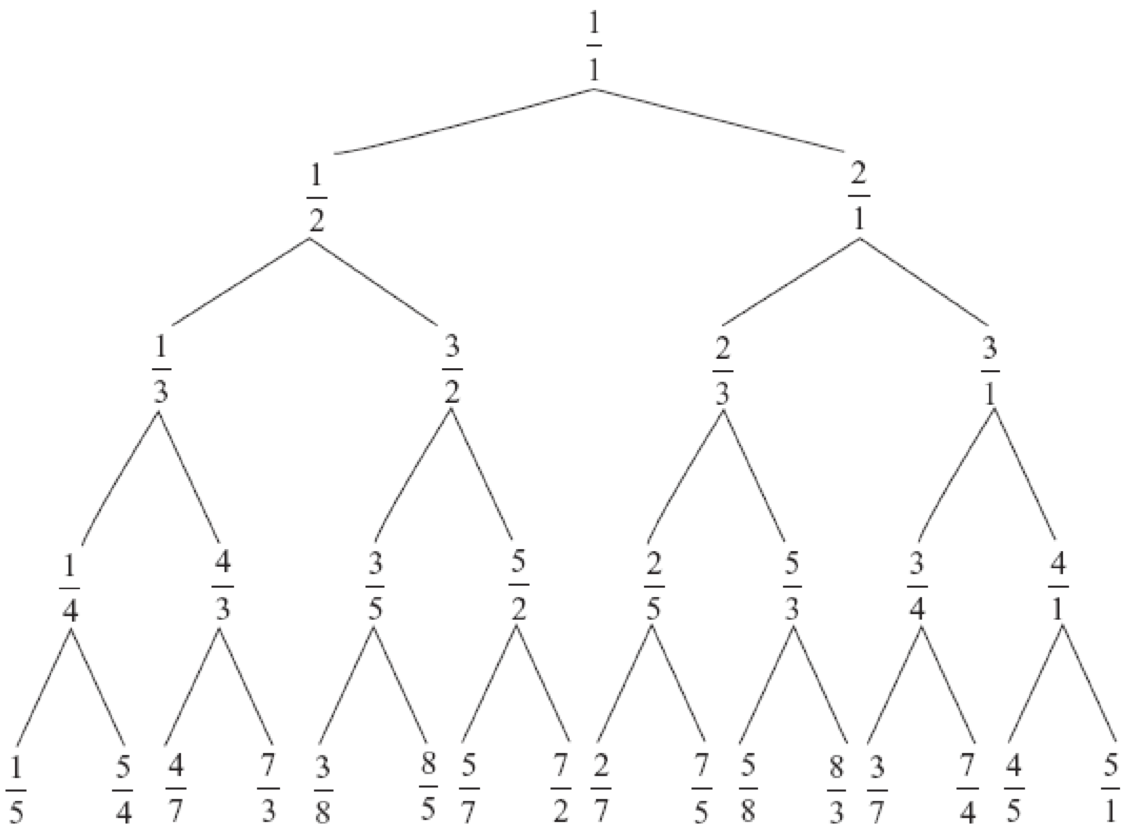

Figure 2. Fraction Tree

3. Levels of the Fraction Tree

Each row in the Fraction Tree beginning with row 0, is defined to be the level of the Fraction Tree. Thus from Figure 2, we see that $\frac{1}{1}$ belong to level 0, the numbers $\frac{1}{2},\frac{2}{1}$ belong to level 1, the numbers $\frac{1}{3},\frac{3}{2},\frac{2}{3},\frac{3}{1}$ belong to level 2, $\frac{1}{4},\frac{4}{3},\frac{3}{5},\frac{5}{2},\frac{2}{5},\frac{5}{3},\frac{3}{4},\frac{4}{1}$ belong to level 3 and so on.

Further we notice that the level n always begins with the fraction $\frac{1}{n+1}$ and ends with $\frac{n+1}{1}$ (3.1)

If we count the position of each fraction in the tree beginning with the Apex number $\frac{1}{1}$ and going row-wise from left to right and from top to bottom, then from (3.1) we notice that $\frac{1}{n+1}$ is the 2^n th term and $\frac{n+1}{1}$ is the $(2^{n+1}-1)$ th term in the Fraction Tree (3.2). We now see a following basic theorem regarding terms of levels in the Fraction Tree.

3.1. Theorem 1

The number of terms in numerator and denominator of numbers in level n of Fraction Tree is 2^n each summing up to 3^n, where n = 0, 1, 2, 3, 4, . . . (3.3)

Proof: Since any particular number in a given level gives rise to a left and right child (two numbers) it is clear that the total numbers in any level is exactly double that of in the previous level. Since level 0 has only one number namely $\frac{1}{1}$, it follows that the total numbers in level n of fraction tree is 2^n**.** We also notice that the numbers in the denominators are precisely the mirror images of those in the numerators. Thus there will be 2^n terms in both numerator and denominator in any level of Fraction Tree.

Moreover, we notice from Figure 2, that the numbers in the denominators are mirror images of the numbers in the numerators of a given level, the sum of numbers in the denominators must be same as that of sum of the numbers in the numerators.

Now, notice that the sum of the numbers in the numerator of each level can be represented in alternate ways as shown below:

Sum of Numbers in level 0: 1 = 1 = 3^0.

Sum of Numbers in level 1: 1 + 2 = 1 + 1 + 1 = 3^1.

Sum of Numbers in level 2: 1 + 3 + 2 + 3 = 1 + 2 + 3 + 2 + 1 = 3^2.

Sum of Numbers in level 3: 1 + 4 + 3 + 5 + 2 + 5 + 3 + 4 = 1 + 3 + 6 + 7 + 6 + 3 + 1 = 3^3.

Sum of Numbers in level 4: 1 + 5 + 4 + 7 + 3 + 8 + 5 + 7 + 2 + 7 + 5 + 8 + 3 + 7 + 4 + 5 = 1 + 4 + 10 + 16 + 19 + 16 + 10 + 4 + 1 = 3^4.

In writing the sum of numerator (denominator) numbers of each level in alternate way as shown above, we see the numbers in new representation, form the coefficients of trinomial expansion $(a+b+c)^n$. Similar to Pascal's Triangle which represents coefficients of the binomial expansion $(a+b)^n$, the coefficients of the terms in the trinomial expansion $(a+b+c)^n$ forms a triangle called Trinomial Triangle shown in Figure 3.

Note that each entry of trinomial triangle in Figure 3 is the sum of three entries directly above it. The sum of numbers in any row of the trinomial triangle in Figure 3 is given by $(1+1+1)^n = 3^n$. Since the sum of the numbers in each level of the trinomial triangle is just a rearrangement of sum of numbers in the numerator/denominator of the numbers in any level of Number Tree of Figure 2, it follows that the sum of 2^n numbers present in numerator/denominator of level n of Fraction Tree must be equal to the sum of numbers in any level of the trinomial triangle which is 3^n . This completes the proof.

1
1 1 1
1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 19 16 10 4 1
1 5 15 30 45 51 45 30 15 5 1
1 6 21 50 90 126 141 126 90 50 21 6 1
1 7 28 77 161 266 357 393 357 266 161 77 28 7 1
1 8 36 112 266 504 784 1016 1107 1016 784 504 266 112 36 8 1
1 9 45 156 414 882 1554 2304 2907 3139 2907 2304 1554 882 414 156 45 9 1

Figure 3. Trinomial Triangle

4. Binary Expansions

It is quite well known that any positive integer N in denary (base 10) has a unique binary (base 2) expansion containing 0's and 1's according to their place values. For example 12 in denary system is represented in binary expansion as 1100. We make use of the concept of expressing a denary positive integer N as equivalent to its binary form with strings of 0's and 1's to explore more properties of the Fraction Tree. We first prove an important theorem.

4.1. Theorem 2

For every natural number N, there exists a unique fraction in the Fraction Tree.

Proof: First we make the convention that 0 stands for left child and 1 stands for right child for a given fraction in the fraction tree of Figure 2. Now if N is a given natural number, then expressing it in equivalent binary expansion form we get strings of 0's and 1's. The last digit of N would be 0 if N is even and it would be 1 if N is odd. After writing out the binary expansion of N with strings of 0's and 1's of the form say 111000…1100… we can begin by reading this number from left to right beginning with 1 representing the Apex number $\frac{1}{1}$ of the tree and then traversing in the fraction tree to the left or right according as we get 0 or 1 in the binary expansion corresponding to N. This process will eventually lead us to a fraction in the Fraction Tree. This fraction must be unique because the binary expansion of any natural number N is unique.

Moreover, the number of digits of N in its binary expansion form with k digits will place N in the level k of the fraction tree. In view of (3.2), we know that level k numbers will correspond to values of N from 2^k to $2^{k+1}-1$ (both inclusive).

In particular, if $N = 2^k$ then its binary expansion form would be 1000…000 where 1 is in the (k+1)st place and the rest k strings are all zero. Thus beginning with 1 for the Apex number $\frac{1}{1}$ of the Fraction Tree and using the convention 0 for left child, we should begin at $\frac{1}{1}$ and traverse to the left child successively for k terms ending with the fraction $\frac{1}{k+1}$.

So the 2^kth term of the Fraction Tree must be $\frac{1}{k+1}$.

If N = 2k+1 – 1 then the binary expansion form would be 11111…11 where the (k+1) strings are all 1. Hence beginning with 1 for the Apex number and using the convention 1 for the right child, we should begin at $\frac{1}{1}$ and traverse to the right child successively for k terms ending with the fraction $\frac{k+1}{1}$.

Since any natural number N lies in $\left[2^k, 2^{k+1}-1\right]$ and the binary expansion of any natural number is unique, any natural number will correspond to a unique fraction at level k of the Fraction Tree. Thus the scheme of traversing either to left (for 0) and right (for 1) beginning at Apex $\frac{1}{1}$ of the tree, will eventually lead us to locate the Nth term in the Fraction Tree.

This completes the proof.

4.2. Illustration of Theorem 2

In view of explaining the scheme provided in Theorem 1, we provide certain examples to locate the Nth term in the fraction tree for a given natural number N.

4.2.1. If N = 11 say, then we can immediately locate the 11^{th} term in the tree by first expressing 11 in its equivalent binary expansion form as 1011. Now in 1011, reading left to right, beginning with1for the number $\frac{1}{1}$ we have to first traverse left (because of 0) and traverse to right twice (because of 11) to finally reach the fraction $\frac{5}{2}$ This scheme is explained in Figure 4.

4.2.2. If N = 26, then we can locate the 26^{th} term in the tree by first writing 26 in binary expansion form as 11010. Now reading the number 11010 from left to right beginning with 1 for $\frac{1}{1}$ and traversing right (because of 1), then traversing left (because of 0) and then traversing right (because of 1) and finally traversing to left (because of 0), we arrive at the fraction $\frac{5}{8}$. This scheme is explained in Figure 4.

We can verify from Figure 2, that the 11^{th} and 26^{th} terms of the Fraction Tree beginning from the Apex and reading level wise from left to right we get the numbers $\frac{5}{2}, \frac{5}{8}$ indeed.

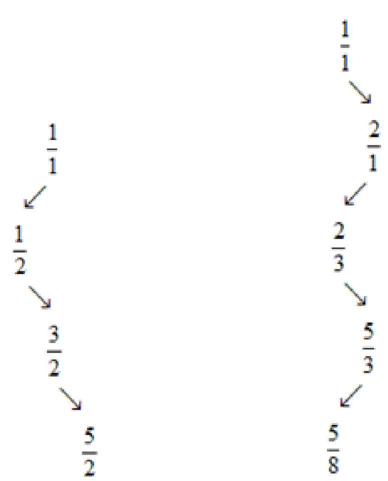

Figure 4. Location of Terms in Fraction Tree

We now prove the following theorem which represents converse of Theorem 2.

5. Theorem 3 (Connection between Fractions of Fraction Tree and Natural Numbers)

For every fraction in the Fraction Tree, there corresponds a unique natural number.

Proof: If $\frac{p}{q}$ is a given fraction in the Fraction Tree, first we check if it is less than 1 or greater than 1. If $\frac{p}{q}$ is less than 1, it should be a left child in some particular level of the fraction tree. Similarly, if $\frac{p}{q}$ is more than 1, it must be a right child at some particular level of the fraction tree. Now, knowing if the given fraction is a left or right child we can find a path all the way to traverse back from $\frac{p}{q}$ to the apex number $\frac{1}{1}$. Since the Fraction Tree represents a tree structure, (being connected and acyclic) we note that every such path is unique. If in case, there is more than one path to lead to the Apex, then a particular fraction at some level would have two different parents leading to two different Apex numbers which is not the case as our Fraction Tree has only one Apex number namely $\frac{1}{1}$. We now provide a method to identify the unique parent for a given left child or a right child. For this, we use the convention 0 for left child fraction and 1 for right child fraction. Using this convention, we can determine the unique natural number for a given fraction in the fraction tree through the following three cases.

Case 1: (All Left Child)

If the given fraction is of the form $\frac{1}{k+1}$ then it is a left child and the parent of this fraction would be $\frac{1}{k}$. Since this is also less than 1, the parent of this left child fraction would be another left child fraction $\frac{1}{k-1}$. Continuing this way, we can eventually reach the Apex number $\frac{1}{1}$ from the given fraction $\frac{1}{k+1}$ by traversing through $k + 1$ numbers. We notice that the path from $\frac{1}{k+1}$ to the Apex number $\frac{1}{1}$ is located on the extreme left of the fraction tree as can be seen from Figure 2. Now starting with $\frac{1}{k+1}$ the path to reach the Apex is the sequence of all left child fractions until we reach the Apex number. This path is given by $\frac{1}{k+1} \to \frac{1}{k} \to \frac{1}{k-1} \to \cdots \to \frac{1}{3} \to \frac{1}{2} \to \frac{1}{1}$. Hence the binary expansion form for this path would be 1000…000 where the leading 1 is followed by k zeros. The natural number corresponding to the binary number 1000…000 is given by $1(2^k) + 0(2^{k-1}) + 0(2^{k-2}) + \cdots + 0(2) + 0(1) = 2^k$. Thus the fraction $\frac{1}{k+1}$ corresponds to 2^k th term in level k of the Fraction Tree.

Case 2: (All Right Child)

If the given fraction is of the form $\frac{k+1}{1}$ then its parents at subsequent levels of the Fraction Tree would also be right child until we reach the Apex number $\frac{1}{1}$. Such a path would be of the form $\frac{k+1}{1} \to \frac{k}{1} \to \frac{k-1}{1} \to \cdots \to \frac{3}{1} \to \frac{2}{1} \to \frac{1}{1}$ located on the extreme right of the Fraction Tree as can be seen from Figure 2. The binary expansion form for this path would be 1111…111 consisting of astring of (k + 1) 1's. The natural number corresponding to the binary number 1111…111l is given by $1(2^k) + 1(2^{k-1}) + 1(2^{k-2}) + \cdots + 1(2) + 1(1) = 2^{k+1} - 1$. Thus the fraction $\frac{k+1}{1}$ corresponds to $(2^{k+1} - 1)$ th term in level k of the fraction tree.

Case 3: (Internal Child)

If we are given any fraction say $\frac{p}{q}$ greater than 2^k but less than $2^{k+1} - 1$ then it would be located in the interior of level k in the Fraction Tree. We now provide a recipe for back-tracking the path from such a fraction $\frac{p}{q}$ up to the Apex number $\frac{1}{1}$.

If the given fraction $\frac{p}{q}$ at the interior of level k is less than 1 (left child), then its parent at level (k – 1) will be of the form $\frac{p}{q-p}$ (5.1)

Similarly, if the given fraction $\frac{p}{q}$ at the interior of level k is greater than 1 (right child), then its parent at level (k – 1) will be of the form $\frac{p-q}{q}$ (5.2).

We see that this is precisely one less than $\frac{p}{q}$.

If the parent obtained through this scheme provides a left child then we may use (5.1) or if it provides a right child we may use (5.2) to determine the parents at the previous levels in the Fraction Tree. At each previous level when we encounter a parent which happens to left child then we assign 0 and if it happens to be right child we assign 1 until we reach the Apex number.

Thus, using this scheme, we can find a unique path from the given fraction $\frac{p}{q}$ at the interior of level k all the way up to the Apex number $\frac{1}{1}$. Knowing the sequence of 0's and 1's that were obtained in back-tracking process we get the binary expansion form of the given fraction. Using this binary expansion we can easily determine the equivalent denary number which provides the required natural number. Hence in view of three cases discussed, we notice that every fraction in the fraction tree corresponds to a unique natural number.

This completes the proof.

5.1. Illustration of Theorem 3

In view of a better understanding of the recipe provided in Theorem 3, we provide the following illustrations.

5.1.1. If we consider the fraction $\frac{2}{7}$ then we will find the corresponding natural number N such that the N th term of the Fraction Tree is $\frac{2}{7}$. We first observe that $\frac{2}{7}$ is less than 1. Hence, it should be a left child at some level of the Fraction Tree. Hence we should assign 0 for this number which forms the last digit in the binary expansion form corresponding to $\frac{2}{7}$. Being left child its parent at the previous level (according to (5.1)) is $\frac{2}{5}$. Since this new number is also less than 1 it should be left child of some parent at the previous level. So we assign 0 corresponding to $\frac{2}{5}$. Now again according to equation (5.1), the parent of $\frac{2}{5}$ is $\frac{2}{3}$. Since this is also less than 1, this should be a left child and so we assign 0 to it. The parent of $\frac{2}{3}$ is $\frac{2}{1}$. Since $\frac{2}{1}$ is larger than 1, this must be a right child. So we assign 1 to $\frac{2}{1}$. Now the parent of $\frac{2}{1}$ by (5.2) is $\frac{1}{1}$,which is the Apex number of the Fraction Tree. Hence we should assign 1 to it. Since we have reached the Apex number this back-tracking process should terminate. The back-tracking path is thus $\frac{2}{7} \to \frac{2}{5} \to \frac{2}{3} \to \frac{2}{1} \to \frac{1}{1}$. Now gathering the 0's and 1's in this back-tracking process we get 11000. The denary equivalent of the binary expansion 11000 is 24. Hence the fraction $\frac{2}{7}$ corresponds to the natural number N = 24. We can verify from Figure 2, that the 24th term in the fraction tree is precisely $\frac{2}{7}$.

5.1.2. If we consider the fraction $\frac{22}{7}$ then we try to find the corresponding natural number. Observing that $\frac{22}{7}$ is greater than 1, this must be a right child at some particular level of the Fraction Tree. Hence we assign 1 to it. Being right child (by equation (5.2)), its parent in the previous level is $\frac{15}{7}$. Since this is also greater than 1 (right child) we assign 1 to it. Continuing in the same way we see that the parent of $\frac{15}{7}$ is $\frac{8}{7}$. This number being greater than 1 corresponds to an assignment of 1. Now the parent of $\frac{8}{7}$ is $\frac{1}{7}$. Since $\frac{1}{7}$ is less than 1, it will be a left child at some level of the Fraction Tree. So we assign 0 to this number. Applying the same principle, the parent of $\frac{1}{7}$ (according to (5.1)) must be $\frac{1}{6}$. This gives an assignment of 0 and determining the parents of each new number we get $\frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{1}$. Hence the binary expansion for the number $\frac{22}{7}$ is 1000000111. The denary equivalent of the binary expansion 1000000111 is 519. Thus the given fraction $\frac{22}{7}$ corresponds to the natural number N = 519 and we can verify that the 519th term of the Fraction Tree is indeed $\frac{22}{7}$.

For knowing more about rational numbers and other special types of numbers see [5–9]. We now establish the following significant theorem.

6. Theorem 4(Bijection between Set of all Natural Numbers and Positive Rational Numbers)

There exists a one-one correspondence between a set of all natural numbers and set of all positive rational numbers.

Proof: First we note that every number in the Fraction Tree is a unique positive rational number. Hence, by Theorem 2, for every natural number we get a unique positive rational number in the Fraction Tree. Similarly, by Theorem 3, for every positive rational number in the Fraction Tree, there corresponds a unique natural number. Hence for every natural number there exists a unique positive rational number (at some level of the Fraction Tree) and conversely for any positive rational number there exist a unique natural number that represent the cardinality of the location of the positive rational number in the Fraction Tree.

This completes the proof.

We now present the connection of terms of the Fraction Tree with the most famous and significant Fibonacci sequence.

7. Fraction Tree and Fibonacci Sequence

We consider the following definition and conventions before establishing the required relationship between Fraction Tree and Fibonacci sequence.

7.1. Definition

The sequence of positive integers is defined recursively by $F_{n+2} = F_{n+1} + F_n \quad (7.1)$ where $F_0 = 1, F_1 = 1$ and $n \geq 0$ is called Fibonacci sequence named after Italian mathematician Fibonacci. The first few terms of the Fibonacci sequence obtained through (7.1) are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, …

7.2. Basic Conventions and Connections

We denote the left child of any fraction by L and right child of any fraction by R of the Fraction Tree in Figure 2. Beginning with the Apex number $\frac{1}{1}$, we consider the traversals of the form LRLRLR… made through L and R successively. This provides us with the following interesting theorem.

7.3. Theorem 5

In the Fraction Tree, the sequence of traversals **LRLR**… always produces a fraction, that is ratio of two consecutive Fibonacci numbers.

Proof: We make use of the production rule of any given fraction as defined in Figure 1 to prove this theorem. First, beginning with the Apex number, $\frac{1}{1} = \frac{F_1}{F_0}$ the first term **L** of the sequence of traversals will lead us to the left child represented by the number $\frac{F_1}{F_0 + F_1}$. But by (7.1), we see that $F_0 + F_1 = F_2$. Hence beginning from $\frac{F_1}{F_0}$ the first term **L** of the sequence of traversals lead us to the fraction $\frac{F_1}{F_2}$. Now considering the first two terms **LR** of the sequence of traversals we will consider $\frac{F_1}{F_2}$ which represent for the first **L** and then produce a right child for **R** by a new fraction (according to the production rule) $\frac{F_1 + F_2}{F_2}$. But by (7.1), this is $\frac{F_3}{F_2}$.

If we consider **LRL** then we get a new fraction $\frac{F_3}{F_2 + F_3} = \frac{F_3}{F_4}$. Similarly, if we consider **LRLR**, we get a new fraction $\frac{F_3 + F_4}{F_4} = \frac{F_5}{F_4}$.

We use Mathematical Induction, to prove that the fractions obtained through the terms of the sequence of traversals are always the ratio of two consecutive Fibonacci numbers.

First we define the length of the sequence of traversals as number of **L**’s and **R**’s present in it. For example, the sequence of traversal of length 1 is just **L**. Similarly sequence of traversal of length 2 is **LR**. With this convention, we see that the sequences of traversals of length 1 and 2 produces ratio of consecutive Fibonacci numbers $\frac{F_1}{F_2}$ and $\frac{F_3}{F_2}$ respectively.

Now, by Induction Hypothesis, we assume that the sequence of traversals of length *k*, where *k* is some natural number produces a number which is ratio of two consecutive Fibonacci numbers.

If *k* is odd, the fraction would be $\frac{F_k}{F_{k+1}}$ and if *k* is even, it would be $\frac{F_{k+1}}{F_k}$. We see that both of these numbers are ratios of two consecutive Fibonacci numbers satisfying the production rule given in Figure 1.

Now if we consider the sequence of traversals of length *k* + 1, then we get the following two cases depending upon if *k* is odd or even.

If *k* is even, then *k* + 1 must be odd and so the sequence of traversals will end with **L**. Hence the new fraction corresponding to sequence of traversals of length *k* + 1 from the sequence of length *k* is given by $\frac{F_{k+1}}{F_{k+1} + F_k} = \frac{F_{k+1}}{F_{k+2}}$ which is ratio of (*k* + 1) th Fibonacci number to that of (*k* + 2) th Fibonacci number.

If *k* is odd, then *k* + 1 must be even and so the sequence of traversals will end with **R**. Hence the new fraction corresponding to sequence of traversals of length *k* + 1 from the sequence of length *k* is given by $\frac{F_k + F_{k+1}}{F_{k+1}} = \frac{F_{k+2}}{F_{k+1}}$ which is ratio of (*k* + 2) th Fibonacci number to that of (*k* + 1) th Fibonacci number.

Thus if the result is true for the sequence of traversals of length *k* then it is also true for length *k* + 1.

Hence by Induction principle, the result must be true for a sequence of traversals of length *k* for all natural numbers *k*. This completes the proof.

8. Fraction Tree and Continued Fraction

We finally present a method to locate a given term in the fraction tree using simple continued fraction expansion. We mention it here just for the sake of providing the connection between Fraction Tree and Continued Fraction.

We know that any given natural number has a unique binary expansion (base two) with strings of 0's and 1's. If *N* is a given natural number, then we will first express *N* in equivalent binary expansion form shown below:

$$N = \overbrace{1\cdots 1}^{a_k}\overbrace{0\cdots 0}^{a_{k-1}}\overbrace{1\cdots 1}^{a_{k-2}}\ldots\overbrace{0\cdots 0}^{a_1}\overbrace{1\cdots 1}^{a_0}$$

We note that $a_0 = 0$, if *N* is even. The terms a_i represent the number of zeros and number of ones in the respective strings in the binary expansion of *N* counting from right to left.

The *N*th fraction in the Fraction Tree f_N of Figure 2 is given by the following simple continued fraction expression

$$f_N = [a_0; a_1, a_2, \cdots, a_k] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}} \quad (8.1)$$

Equation (8.1) helps us to locate the *N*th term in the Fraction Tree immediately by writing out its binary expansion. For example, we will determine the 50th term of the fraction tree.

First, we observe that 50 in binary expansion is 110010. Concerning equation (8.1), we get

$$f_{50} = [0;1,1,2,2] = 0 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2}}}} = 0 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{2}{5}}} = 0 + \cfrac{1}{1+\cfrac{5}{7}} = \frac{7}{12}$$

Similarly we wish to determine the 101st term in the Fraction Tree. First we note that the binary expansion of 101 is 1100101. Thus from equation (8.1), we get

$$f_{101} = [1;1,1,2,2] = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2}}}} = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{5}{2}}}} = 1 + \cfrac{1}{1+\cfrac{5}{7}} = \frac{19}{12}$$

In fact, equation (8.1) actually provides the one-one correspondence between the set of all natural numbers and set of all positive rational numbers which are entries of Fraction Tree. So equation (8.1) is not only useful for locating a particular term in the Fraction Tree, but in doing so it helps us to establish the one-one correspondence between natural numbers and set of all positive rational numbers, the same conclusion which we have arrived in Theorem 4 of section 6.

9. Conclusions

We observed that the entries of the Fraction Tree displayed in Figure 2, contain all positive rational numbers. Using the simple construction rule we proved two interesting properties of the through theorem 1 in section 3. Using Binary expansions for a given natural number, we provided two recipes in theorems 2 and 3 through which we have established the fact that the set of all positive rational numbers is countable, which is equivalent to stating that the set of all positive rational numbers can be put in to one – one correspondence with natural numbers in theorem 4 of section 6. This same conclusion was observed in section 8 using continued fractions. In theorem 5 of section 7, we proved that the sequence of traversals **LRLR**… always leads us to the ratio of consecutive Fibonacci numbers. This theorem thus provides the connection between the terms of Fraction Tree and Fibonacci numbers. The interested reader may discover an interesting sequence of traversals like mentioned in theorem 5 to end up with some known sequence of numbers like say ratio of two triangular numbers or ratio of two squares or ratio of two primes etc. This will provide fruitful research insight. Connecting various branches of mathematics most elegantly is a significant part of this work.

One can easily generalize the construction rule given in Figure 1 regarding the left and right children and try to explore more enriching properties of the tree constructed through that particular construction. This will be a very exciting scope for knowing about numbers and their exotic connections spanning different branches of mathematics. Stern – Brocot actually introduced this fascinating Fraction Tree for finding applications in gear systems which is very much used even today. Thus the Fraction Tree discussed in this work has awide range of applications in designing gear systems in automobile industries. For knowing applications of Stern – Brocot Tree in clock making see [10].

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