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The Relative (Co)homology Theory through Operator Algebras

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Abstract This paper introduces a new idea in the unital involutive Banach algebras and its closed subset. This paper aims to study the cohomology theory of operator algebra. We will study the Banach algebra as an applied example of operator algebra, and the Banach algebra will be denoted by $\mathcal{A}$. The definitions of cyclic, simplicial, and dihedral cohomology group of $\mathcal{A}$ will be introduced. We presented the definition of $\mathcal{B}$-relative dihedral cohomology group that is given by: $\mathcal{HD}^{\mathrm{n}}[(\mathcal{A},\mathcal{B}),\mathcal{X}] = {\mathrm{Z}^{\mathrm{n}}[(\mathcal{A},\mathcal{B}),\mathcal{X}]}/{\mathrm{B}^{\mathrm{n}}[(\mathcal{A},\mathcal{B}),\mathcal{X}]}$, and we will show that the relation between dihedral and $\mathcal{B}$-relative dihedral cohomology group $0 \to \mathcal{HD}^0[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \mathcal{HD}^0(\mathcal{A},\mathcal{X}) \to \mathcal{HD}^0({\mathcal{A}}/{\mathcal{B}},\mathcal{X}) \to \mathcal{HD}^1[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \mathcal{HD}^1(\mathcal{A},\mathcal{X}) \to \cdots \to \mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \mathcal{HD}^n(A,\mathcal{X}) \to \mathcal{HD}^n({\mathcal{A}}/{\mathcal{B}},\mathcal{X}) \to \mathcal{HD}^{n+1}[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \cdots$ can be obtained from the sequence $0 \to \mathcal{CD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \mathcal{CD}^n(\mathcal{A},\mathcal{X}) \to \mathcal{CD}^n({\mathcal{A}}/{\mathcal{B}},\mathcal{X}) \to 0$. Among the principal results that we will explain is the study of some theorems in the relative dihedral cohomology of Banach algebra as a Connes-Tsygan exact sequence, since the relation between the relative Banach dihedral and cyclic cohomology group ($\mathcal{HD}^{\mathrm{n}}[(\mathcal{A},\mathcal{B}),\mathcal{X}]$ and $\mathcal{HC}^{\mathrm{n}}[(\mathcal{A},\mathcal{B}),\mathcal{X}]$) of $\mathcal{A}$ will be proved as the sequence $\to {}^{-}\mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}] \xrightarrow{i^*} \mathcal{HC}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}] \xrightarrow{j^*} \mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to {}^{-}\mathcal{HD}^{n+1}[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \cdots$. Also, we studied and proved some basic notations in the relative cohomology of Banach algebra with unity and defined its properties. So, we showed the Morita invariance theorem in a relative case with maps $\mathrm{tr}^*: \mathcal{HD}^*(\mathcal{M}_{\mathrm{r}}(\mathcal{B}),\mathcal{M}_{\mathrm{r}}(\mathcal{M})) \to \mathcal{HD}^*(\mathcal{B},\mathcal{M})$ and $\mathrm{inc}^*: \mathcal{HD}^*(\mathcal{B},\mathcal{M}) \to \mathcal{HD}^*(\mathcal{M}_{\mathrm{r}}(\mathcal{B}),\mathcal{M}_{\mathrm{r}}(\mathcal{M}))$, and proved the Connes-Tsygan exact sequence that relates the relative cyclic and dihedral (co)homology of $\mathcal{A}$. We proved the Mayer-Vietoris sequence of $\mathcal{HD}^{\mathrm{n}}[(\mathcal{A},\mathcal{B}),\mathcal{X}]$ in a new form in the Banach $\mathcal{B}$-relative dihedral cohomology: $\cdots \to \mathcal{HD}^{\mathrm{n}-1}(\mathcal{L} \cap \mathcal{N},\mathcal{K} \cap \mathcal{J})_{\mathcal{X}} \xrightarrow{\partial_*} \mathcal{HD}^{\mathrm{n}}(\mathcal{A},\mathcal{B})_{\mathcal{X}} \xrightarrow{\mathrm{k}_*-\mathrm{l}_*} \mathcal{HD}^{\mathrm{n}}(\mathcal{L},\mathcal{K})_{\mathcal{X}} \oplus \mathcal{HD}^{\mathrm{n}}(\mathcal{N},\mathcal{J})_{\mathcal{X}} \xrightarrow{(\mathrm{i}_*,\mathrm{j}_*)} \mathcal{HD}^{\mathrm{n}}(\mathcal{L} \cap \mathcal{N},\mathcal{K} \cap \mathcal{J})_{\mathcal{X}} \xrightarrow{\partial_*} \mathcal{HD}^{\mathrm{n}+1}(\mathcal{A},\mathcal{B})_{\mathcal{X}} \to \cdots$. It should be borne in mind that the study of the cohomology theory of operator algebra is concerned with studying the spread of Covid 19.

Keywords Cyclic Homology, Dihedral Cohomology, Exact Sequence, Mayer-Vietoris Sequence, Operator Algebra, Relative Cohomology

MSC classes: 20J06, 57M27.

1. Introduction

The cohomology of Banach algebra was studied by Johnson [22] who showed that the Banach algebra $\mathcal{A}$ is characterized by the Hochschild cohomology group. He used the vanishing of the Hochschild cohomology group of $\mathcal{A}$ to determine a significant class of Banach algebra called amenable algebra. Kadison [23], Sinclair [24], and Ringrose [25] extended the Banach algebra to the operator algebra (C*-algebra- von Neumann algebra). The effective tool which computes the cyclic (co)homology is the Connes-Tsygan exact sequence that relates the simplicial (co)homology to the cyclic (co)homology of much algebra. If we have the relation between the cohomology group

$\mathcal{H}_n(\mathcal{A},\mathcal{A}^*)$ of Banach algebra $\mathcal{A}$ and the corresponding relative $\mathcal{H}_n\mathcal{B}(\mathcal{A},\mathcal{A}^*)$, such $\mathcal{A}^*$ is the dual Banach space of $\mathcal{A}$, and then the relation between relative dihedral cohomology groups of $\mathcal{A}$ and dihedral cohomology groups of $\mathcal{A}$ can be established. Helemskii [2] studied Banach algebra via the Banach homology theory. According to his investigation, he defined the biflat and bi-projective Banach algebra. Indeed, for a bounded $\mathcal{A}$ -bimodule morphism $\rho: \mathcal{A} \to (\mathcal{A} \otimes_p \mathcal{A})^{**} (\rho : \mathcal{A} \to \mathcal{A} \otimes_p \mathcal{A})$, $\mathcal{A}$ is called biflat, such for the right inverse ρ of π, $\pi^{**} \circ \rho$ is canonical embedding from $\mathcal{A}$ into $\mathcal{A}^{**}$. He showed that: $L_1(G)$ is biflat Banach algebra if G is amenable and $L_1(G)$ is bi-projective if G is compact. Kanuith, [3], used this idea and defined a new notion of amenability of $\mathcal{A}$ depending on the character of Banach algebra. For the operator algebra, its simplicial cohomology has been studied in [22], [23], and [11]. The cyclic (co)homology of operator algebras was studied in [24], [1], and [2]. In [15], Karasauskas introduced the dihedral homology group of algebra with characteristic zero. The dihedral (co)homology of an involutive and unital algebra over commutative ring was studied by Loday [17] and others. Alaa introduced important results on the dihedral homology of operator algebras [5]. Karasauskas, Lapin, and Solovev [15] studied the dihedral homology and cohomology theory, as well as the connections between the other homology and dihedral homology. They introduced the Cheam characteristics of dihedral and studied the relations between the dihedral homology and Hermitian K-theory.

The cyclic and dihedral cohomology of operator algebra was studied in [6] and [7]. The triviality and non-triviality of dihedral cohomology groups of operator algebra were studied in [8]. The ideal amenability in Banach algebra was studied in [21]. Alaa and Gouda studied the dihedral cohomology theory of Banach algebras and introduced many working.

This paper will describe the calculation of the relative Banach cyclic, relative Reflexive, and relative dihedral cohomology groups of operator algebra. The cyclic cohomology groups of $\mathcal{A}$ which relative to the closed sub-algebra $\mathcal{B}$ of $\mathcal{A}$ is denoted by $\mathcal{HC}^n_{\mathcal{B}}(\mathcal{A})$. Here, we present a relation between the relative Banach dihedral (reflexive) cohomology group $\mathcal{HD}^n(\mathcal{A})$ of operator algebra $\mathcal{A}$ with the Banach $\mathcal{B}$-relative cyclic cohomology group $\mathcal{HC}^n_{\mathcal{B}}(\mathcal{A})$ of $\mathcal{A}$. Morita invariance theorem is shown in relative case. The important result is studying the Connes-Tsygan exact sequence as a relation between the relative cyclic (co)homology of operator algebra with their simplicial (co)homology and relative dihedral cohomology. We introduce the sequence that associates the Banach dihedral cohomology group with the Banach $\mathcal{B}$-relative dihedral cohomology group.

In the second section: we recalled the definitions and background notations in the cohomology theory. We presented Banach`s relative dihedral cohomology of operator algebra (Banach algebra).

In the third section: we study and prove some basic notations in the relative cyclic cohomology and define some properties. Moreover, we proved some relations in theorems (3.9), (3.12). We gave connections in Banach relative dihedral cohomology groups, proved the new results, and gave examples. Finally, we introduced the Mayer-Vietoris sequence of $\mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}]$ in a new form.

2. Cohomology Theory of Operator Algebra

This section recalled background notations of the cohomology theory of operator algebra. Here we will study the cohomology theory of Banach algebra as an applied example of operator algebra, and the Banach algebra will be denoted by $\mathcal{A}$. The definitions of simplicial, cyclic, and dihedral cohomology group of $\mathcal{A}$ and the theorems, which relate among them, will be introduced.

Definition (2.1): [4]

The Banach algebra $\mathcal{A}$ is called unity Banach algebra if it has the unit e $s.h.$ $x_1 \in \mathcal{A}$ the inverse of x_1 is $x_2 \in \mathcal{A}$ $s.h.$ $x_1x_2 = x_2x_1 = e$. And the involution of any element $x \in \mathcal{A}$ is x^* $s.h.$ $(x^*)^* = x$. (s.h. is abbreviation for "such that").

Definition (2.2): [18]

For the Banach algebra $\mathcal{A}$, if $x_1, x_2 \in \mathcal{A}$ and satisfy that $x_1x_2 = x_2x_1$, then $\mathcal{A}$ is called the commutative Banach algebra.

Definition (2.3):

Let $\mathcal{A}$ be unital algebra, the invertible element $x_1 \in \mathcal{A}$ satisfy that $\forall x_2 \in \mathcal{A} \Rightarrow x_1x_2 = x_2x_1 = 1$ and x_2 is unique and then $x_2 = x_1^{-1}$. The group under multiplication is

$$GL(\mathcal{A}) = \{x \in \mathcal{A}: \ x \text{ invertiple}\}. \tag{1}$$

We can define the spectrum of $x \in \mathcal{A}$ as

$$sp(x) = sp_{\mathcal{A}}(x)$$
$$= \{\lambda \in \mathbb{C}: \lambda I - x \notin GL(\mathcal{A})\}.$$

We write λ instead of λI. Then the complement of the spectrum defines the resolvent, since the resolvent function is $\mathcal{R}(\lambda) = (\lambda - x)^{-1}$.

Lemma (2.4): [3]

Suppose that $\mathcal{A}$ is unital Banach algebra, $\mathscr{b} \in \mathcal{A}$ $s.h.$ $\|1-\mathscr{b}\| \leq 1 \Rightarrow x \in GL(\mathcal{A})$, and $\mathscr{b}^{-1} = \sum_{n=0}^{\infty}(1-x)^n$, then

$$\|\mathscr{b}^{-1}\| \leq \frac{1}{1-\|1-\mathscr{b}\|} \text{ and } \|1-\mathscr{b}^{-1}\| \leq \frac{\|1-\mathscr{b}\|}{1-\|1-\mathscr{b}\|}. \tag{2}$$

Definition (2.5):

For the Banach algebra $\mathcal{A}$ and open subset $Q \in \mathbb{C}$, then the function $\mathscr{f}: Q \longrightarrow \mathcal{A}$ is analytic. If $\mathcal{O}(\lambda_0)$ is the open neighborhood, then

$$\forall\, \lambda_0 \in \mathcal{Q} \quad \exists\, \mathcal{O}(\lambda_0) \quad s.h.$$
$$f(\lambda) = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_0)^n.$$

We get $f(\lambda)$ converges $\forall \lambda \in \mathcal{O}(\lambda_0)$. If $\lambda_0 = \infty$, then f is analytic if

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^{-n}.$$

Definition (2.6): [9]

The involution x_1^* of $x_1 \in \mathcal{A}$ is satisfying that:

$$(x_1^*)^* = x_1,\ (x_1 x_2)^* = x_2^* x_1^*, (x_1 + \lambda x_2)^* = x_1^* + \bar{\lambda} x_2^*.$$

If $\mathcal{A}^*$-algebra is the involutive algebra $\mathcal{A}$, and $\mathcal{B}$ is involutive Banach *-algebra, then the homomorphism $\varphi: \mathcal{A} \to \mathcal{B}$ (the map between two *-algebras) is *-homomorphism since $\varphi(x_1^*) = \varphi(x_1)^*$. If $\|x_i^* x_i\| = \|x_i\|^2 \quad \forall\, x_i \in \mathcal{A}$, then the C^*-algebra is Banach *-algebra.

Definition (2.7): [10]

For any two pre-simplicial maps $f, g: \mathcal{C} \to \grave{\mathcal{C}}$, then the collection $h_i: \mathcal{C}_n \to \grave{\mathcal{C}}_{n+1}$ $s.h.$ $i = 0, \dots, n$ is defined the pre-simplicial homotopy, and the face maps $d_i: \mathcal{C}_n \to \mathcal{C}_{n-1}$ are satisfying that:

$$\begin{cases} d_i h_i = h_{i-1} d_i & if\ \ i < j, \\ d_i h_i = d_i h_{i-1} & if\ 0 < i < n, i = j, j+1, \\ d_i h_j = h_j d_{i-1} & if\ i > j+1, \\ d_0 h_0 = f \quad and & d_{n+1} h_n = g \end{cases} \tag{3}$$

We showed the notations which we used in our study. If $\mathcal{A}$ is Banach algebra, $\mathcal{S} \subset \mathcal{A}$ is closed. The cyclic cohomology group of $\mathcal{A}$ is denoted by $\mathcal{HC}^n(\mathcal{A})$, but the dihedral cohomology group of $\mathcal{A}$ is $\mathcal{HD}^n(\mathcal{A})$. The notations $\mathcal{HC}_{\mathcal{S}}^n(\mathcal{A}), \mathcal{HD}_{\mathcal{S}}^n(\mathcal{A})$ are the $\mathcal{S}$-relative cyclic and dihedral cohomology group of $\mathcal{A}$, respectively. The id is the identity operator and $\widehat{\otimes}$ is the tensor product. The notation $\mathcal{HD}^n(\mathcal{A})_{\mathcal{X}}$ was written as the notation of the dihedral cohomology of $\mathcal{A}$ with elements in $\mathcal{X}$.

Definition (2.8): [12]

Consider the Banach algebra $\mathcal{A}$ and Banach $\mathcal{A}$-bimodule $\mathcal{X}$. Then the cochain complex $\mathcal{C}^n(\mathcal{A}, \mathcal{X})$ is,

$$0 \to \mathcal{C}^0(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^0} \dots \to \mathcal{C}^n(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^n} \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{X}) \to \dots \tag{4}$$

such that $\mathcal{C}^0(\mathcal{A}, \mathcal{X}) = \mathcal{X}$ and $\delta^n: \mathcal{C}^n(\mathcal{A}, \mathcal{X}) \to \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{X})$ is the coboundary and characterized as

$$(\delta^n f)(a_1, \dots, a_{n+1}) = a_1 f(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1}) + (-1)^{n+1} f(a_1, \dots, a_n) a_{n+1}, \tag{5}$$

and the kernel $Z^n(\mathcal{A}, \mathcal{X})$ of δ^n are the n-cocyclies. The image $B^n(\mathcal{A}, \mathcal{X})$ of δ^{n-1} is the coboundaries, $\delta^{n+1} \circ \delta^n = 0$. The Banach cohmology group of $\mathcal{A}$ is given by;

$$\mathcal{H}^n(\mathcal{A}, \mathcal{X}) = Z^n(\mathcal{A}, \mathcal{X}) \big/ B^n(\mathcal{A}, \mathcal{X}). \tag{6}$$

Definition (2.9): [15]

Let $\mathcal{M}_*$ be the Banach $\mathcal{A}$-bimodule, then the double $\mathcal{M} = (\mathcal{M}_*)^*$ is the Banach $\mathcal{A}$-bimodule. If $\mathcal{X}$ to be the normed Banach cohomology $\mathcal{H}^1(\mathcal{A}, \mathcal{M}) = \{0\}$, then $\mathcal{A}$ is called amenable. If $\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = \{0\}$, then $\mathcal{A}$ is contractible.

Definition (2.10): [16]

For the closed sub-algebra $\mathcal{S}$ of $\mathcal{A}$, we can define the closed subspace $\mathcal{C}_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X})$ of $\mathcal{C}^n(\mathcal{A}, \mathcal{X})$ with δ as the sequence,

$$0 \to \mathcal{C}_{\mathcal{S}}^0(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^0} \dots \to \mathcal{C}_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X}) \xrightarrow{\delta^n} \mathcal{C}_{\mathcal{S}}^{n+1}(\mathcal{A}, \mathcal{X}) \to \dots \tag{7}$$

since for all $a_1, a_2, \dots, a_n \in \mathcal{A}, s \in \mathcal{S}, 1 \le i \le n$ satisfy that,

$$\rho(s a_1, a_2, \dots, a_n) = s \rho(a_1, a_2, \dots, a_n),$$
$$\rho(a_1, a_2, \dots, a_{i-1}, a_i s, a_{i+1}, \dots, a_n) = \rho(a_1, a_2, \dots, a_i, s a_{i+1}, a_{i+2}, \dots, a_n),$$

since $\rho(a_1, a_2, \dots, a_n s) = \rho(a_1, a_2, \dots, a_n) s$. The complex $\mathcal{C}_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X})$ is the $\mathcal{S}$-relative n-cochains and the $\mathcal{S}$-relative n-cocycles $Z_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X})$ are the kernel of δ^n. The $\mathcal{S}$-relative co-boundaries are defined as the image of $\delta^{n-1}: \mathcal{C}_{\mathcal{S}}^{n-1}(\mathcal{A}, \mathcal{X}) \to \mathcal{C}_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X})$ and denoted by $B_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X})$. Then the $\mathcal{S}$-relative cohomology group of $\mathcal{C}_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X})$is;

$$H_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X}) = Z_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X}) \big/ B_{\mathcal{S}}^n(\mathcal{A}, \mathcal{X}). \tag{8}$$

Proposition (2.11): [20]

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two unital Banach algebras. The unital Banach $\mathcal{A}_1, \mathcal{A}_2$ -bimodule $\mathcal{Y}$ and the natural triangular matrix $\mathcal{W} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{Y} \\ 0 & \mathcal{A}_2 \end{bmatrix}$ with norm $\|\cdot\|_{\mathcal{W}}$ which is equivalent to $\|\cdot\|_{\mathcal{A}_i}$ on $\mathcal{A}_i, i = 1,2$, $\|\cdot\|_{\mathcal{Y}}$ on $\mathcal{Y}$ and $\mathcal{X}$ is Banach $\mathcal{W}$-bimodule. If $e_{11} \mathcal{X} e_{22} = \{0\}\ or\ e_{11} \mathcal{W} e_{22} = \{0\}$. Then we get the form,

$$\mathcal{H}^n(\mathcal{W}, \mathcal{X}) = \mathcal{H}^n(\mathcal{A}_1, e_{11} \mathcal{X} e_{11}) \oplus \mathcal{H}^n(\mathcal{A}_2, e_{22} \mathcal{X} e_{22}).$$

Proposition (2.12): [18]

For the Banach algebra $\mathcal{A}$, and closed sub-algebra $\mathcal{B}$ of $\mathcal{A}$ that be amenable. Let $\mathcal{M}$be the double $\mathcal{A}$-bimodule $\forall n \ge 1$ and if $\rho \in \mathcal{C}^n(\mathcal{A}, \mathcal{M})$ such that $(\delta^n \rho)(a_1, \dots, a_{n+1}) = 0$ since the one of $a_1, \dots, a_{n+1} \in \mathcal{B}$. Then there is $\xi \in \mathcal{C}^{n-1}(\mathcal{A}, \mathcal{M})$ $s.h.$ $\forall\, a_1, \dots, a_n \in \mathcal{B}$.

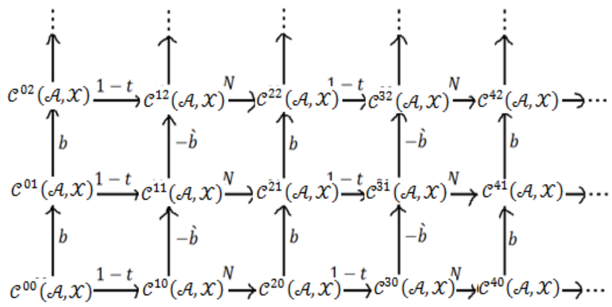
Lemma (2.13): [13]

For the Banach algebra $\mathcal{A}$ and the Banach $\mathcal{A}$-bimodule $\mathcal{X}$. If $\mathcal{B}$ is the closed sub-algebra of $\mathcal{A}$, $\rho \in \mathcal{C}^n(\mathcal{A},\mathcal{X})\ s.h.\ (\delta^n\rho)(a_1,\dots,a_{n+1}) = 0\ \ \forall\, n \geq 1$

for that one of $a_1,\dots,a_{n+1} \in \mathcal{B}$ and $\rho(a_1,\dots,a_n) = 0$ and one of $a_1,\dots,a_n \in \mathcal{B}$. Then, $\rho \in \mathcal{C}_B^n(\mathcal{A},\mathcal{X})$.

Definition (2.14): [19]

Consider the unital Banach algebra $\mathcal{A}$ with coefficients in $\mathcal{A}$-bimodule $\mathcal{X}$, then the cyclic cohomology group of $\mathcal{A}$ is defined as the cohomology of the co-complex $Tot\ \mathcal{CC}^{**}(\mathcal{A})$



as

$$\mathcal{HC}^n(\mathcal{A},\mathcal{X}) = \mathcal{H}^n\big(Tot\mathcal{CC}^{**}(\mathcal{A},\mathcal{X})\big), \quad (9)$$

where $\mathcal{CC}^{**}(\mathcal{A},\mathcal{X})$ is the bicomplex with the horizontal maps $(1-t)^*$ *and* $N^*: \mathcal{CC}^{\alpha\,\beta} \to \mathcal{CC}^{\alpha+1\,\beta}$ and the vertical maps b^* *and* $b^{`*}: \mathcal{CC}^{\alpha\,\beta} \to \mathcal{CC}^{\alpha\,\beta+1}$.

Definition (2.15):

If $\mathcal{A}$ is the unital Banach algebra, then we can define the $\mathcal{S}$-relative cyclic cohomology $\mathcal{HC}_S^n(\mathcal{A},\mathcal{X})$ of the following cochain exact complex, since of the cochain exact complex;

$$0 \to \mathcal{CC}_S^0(\mathcal{A},\mathcal{X}) \xrightarrow{\delta^0} \dots \to \mathcal{CC}_S^n(\mathcal{A},\mathcal{X}) \xrightarrow{\delta^n} \mathcal{CC}_S^{n+1}(\mathcal{A},\mathcal{X}) \to \dots,$$

and the cyclic relative cohomology of $\mathcal{S}$ in $\mathcal{A}$ takes the forum, $\mathcal{HC}_S^n(\mathcal{A},\mathcal{X}) = {Z_S^n(\mathcal{A},\mathcal{X})}\big/{B_S^n(\mathcal{A},\mathcal{X})}$. Where

$Z_S^n(\mathcal{A},\mathcal{X})$ is the kernel of δ^n and $B_S^n(\mathcal{A},\mathcal{X})$ is the image of δ^{n+1} such defined with the maps $\mathfrak{f}(da_0,a_1,\dots,a_n) = \mathfrak{f}(a_0,a_1,\dots,a_n d)$ and $t_n: \mathcal{C}_S^n(\mathcal{A},\mathcal{X}) \to \mathcal{C}_S^n(\mathcal{A},\mathcal{X})$ and satisfy that;

$$t_n\mathfrak{f}(a_0,a_1,\dots,a_n) = (-1)^n\mathfrak{f}(a_n,a_1,\dots,a_0)$$

$$t^n\mathfrak{f}(da_0,a_1,\dots,a_n) = (-1)^n\mathfrak{f}(a_n,a_1,\dots,da_0)$$

$$= (-1)^n\mathfrak{f}(a_1,\dots,a_n d,a_0) = t^n\mathfrak{f}(a_0,a_1,\dots,a_n d).$$

Theorem (2.16):

For the unital Banach algebra $\mathcal{A}$ and the $\mathcal{A}$-bimodule $\mathcal{X}$, the relation between $\mathcal{HH}^n(\mathcal{A},\mathcal{X})$ and $\mathcal{HC}^n(\mathcal{A},\mathcal{X})$ is given as;

$$\cdots \to \mathcal{HH}^n(\mathcal{A},\mathcal{X}) \xrightarrow{B} \mathcal{HC}^{n-1}(\mathcal{A},\mathcal{X}) \xrightarrow{S} \mathcal{HC}^{n+1}(\mathcal{A},\mathcal{X}) \xrightarrow{I} \mathcal{HH}^{n+1}(\mathcal{A},\mathcal{X}) \xrightarrow{B} \cdots. \quad (10)$$

Proof:

For the bicomplex $\mathcal{CC}(\mathcal{A},\mathcal{X})^{\{2\}}$, which contains first two columns of $\mathcal{CC}(\mathcal{A},\mathcal{X})$ and $\mathcal{C}[2,0]_{pq} = \mathcal{C}_{p-2,q}$. Then we have the exact sequence;

$$0 \to \mathcal{CC}(\mathcal{A},\mathcal{X})^{\{2\}} \to \mathcal{CC}(\mathcal{A},\mathcal{X}) \to \mathcal{CC}(\mathcal{A},\mathcal{X})[2,0] \to 0,$$

for the long exact sequence of this sequence, we get

$$\cdots \to \mathcal{HH}^n(\mathcal{A},\mathcal{X}) \xrightarrow{B} \mathcal{HC}^{n-1}(\mathcal{A},\mathcal{X}) \xrightarrow{S} \mathcal{HC}^{n+1}(\mathcal{A},\mathcal{X}) \xrightarrow{I} \mathcal{HH}^{n+1}(\mathcal{A},\mathcal{X}) \xrightarrow{B} \cdots.$$

Theorem (2.17): [17]

Let $\mathcal{S}$ be the closed sub-algebra of $\mathcal{A}$ which is unital and involutive Banach and $\mathcal{X}$ is the $\mathcal{A}$-bimodule. The sequence that relates the $\mathcal{S}$-relative Hochschild and cyclic cohomology group is formed as;

$$\cdots \to \mathcal{HH}_S^n(\mathcal{A},\mathcal{X}) \xrightarrow{B} \mathcal{HC}_S^{n-1}(\mathcal{A},\mathcal{X}) \xrightarrow{S} \mathcal{HC}_S^{n+1}(\mathcal{A},\mathcal{X}) \xrightarrow{I} \mathcal{HH}_S^{n+1}(\mathcal{A},\mathcal{X}) \xrightarrow{B} \cdots. \quad (11)$$

Definition (2.18):

For the Banach $\mathcal{A}$-bimodule $\mathcal{X}$, such $\mathcal{A}$ is involutive Banach algebra. The dihedral cohomology group $\mathcal{HD}^n\big(\mathcal{C}(\mathcal{A},\mathcal{X})\big) = \mathcal{HD}^n(\mathcal{A},\mathcal{X})$ of the cochain complex

$$0 \to \mathcal{CD}^0(\mathcal{A},\mathcal{X}) \xrightarrow{\delta^0} \dots \to \mathcal{CD}^n(\mathcal{A},\mathcal{X}) \xrightarrow{\delta^n} \mathcal{CD}^{n+1}(\mathcal{A},\mathcal{X}) \to \dots,$$

is the sequence;

$$0 \to \mathcal{HD}^0(\mathcal{A},\mathcal{X}) \xrightarrow{\delta^0} \dots \to \mathcal{HD}^n(\mathcal{A},\mathcal{X}) \xrightarrow{\delta^n} \mathcal{HD}^{n+1}(\mathcal{A},\mathcal{X}) \to \dots,$$

with maps:

$$\sigma_n^i: [n+1] \to [n],\ \ \delta_n^i: [n-1] \to [n],\ \ \rho_n: [n] \to [n] \text{ and } \tau_n: [n] \to [n],$$

that satisfy:

$$\begin{cases} \delta_{n+1}^j \delta_n^i = \delta_{n+1}^i \delta_n^{j-1} & if\ i < j \\ \sigma_n^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j-1} & if\ i \leq j \\ \sigma_n^j \delta_{n+1}^i = \begin{cases} \delta_{n-1}^i \sigma_{n-2}^{j-1} & if\ \ i < j \\ Id_{[n]} & if\ i = j\ or\ j+1 \\ \delta_{n+1}^{i-1} \sigma_n^j & if\ i > j+1 \end{cases} \end{cases}, \quad (12)$$

$$\begin{cases} \tau_n \delta_n^i = \delta_n^{i-1} \tau_{n-1} & 1 \leq i \leq n \\ \tau_n \sigma_n^j = \sigma_n^{j-1} \tau_{n+1} & 1 \leq j \leq n \\ \tau_n^{n+1} = Id_{[n]} \end{cases} \quad (13)$$

$$\begin{cases} \rho_n \delta_n^i = \delta_n^{i-1} \rho_{n-1} & 0 \leq i \leq n \\ \rho_n \sigma_n^j = \sigma_n^{j-1} \rho_{n+1} & 0 \leq j \leq n \end{cases}, \quad (14)$$

$$\rho_n^2 = Id_{[n]}, \quad (15)$$

$$\tau_n \rho_n = \rho_n \tau_n^{-1}. \quad (16)$$

Theorem (2.19):

Consider the unital involutive Banach algebra $\mathcal{A}$ with coefficients in $\mathcal{A}$-bimodule$\mathcal{X}$. The relation between cyclic and dihedral cohomology group ($\mathcal{HC}^n(\mathcal{A},\mathcal{X})$ and ${}^{\alpha}\mathcal{HD}^{n+1}(\mathcal{A},\mathcal{X})$, $\alpha = \pm 1$) is;

$$\cdots \to {}^{-}\mathcal{HD}^n(\mathcal{A},\mathcal{X}) \xrightarrow{i^*} \mathcal{HC}^n(\mathcal{A},\mathcal{X}) \xrightarrow{j^*} \mathcal{HD}^n(\mathcal{A},\mathcal{X}) \to {}^{-}\mathcal{HD}^{n+1}(\mathcal{A},\mathcal{X}) \to \cdots. \tag{17}$$

Proof:

Let $\mathcal{C}(\mathcal{A},\mathcal{X})$be the total bi-complex and $\mathcal{D}(\mathcal{A},\mathcal{X})$ be the tri-complex. Then the short exact sequence,

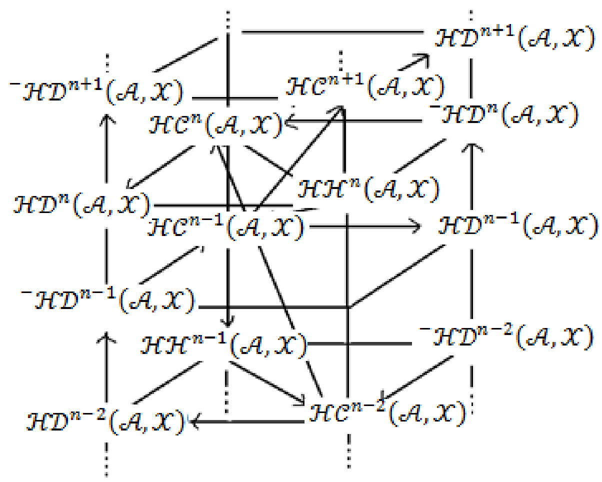
$$0 \to Tot\ \mathcal{C}(\mathcal{A},\mathcal{X}) \to Tot\ \mathcal{D}(\mathcal{A},\mathcal{X}) \to Tot\ {}^{-}\mathcal{D}(\mathcal{A},\mathcal{X}) \to 0,$$

Has the long exact sequence

$$\cdots \to {}^{-}\mathcal{HD}^n(\mathcal{A},\mathcal{X}) \xrightarrow{i^*} \mathcal{HC}^n(\mathcal{A},\mathcal{X}) \xrightarrow{j^*} \mathcal{HD}^n(\mathcal{A},\mathcal{X}) \to {}^{-}\mathcal{HD}^{n+1}(\mathcal{A},\mathcal{X}) \to \cdots.$$

Theorem (2.20): [17]

For $\mathcal{A}$-bimodule $\mathcal{X}$, consider the involutive and unital Banach algebra $\mathcal{A}$ with coefficients $\mathcal{X}$. The relation among $\mathcal{HH}^n(\mathcal{A},\mathcal{X}), \mathcal{HC}^n(\mathcal{A},\mathcal{X})$ and $\mathcal{HD}^n(\mathcal{A},\mathcal{X})$ is the sequence;



Theorem (2.21):

For the Banach algebra $\mathcal{A}$, then we have $\mathcal{H}^n(\mathcal{A}\otimes\mathcal{E}) \cong \mathcal{H}^n(\mathcal{A})$, $\forall n \geq 1$.

Proof:

Consider $\mathcal{X} = \mathcal{A}\otimes\mathcal{E}$ and $\mathcal{X}_i = \mathcal{A}\otimes\mathcal{E}_i$ since $i = 1,2$. Then the traid $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X})$ is exact. Let

$$\mathcal{V}_1 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, a \in \mathcal{A}\right\}, \mathcal{V}_2 = \left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a & 0 & 0 \end{pmatrix}, a \in \mathcal{A}\right\}$$

and the projections in $\mathcal{X}_1 \cap \mathcal{X}_2$ be defined as

$$e_1 = e_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$$

since the completely bounded surjective homorphisms are $\wp: \mathcal{X} \longrightarrow \mathcal{X}_1$, $q; \mathcal{X} \longrightarrow \mathcal{X}_2$. The symbol $\oplus$ means the direct sum of $\mathcal{A}$. From the long exact sequence;

$$\mathcal{H}^1(\mathcal{X}_1)\oplus\mathcal{H}^1(\mathcal{X}_2) \xrightarrow{i_1} (\mathcal{H}^1(\mathcal{X}_1 \cap \mathcal{X}_2) \xrightarrow{\wp_2} \cdots$$
$$\to \mathcal{H}^{n-1}(\mathcal{X}_1)\oplus\mathcal{H}^{n-1}(\mathcal{X}_2) \xrightarrow{i_{n-1}} \mathcal{H}^{n-1}(\mathcal{X}_1 \cap \mathcal{X}_2)$$
$$\xrightarrow{\wp_n} \mathcal{H}^n(\mathcal{X}) \xrightarrow{q_n} \mathcal{H}^n(\mathcal{X}_1)\oplus\mathcal{H}^n(\mathcal{X}_2)$$
$$\xrightarrow{i_n} \mathcal{H}^n(\mathcal{X}_1 \cap \mathcal{X}_2) \to \cdots. \tag{18}$$

Then we can compute $\mathcal{H}^n(\mathcal{X})$ since the surjective is

$$i_n: \mathcal{H}^n(\mathcal{X}_1)\oplus\mathcal{H}^n(\mathcal{X}_2) \longrightarrow \mathcal{H}^n(\mathcal{X}_1 \cap \mathcal{X}_2) \quad \forall n \geq 1$$

The restriction of cocycles on $\mathcal{X}_1$ to $\mathcal{X}_1 \cap \mathcal{X}_2$ is produced of $\varphi_1\oplus\varphi_1\oplus\varphi_2$, but the restriction of cocycles on $\mathcal{X}_2$ to $\mathcal{X}_1 \cap \mathcal{X}_2$ is produced of $\varphi_3\oplus\varphi_4\oplus\varphi_3$ and the range of i_n is

$$\{[(\varphi_1 - \varphi_3)\oplus(\varphi_1 - \varphi_4)\oplus(\varphi_2 - \varphi_3)]: \varphi_i \in Z^n(\mathcal{A})\}, \tag{19}$$

but $\forall n \geq 2$ $\wp_n$ is zero map and q_n is injective and thus

$$\mathcal{H}^n(\mathcal{X}) \cong \mathrm{Im}q_n = \mathrm{Ker}i_n. \tag{20}$$

Consider the unitary permutation $\mathcal{U}$ $s.h.$ $\mathcal{U}^*\mathcal{E}_1\mathcal{U} = \mathcal{E}_2$. Then there exists an isomorphism $\pi: \mathcal{A}\otimes\mathcal{E}_1 \to \mathcal{A}\otimes\mathcal{E}_2$ which induces the isomorphism

$$[\psi] \longrightarrow [\psi_\pi], \ \mathcal{H}^n(\mathcal{X}_1) \longrightarrow \mathcal{H}^n(\mathcal{X}_2),$$

and is defined by: $\forall y_i \in \mathcal{X}_2$;

$$\psi_\pi(y_1, \dots, y_n) = \pi\psi\big(\pi^{-1}(y_1), \dots, \pi^{-1}(y_n)\big).$$

That follows (20) and we get $i_n \cong \mathcal{H}^n(\mathcal{A})$. Then, we get the required.

Theorem (2.22):

For the cohomology of $\mathcal{A}\otimes\mathcal{A}_4$, we have $\forall\ n \geq 1$,

$$\mathcal{H}^n(\mathcal{A}\otimes\mathcal{A}_4) \cong \mathcal{H}^n(\mathcal{A})\oplus\mathcal{H}^{n-1}(\mathcal{A}). \tag{21}$$

Proof:

For $n \geq 2$, consider the following subspaces and subalgebras of $\mathcal{M}_4$,

$$\mathcal{L}_1 = \left\{\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & 0 & * \end{pmatrix}\right\}, \quad \mathcal{L}_2 = \left\{\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix}\right\}$$

$$\mathcal{N}_1 = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}\right\}, \quad \mathcal{N}_2 = \left\{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & 0 & 0 \end{pmatrix}\right\}$$

$$\mathcal{D}_1 = \left\{\begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}\right\}$$

since the stars are arbitrary complex numbers. Let $\mathcal{X} = \mathcal{A}\otimes\mathcal{A}_4$, $\mathcal{X}_1 = \mathcal{A}\otimes\mathcal{L}_1$, $\mathcal{X}_2 = \mathcal{A}\otimes\mathcal{L}_2$, $\mathcal{X}_1 \cap \mathcal{X}_2 = \mathcal{A}\otimes\mathcal{D}_4$, $\mathcal{V}_1 = \mathcal{A}\otimes\mathcal{N}_1$ and $\mathcal{V}_2 = \mathcal{A}\otimes\mathcal{N}_2$. The projections e_1, e_2 and e_3 are

$$e_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Then from the portion of Mayer-vietories sequence of $(\mathcal{X}_1, \mathcal{X}_2, \mathcal{X})$;

$$\mathcal{H}^{n-1}(\mathcal{X}_1) \oplus \mathcal{H}^{n-1}(\mathcal{X}_2) \xrightarrow{i_{n-1}} \mathcal{H}^{n-1}(\mathcal{X}_1 \cap \mathcal{X}_2) \xrightarrow{\mathcal{p}_n}$$

$$\mathcal{H}^n(\mathcal{X}) \xrightarrow{\mathcal{q}_n} \mathcal{H}^n(\mathcal{X}_1) \oplus \mathcal{H}^n(\mathcal{X}_2) \xrightarrow{i_n} \mathcal{H}^n(\mathcal{X}_1 \cap \mathcal{X}_2) \quad (22)$$

where $\mathcal{X}_1 = (\mathcal{A} \oplus \mathcal{E})$, then we have $\mathcal{H}^n(\mathcal{X}_1) \cong \mathcal{H}^n(\mathcal{A}) \oplus \mathcal{H}^n(\mathcal{A})$. Then the image of cocycles under i_n is $\{[\varphi] \oplus [\psi] \oplus [\psi] \oplus [\psi] : \varphi, \psi \in Z^n(\mathcal{A})\}$.

Thus, for applying i_i on $\mathcal{H}^n(\mathcal{X}_2)$, we get $\{[\zeta] \oplus [\eta] \oplus [\zeta] \oplus [\zeta] : \zeta, \eta \in Z^n(\mathcal{A})\}$. Then the image of i_n is

$$\{[\varphi - \zeta] \oplus [\psi - \eta] \oplus [\psi - \zeta] \oplus [\psi - \zeta] : \varphi, \psi, \zeta, \eta \in Z^n(\mathcal{A})\} \quad (23)$$

Then; $Ker\ i_n = \{[\varphi \oplus \varphi \oplus I_3] \oplus [\varphi \oplus \varphi \oplus I_3]_\pi ; \varphi \in Z^n(\mathcal{A})\}$, $Ker\ i_n \cong \mathcal{H}^n(\mathcal{A})$ from (19) we have,

$$\mathcal{H}^n(\mathcal{X}) \cong Im \mathcal{q}_n \oplus Ker\ \mathcal{q}_n = Ker\ i_n \oplus Im\ \mathcal{p}_n$$

$$\cong Ker\ i_n \oplus \mathcal{H}^{n-1}(\mathcal{X}_1 \cap \mathcal{X}_2) / Ker\ \mathcal{p}_n$$

$$= Ker\ i_n \oplus \mathcal{H}^{n-1}(\mathcal{X}_1 \cap \mathcal{X}_2) / Im\ i_{n-1}. \quad (24)$$

From (24);

$$Im\ i_{n-1} = \{[\alpha] \oplus [\beta] \oplus [\gamma] \oplus [\gamma] : \alpha, \beta, \gamma \in Z^{n-1}(\mathcal{A})\},$$

$$\mathcal{H}^{n-1}(\mathcal{X}_1 \cap \mathcal{X}_2) = Im\ i_{n-1} \oplus \{0 \oplus 0 \oplus 0 \oplus [\varphi] : \varphi Z^{n-1}(\mathcal{A})\}.$$

Thus $\mathcal{H}^{n-1}(\mathcal{X}_1 \cap \mathcal{X}_2) / Im\ i_{n-1} \cong \mathcal{H}^n(\mathcal{A})$ and from (21);

$$\mathcal{H}^n(\mathcal{A} \otimes \mathcal{A}_4) \cong \mathcal{H}^n(\mathcal{A}) \oplus \mathcal{H}^{n-1}(\mathcal{A}).$$

3. Main Results

Here, we obtained the main result of this paper. The relation between the relative Banach dihedral and cyclic cohomology group ($\mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}]$ and $\mathcal{HC}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}]$) of $\mathcal{A}$ will be proved. We showed the Morita invariance theorem in the relative case and proved the Connes-Tsygan exact sequence that relates the relative cyclic and dihedral (co)homology of $\mathcal{A}$. Here we used [14], [17], [18] and [19].

Definition (3.1):

Let $\mathcal{HD}^n(\mathcal{A}, \mathcal{X})$ be the dihedral cohomology of $\mathcal{A}$ with coefficients in $\mathcal{X}$. If $\mathcal{B}$ is the closed sub-algebra of $\mathcal{A}$, then the Banach $\mathcal{B}$-relative dihedral cohomology group $\mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}]$ of $\mathcal{A}$, then the sequence of dihedral cohomology of the exact complex;

$$0 \to \mathcal{CD}^0[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \xrightarrow{\delta^0} \cdots \to \mathcal{CD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \xrightarrow{\delta^n} \mathcal{CD}^{n+1}[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \cdots,$$

is the sequence,

$$0 \to \mathcal{HD}^0[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \xrightarrow{\delta^0} \cdots \to \mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \xrightarrow{\delta^n} \mathcal{HD}^{n+1}[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \cdots.$$

The $\mathcal{B}$-relative dihedral cohomology group is;

$$\mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] = Z^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] / B^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}].$$

since $Z^n[(A, S), X]$ is the kernel of δ^n and $B^n[(A, S), X]$ is the image of δ^{n+1} and satisfy the relations (12), (13), (14), (15) and (16).

The sequence which related between $C^n(\mathcal{A}, \mathcal{X})$ and $\mathcal{CD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}]$ is;

$$0 \to \mathcal{CD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \mathcal{CD}^n(\mathcal{A}, \mathcal{X}) \to \mathcal{CD}^n(\mathcal{A}/\mathcal{B}, \mathcal{X}) \to 0,$$

and the relation between dihedral and $\mathcal{B}$-relative dihedral cohomology group is;

$$0 \to \mathcal{HD}^0[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \mathcal{HD}^0(\mathcal{A}, \mathcal{X}) \to \mathcal{HD}^0(\mathcal{A}/\mathcal{B}, \mathcal{X}) \to \mathcal{HD}^1[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \mathcal{HD}^1(\mathcal{A}, \mathcal{X}) \to \cdots \to \mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \mathcal{HD}^n(A, \mathcal{X}) \to \mathcal{HD}^n(\mathcal{A}/\mathcal{B}, \mathcal{X}) \to \mathcal{HD}^{n+1}[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \cdots.$$

Example (3.2):

Consider two unital Banach algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, and $\mathcal{Y}$ is the unital Banach $\mathcal{A}_1$, $\mathcal{A}_2$ -bimodule. For natural triangular matrix algebra $\mathcal{W} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{Y} \\ 0 & \mathcal{A}_2 \end{bmatrix}$ defined with norm $\|\cdot\|_\mathcal{W}$ which is equivalent to $\|\cdot\|_{\mathcal{A}_i}$ on $\mathcal{A}_i$, $i = 1,2$ and $\|\cdot\|_\mathcal{Y}$ on $\mathcal{Y}$, then $\forall\ n \geq 0$, we obtained

$$\mathcal{HD}^n(\mathcal{W}) = \mathcal{HD}^n(\mathcal{A}_1) \oplus \mathcal{HD}^n(\mathcal{A}_2). \quad (25)$$

Theorem (3.3):

For the closed sub-set $\mathcal{B}$ of the unital Banach algebra $\mathcal{A}$ and $\mathcal{A}$ -bimodule $\mathcal{X}$, then the relation between $\mathcal{HC}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}]$ and $\mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}]$ is;

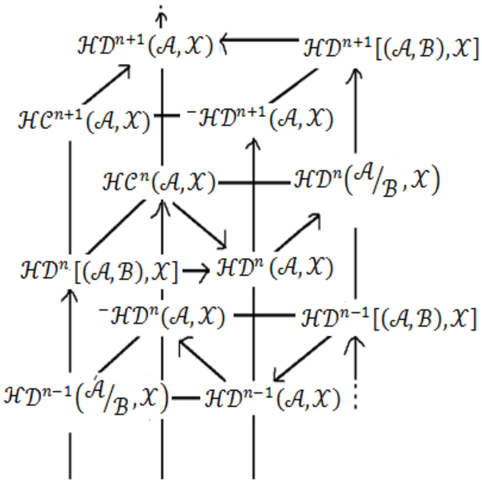
$$\cdots \to {}^{-}\mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \xrightarrow{i^*} \mathcal{HC}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \xrightarrow{j^*} \mathcal{HD}^n[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to {}^{-}\mathcal{HD}^{n+1}[(\mathcal{A}, \mathcal{B}), \mathcal{X}] \to \cdots \quad (26)$$

Proof:

We can prove this theorem as similar as in theorem (2.19).

Theorem (3.4):

Let $\mathcal{B}$ be closed sub-set of the involutive Banach algebra $\mathcal{A}$ and $\mathcal{A}$ -bimodule $\mathcal{X}$. If we instead ${}^{\alpha}\mathcal{HD}^n(\mathcal{A},\mathcal{X})$ by $\mathcal{HD}^n(\mathcal{A},\mathcal{X})$, then we can relate among $\mathcal{H}^n(\mathcal{A},\mathcal{X}), \mathcal{HC}^n(\mathcal{A},\mathcal{X})$ and $\mathcal{HD}^n(\mathcal{A},\mathcal{X})$ as;



Proof:

By the exact sequences

$$0 \to \mathcal{HD}^0[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \mathcal{HD}^0(\mathcal{A},\mathcal{X}) \to \mathcal{HD}^0(\mathcal{A}/\mathcal{B},\mathcal{X}) \to \mathcal{HD}^1[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \mathcal{HD}^1(\mathcal{A},\mathcal{X}) \to \cdots \to \mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \mathcal{HD}^n(\mathcal{A},\mathcal{X}) \to \mathcal{HD}^n(\mathcal{A}/\mathcal{B},\mathcal{X}) \to \mathcal{HD}^{n+1}[(\mathcal{A},\mathcal{B}),\mathcal{X}] \to \cdots,$$

and

$$\cdots \to {}^{-}\mathcal{HD}^n(\mathcal{A},\mathcal{X}) \xrightarrow{i^*} \mathcal{HD}^n(\mathcal{A},\mathcal{X}) \xrightarrow{j^*} \mathcal{HD}^n(\mathcal{A},\mathcal{X}) \to {}^{-}\mathcal{HD}^{n+1}(\mathcal{A},\mathcal{X}) \to \cdots.$$

Then we get the required.

Definition (3.5):

Consider the involutive Banach algebra $\mathcal{A}$ with coefficients in $\mathcal{X}$, the subset $\mathcal{B}$ of $\mathcal{A}$ and bimodule $\mathcal{M}$. Let the matrix $\mathcal{M}_r(\mathcal{M})$ be the degree of $r \times r$. Then the inclusion is

$inc: \mathcal{M}_r(\mathcal{M}) \to \mathcal{M}_{r+1}(\mathcal{M})$, as the form

$$\alpha \mapsto \begin{bmatrix} & & 0 \\ & \alpha & \bullet \\ 0 & \bullet \quad 0 & 0 \end{bmatrix},$$

since the trace map $tr: \mathcal{M}_r(\mathcal{M}) \to \mathcal{M}$ is given by $tr(\alpha) = \sum_{i=1}^{r} \alpha_{ii}$ and the generalized trace map $tr: \mathcal{M}_r(\mathcal{M}) \otimes \mathcal{M}_r(\mathcal{B})^{\otimes n} \to \mathcal{M} \otimes \mathcal{B}^{\otimes n}$ as $tr(\alpha \otimes \beta \otimes \cdots \otimes \eta) = \sum(\alpha_{i_0 i_1} \otimes \beta_{i_1 i_2} \otimes \ldots \otimes \eta_{i_n i_0})$.

Theorem (3.6):

Consider the involutive Banach algebra $\mathcal{A}$ with coefficients in $\mathcal{X}$ and $\mathcal{B} \subset \mathcal{A}$. Then we have two isomorphisms that are inverse to each other,

$$tr^*: \mathcal{HD}^*(\mathcal{M}_r(\mathcal{B}), \mathcal{M}_r(\mathcal{M})) \to \mathcal{HD}^*(\mathcal{B},\mathcal{M}), \quad (27)$$

$$inc^*: \mathcal{HD}^*(\mathcal{B},\mathcal{M}) \to \mathcal{HD}^*(\mathcal{M}_r(\mathcal{B}), \mathcal{M}_r(\mathcal{M})). \quad (28)$$

Proof:

Consider the pre-simplicial homotopy $\mathcal{k} = \sum(-1)^i \mathcal{k}_i$ such that $\mathcal{h}_i: \mathcal{M}_r(\mathcal{M}) \otimes \mathcal{M}_r(\mathcal{B})^{\otimes n} \to \mathcal{M}_r(\mathcal{M}) \otimes \mathcal{M}_r(\mathcal{B})^{\otimes n+1}$ which is defined by;

$$\mathcal{h}_i(a^0, \ldots, a^n) = \sum \mathcal{E}_{j1}(a_{jk}^0) \otimes \mathcal{E}_{11}(a_{km}^1) \otimes \ldots \otimes \mathcal{E}_{11}(a_{pq}^i) \otimes \mathcal{E}_{1q}(1) \otimes a^{i+1} \otimes \ldots \otimes a^n$$

for $a^0 \in \mathcal{M}_r(\mathcal{M})$ and the others $a^s \in \mathcal{M}_r(\mathcal{B})$ and $\mathcal{h}_i$ satisfy the first three relations in definition (2.7). If $n = 0$: $\mathcal{h}(a) = \mathcal{E}_{j1}(a_{jk}) \otimes \mathcal{E}_{1k}(1)$. If

$$n = 1: \ h(a, \mathcal{b}) = \mathcal{E}_{j1}(a_{jk}) \otimes \mathcal{E}_{1k}(1) \otimes \mathcal{b} - \mathcal{E}_{j1}(a_{jk}) \otimes \mathcal{E}_{11}(\mathcal{b}_{ki})\mathcal{E}_{1l}(1)$$

$$\Rightarrow \ \mathcal{k}d + d\mathcal{h} = d_0\mathcal{h}_0 - d_{n+1}\mathcal{h}_n.$$

Since $d_{n+1}\mathcal{h}_n = inc \circ tr$ and $id = d_0\mathcal{h}_0$, then id and $inc \circ tr$ are homotopic with each other.

Theorem (3.7):

Consider the Banach algebra $\mathcal{A}$ and closed sub-algebra $\mathcal{B}$, let $\mathcal{L}, \mathcal{N} \subset \mathcal{A}$, $\mathcal{K}, \mathcal{J} \subset \mathcal{B}$ such that $\cup \mathcal{K} \subset \mathcal{A}$, $\cup \mathcal{J} \subset \mathcal{B}$. The Mayer-Vietoris sequence for $\mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}]$ is:

$$\cdots \to \mathcal{HD}^{n-1}(\mathcal{L} \cap \mathcal{N}, \mathcal{K} \cap \mathcal{J})_{\mathcal{X}} \xrightarrow{\partial_*} \mathcal{HD}^n(\mathcal{A},\mathcal{B})_{\mathcal{X}} \xrightarrow{k_* - l_*} \mathcal{HD}^n(\mathcal{L},\mathcal{K})_{\mathcal{X}} \oplus \mathcal{HD}^n(\mathcal{N},\mathcal{J})_{\mathcal{X}} \xrightarrow{(i_*, j_*)} \mathcal{HD}^n(\mathcal{L} \cap \mathcal{N}, \mathcal{K} \cap \mathcal{J})_{\mathcal{X}} \xrightarrow{\partial_*} \mathcal{HD}^{n+1}(\mathcal{A},\mathcal{B})_{\mathcal{X}} \to \cdots. \quad (29)$$

Since

$$i_{\mathcal{A}}: \mathcal{L} \hookrightarrow \mathcal{L} \cap \mathcal{N}, \quad i_{\mathcal{B}}: \mathcal{K} \hookrightarrow \mathcal{K} \cap \mathcal{J}, j_{\mathcal{A}}: \mathcal{N} \hookrightarrow \mathcal{L} \cap \mathcal{N}, \quad j_{\mathcal{B}}: \mathcal{J} \hookrightarrow \mathcal{K} \cap \mathcal{J}, \quad k_{\mathcal{A}}: \mathcal{A} \hookrightarrow \mathcal{L}, k_{\mathcal{B}}: \mathcal{B} \hookrightarrow \mathcal{K}, \quad l_{\mathcal{A}}: \mathcal{A} \hookrightarrow \mathcal{N} \text{ and } l_{\mathcal{B}}: \mathcal{B} \hookrightarrow \mathcal{J}.$$

Proof:

Consider the short exact sequence

$$0 \to \mathcal{CD}^n(\mathcal{A},\mathcal{B}) \xrightarrow{\mathcal{g}} \mathcal{CD}^n(\mathcal{L},\mathcal{K}) \oplus \mathcal{CD}^n(\mathcal{N},\mathcal{J}) \xrightarrow{\mathcal{f}} \mathcal{CD}^n(\mathcal{L} \cap \mathcal{N}, \mathcal{K} \cap \mathcal{J}),$$

with the epimorphism $\mathcal{f}$ and the monomorphism $\mathcal{g}$. Since $\mathcal{f} = k - l$, $\mathcal{g} = (i,j)$ and $\mathcal{f} \circ \mathcal{g} = 0$. Consider that $(\mathcal{L} \cap \mathcal{N}, \mathcal{K} \cap \mathcal{J})(\mathcal{U}) \in \mathcal{CD}^n(\mathcal{A},\mathcal{B})$, then

$$(\mathcal{f} \circ \mathcal{g})(\mathcal{A},\mathcal{B})(\mathcal{U}) = \mathcal{f}((\mathcal{L},\mathcal{K})(\mathcal{U}), (\mathcal{N},\mathcal{J})(\mathcal{U})) = 0.$$

Since

$$((\mathcal{L},\mathcal{K})(\mathcal{U}), (\mathcal{N},\mathcal{J})(\mathcal{U})) \in [\mathcal{CD}^n(\mathcal{L},\mathcal{K}) \oplus \mathcal{CD}^n(\mathcal{N},\mathcal{J})].$$

Then the Mayer-vietoris sequence is the sequence;

$$\cdots \to \mathcal{HD}^{n-1}(\mathcal{L} \cap \mathcal{N}, \mathcal{K} \cap \mathcal{J})_{\mathcal{X}}$$

$$\overset{\partial_*}{\rightarrow} \mathcal{HD}^n(\mathcal{A},\mathcal{B})_{\mathcal{X}} \xrightarrow{k_*-l_*} \mathcal{HD}^n(\mathcal{L},\mathcal{K})_{\mathcal{X}} \oplus \mathcal{HD}^n(\mathcal{N},\mathcal{J})_{\mathcal{X}}$$
$$\xrightarrow{(i_*,j_*)} \mathcal{HD}^n(\mathcal{L}\cap\mathcal{N},\mathcal{K}\cap\mathcal{J})_{\mathcal{X}}$$
$$\overset{\partial_*}{\rightarrow} \mathcal{HD}^{n+1}(\mathcal{A},\mathcal{B})_{\mathcal{X}} \rightarrow \cdots.$$

4. Conclusions

We presented the definition of $\mathcal{B}$-relative dihedral cohomology group that is given by

$$\mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}] = {Z^n[(\mathcal{A},\mathcal{B}),\mathcal{X}]}\Big/{B^n[(\mathcal{A},\mathcal{B}),\mathcal{X}]}.$$

The relation between the relative *Banach* dihedral and cyclic cohomology group ($\mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}]$ and $\mathcal{HC}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}]$) of $\mathcal{A}$ was proved. So, we showed, the *Morita invariance* theorem in the relative case and proved the *Connes-Tsygan* exact sequence that relates the relative cyclic and dihedral (co)homology of $\mathcal{A}$.

We proved the *Mayer-Vietoris* sequence of $\mathcal{HD}^n[(\mathcal{A},\mathcal{B}),\mathcal{X}]$ in a new form in the Banach $\mathcal{B}$-relative dihedral cohomology:

$$\cdots \rightarrow \mathcal{HD}^{n-1}(\mathcal{L}\cap\mathcal{N},\mathcal{K}\cap\mathcal{J})_{\mathcal{X}}$$
$$\overset{\partial_*}{\rightarrow} \mathcal{HD}^n(\mathcal{A},\mathcal{B})_{\mathcal{X}} \xrightarrow{k_*-l_*} \mathcal{HD}^n(\mathcal{L},\mathcal{K})_{\mathcal{X}} \oplus \mathcal{HD}^n(\mathcal{N},\mathcal{J})_{\mathcal{X}}$$
$$\xrightarrow{(i_*,j_*)} \mathcal{HD}^n(\mathcal{L}\cap\mathcal{N},\mathcal{K}\cap\mathcal{J})_{\mathcal{X}}$$
$$\overset{\partial_*}{\rightarrow} \mathcal{HD}^{n+1}(\mathcal{A},\mathcal{B})_{\mathcal{X}} \rightarrow \cdots.$$

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Ethics Approval and Consent to Participate

Not applicable.

Consent for Publication

Not applicable.

Competing Interests

The authors declare that they have no competing interests.

Availability of Data and Materials

Not applicable.

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