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The Radii of Starlikeness for Concave Functions

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Abstract Let S denote the functions' class that is normalized, analytic, as well as univalent in the unit disc given by $D = \{z : |z| < 1\}$. Convex, starlike, as well as close-to-convex functions resemble the main subclasses of S, expressed by C, S^*, as well as K, accordingly. Many mathematicians have recently studied radius problems for various classes of functions contained in S. The determination of the univalence radius, starlikeness, and convexity for specific special functions in S is a relatively new topic in geometric function theory. The problem of determining the radius has been initiated since the 1920s. Mathematicians are still very interested in this, particularly when it comes to certain special functions in S. Indeed, many papers investigate the radius of starlikeness for numerous functions. With respect to the open unit disc D and class S, the class of concave functions $f \in S$, known as $Co(\alpha)$, is defined. It is identified as a normalised analytic function f, which meets the requirement of having the opening angle of $f(D)$ at $\infty \leq \pi\alpha \in (1, 2]$. A univalent function $f : D \to \bar{C}$ is known as concave provided that $f(D)$ is concave. In other words, we have that $C \setminus f(D)$ is also convex. There is no literature to date on determining the radius of starlikeness for concave univalent functions related to certain rational functions, lune, cardioid, and the exponential equation. Hence, by employing the subordination method, we present new findings on determining several radii of starlikeness for different subclasses of starlike functions for the class of concave univalent functions $Co(\alpha)$.

Keywords Radius Problems, Radius of Starlikeness, Concave Functions, Starlike Functions, Analytic Univalent Functions

1 Introduction

Consider that S expresses all univalent functions classes $f \in A$, in which f is analytic on the disc $D = z \in C : |z| < 1$. Here, the functions f in class A are normalised using conditions given by $f(0) = f'(0) - 1 = 0$.

The concave functions class $Co(\alpha)$ consists of all functions $f \in A$ satisfying the conditions given below:

(i) In D, f is analytic corresponding to the normalisation $f(0) = 0$ and $f'(0) = 1$. Moreover, it is given that $f(1) = \infty$.

(ii) f conformally maps D onto a set, in which the C complement is convex.

(iii) The $f(D)$ opening angle is at $\infty \leq \pi\alpha \in (1, 2]$.

We now assume that the closed set $C \setminus f(D)$ is convex given that it is unbounded for $f \in Co(\alpha), \alpha \in (1, 2]$. Furthermore, [4] determined the analytic characterization of functions in $Co(\alpha), \alpha \in (1, 2] : f \in Co(\alpha)$ with regards to

$$Re\left\{\frac{2}{\alpha-1}\left(\frac{(\alpha+1)}{2}\frac{1+z}{1-z} - 1 - \frac{zf''(z)}{f'(z)}\right)\right\} > 0, z \in D.$$

We observe that $Co(2)$ comprises the $Co(\alpha)$ classes, where $\alpha \in (1, 2]$.

Additional investigations of concave functions are pointed out in [1], [2], [3], [5], [6], [8], [9], [24] and [25].

The function f, which is an analytic function, subordinate to g is represented as $f \prec g$ provided that there exists a function identified as $w : D \to D$ having $w(0) = 0$ fulfilling $f(z) = g(w(z))$. Here, provided that g is univalent, it follows that $f \prec g$ is true provided that $f(D) \subseteq g(D)$ as well as $f(0) = g(0)$.

Now, we assume a, b, as well as c are complex numbers having $c \neq 0, -1, -2, \ldots$. Subsequently, we construe the Gaussian hypergeometric function ${}_2F_1(a, b, c; z)$ given as follows

$${}_2F_1(a,b,c;z) = F(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k k!}.$$

In regards to the Gamma function, Γ, the Pochhammer symbol $(\gamma)_k$ is expressed as given below

$$(\gamma)_k = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$$

$$= \begin{cases} 1 & (k=0) \\ \gamma(\gamma+1)...(\gamma+k-1) & (k \in N) \end{cases}.$$

It is observed that the Gaussian hypergeometric function, F, fulfills the hypergeometric differential equation given by

$$z(1-z)F''(z) + [c-(a+b+1)z]F'(z) - abF(z) = 0.$$

Numerous authors have extensively researched the Gaussian hypergeometric function (see [7], [16], [17], [18], and [23]). Particularly, other manifold characteristics, convexity, starlikeness, close-to-convexity, and univalency linked with these hypergeometric functions were assessed with regards to a,b, as well as c characteristics by [11], [18], [19] and [21].

For $a = 3$, $b = 1$, and $c = 2$, then

$$f(z) = \frac{2}{3}F(3,1,2;z) = \frac{2}{3}\left(-1 + \frac{1-\frac{z}{2}}{(1-z)^2}\right)$$

belongs to $Co(2)$ as proved by [24].

The compendium of functions $f \in S$ having the condition $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$, in which $z \in D$ represents the starlike function S^*, represents a subclass of S. For $z \in D$, the functions in S having $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ form the subclass K of convex functions. These classes are described in relation to the Carathéodory function class P, or functions having a positive real part, comprising the analytical functions p having $p(0) = 1$. This equates to $Re(p(z)) > 0$, or equivalent to $p(z) \prec \frac{(1+z)}{(1-z)}$. Therefore, the convex and starlike functions classes constitute of $1 + \frac{zf''(z)}{f'(z)} \in P$ and $f \in A$ with $\frac{zf'(z)}{f(z)} \in P$, respectively. Thus, subordination of $\frac{zf'(z)}{f(z)}$, in which $1 + \frac{zf''(z)}{f'(z)}$ to some function in P was utilized to represent a variety of convex functions as well as starlike subclasses. Furthermore, Ma and Minda [13] employed subordination for function in classes $S^*(\zeta) = \{f \in A : \frac{zf'(z)}{f(z)} \prec \zeta(z)\}$ as well as $K(\zeta) = \{f \in A : 1 + \frac{zf''(z)}{f'(z)} \prec \zeta(z)\}$, in which $\zeta \in P$, symmetric about the x-axis, starlike in regards to 1 and $\zeta'(0) > 0$ to give distortion, rotation, a growth unified treatment as well as coefficient inequalities. For distinct function ζ choices, for instance, $\frac{(1+Az)}{(1+Bz)}, e^z, z + \sqrt{1+z^2}$ and so forth, a number of classes were defined.

The $R_G(F)$ is the G-radius of the class F. It is implied as the greatest number of $R_G \in (0,1)$. Hence, we have $r^{-1}f(rz) \in G$ for $0 < r < R_G$ as well as all $f \in F$ for any two subclasses F and G of A. In this report, influenced by [14] and pertaining to the class of concave functions $Co(\alpha)$, we establish the radii of different subclasses of starlike functions, together with starlikeness pertinent to a particular rational function, the exponential equation, cardioid, and lune.

Then, we administer several Lemmas that will be applied in the primary outcomes.

Lemma 1.1. *[12] For $2(\sqrt{2}-1) < c < 2$, let r_c be given by*

$$r_c = \begin{cases} c - 2(\sqrt{2}-1) & if\ 2(\sqrt{2}-1) < c \le \sqrt{2}, \\ 2-c & if\ \sqrt{2} \le c < 2. \end{cases} \quad (1.1)$$

Then,
$\{w : |w-c| < r_c\} \subset \psi(D)$, *where ψ is presented by*

$$\psi(z) = 1 + \frac{z}{k}\left(\frac{k+z}{k-z}\right) = 1 + \frac{1}{k}z + \frac{2}{k^2}z^2 + \frac{2}{k^3}z^3 + ..., k = \sqrt{2}+1.$$

Lemma 1.2. *[10] For $\sqrt{2}-1 < c \le \sqrt{2}+1$, let $r_c = 1 - |\sqrt{2}-c|$ and $R_c = \sqrt{c^2+1}$. Thus*

$$\{w : |w-c| < r_c\} \subset \{w : |w^2-1| < 2|w|\} \subset \{w : |w-c| < R_c\}. \quad (1.2)$$

Lemma 1.3. *[22] For $1/3 < c < 3$, let r_c be given by*

$$r_c = \begin{cases} \frac{(3c-1)}{3} & (1/3 < c \le 5/3), \\ 3-c & (5/3 \le c < 3), \end{cases} \quad (1.3)$$

and R_c be indicated by

$$R_c = \begin{cases} 3-c, & (1/3 < c \le 11/9), \\ \sqrt{\frac{(3c-1)^3}{27(c-1)}}, & (11/9 \le c < 3). \end{cases}$$

Then

$$\{w : |w-c| < r_c\} \subseteq \Omega_c \subseteq \{w : |w-c| < R_c\},$$

in which Ω_c denotes the section represented by the cardioid $\{x+iy : (9x^2+9y^2-18x+5)^2 - 16(9x^2+9y^2-6x+1) = 0\}$.

Lemma 1.4. *[15] For $1/e < c < e$, let r_c be presented by*

$$r_c = \begin{cases} c - \frac{1}{e}, & (1/e < c \le \frac{e+1/e}{2}), \\ e-c & \frac{(e+1/e)}{2} \le c < e. \end{cases} \quad (1.4)$$

Following from there,

$$\{w : |w-c| < r_c\} \subset \{w : |\log w| < 1\}.$$

2 Main Results

This section contains the results of sharp radius for each different subclass.

2.1 S^*_R-radius of $Co(\alpha)$

The starlike functions class having a rational function was introduced by [12], which is related to the rational function $\psi(z) = 1 + \frac{z}{k}\left(\frac{k+z}{k-z}\right) = 1 + \frac{1}{k}z + \frac{2}{k^2}z^2 + \frac{2}{k^3}z^3 + ..., k = \sqrt{2}+1$, represented by $S^*_R = S^*(\psi(z))$. The authors developed Lemma (1.1) that would be employed in the first theorem.

Theorem 2.1. *The S^*_R radius is clarified as $R_{S^*_R} \approx 0.406930$. Thus, $R_{S^*_R}$ refers to the smallest positive root of the polynomial $4r^2 - 9r + 3$. In addition, the results are sharp.*

Proof. For functions $f \in Co(\alpha)$, we obtain

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-z}\right| \le -\frac{z}{(2z-3)(1-z)}, |z| = r < 1, \quad (2.1)$$

where the smallest positive root of equation $4R^2 - 9R + 3 = 0$ is $R = S^*_R$.

For $0 \le r < 1 = R_{S^*}$, the function

$$h(r) = \frac{1}{1-r} - \frac{r}{(2r-3)(1-r)} = \frac{r-3}{(2r-3)(1-r)},$$

is an increasing function of r whereas S^*_R refers to a subclass of S^*, a parabolic starlike class. Here, we have that $R = S^*_R$ resembles the equation $h(r) = 2$ of the smallest positive root; thus, for $0 \le r < R$, we get

$$-\frac{r}{(2r-3)(1-r)} < 2 - \frac{1}{1-r}. \quad (2.2)$$

Therefore, Equations (2.1) and (2.2) yield

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-r}\right| < 2 - \frac{1}{1-r}; |z| \le r.$$

For $r \in [0, R), C(r) \in [1, C(R)) \subseteq [1, C(0.5)] \approx [1, 2] \subseteq (2(\sqrt{2}-1), 2)$ as the centre $C(r)$ of (2.1) is an increasing function of r. Then, we can deduce from (1.1) that the disc $\{w : |w - c| < 2 - \frac{1}{1-r} \subseteq \psi(D)$.

Here, we present R as the precise S^*_R-radius of the $Co(\alpha)$ class. In this study, we employ the subsequent function $f : D \to C$ recognized by

$$f(z) = \frac{2}{3}\left(-1 + \frac{1 - \frac{z}{2}}{(1-z)^2}\right),$$

because it is a representation of a function from the $Co(\alpha)$ class. This function is a Gaussian hypergeometric function that has enough conditions to be in $Co(\alpha)$ and has been proved by [24]. For this function, we have

$$\frac{zf'(z)}{f(z)} = \frac{z-3}{(2z-3)(1-z)},$$

while using the equation for R, we get

$$r - 3 = 2(2r-3)(1-r).$$

The class S^*_R introduced by [12] fulfils the subordination

$$\frac{zf'(z)}{f(z)} \prec \psi(z), (z \in D).$$

Hence, for sharpness, it is easy to see that for $z = R$, a simple calculation depicts that

$$\frac{zf'(z)}{f(z)} = 2 = \psi(1).$$

□

2.2 $S^*_{\rightmoon}$ -radius of $Co(\alpha)$

Raina and Sokól [20] established the $S^*_{\rightmoon} = S^*(z + \sqrt{1+z^2})$ class in 2015, illustrating that $f \in S^*_{\rightmoon} \iff \frac{zf'(z)}{f(z)}$ sets in the lune area $\{w : |w^2 - 1| < 2|w|\}$, where

$$\{w : |w - c| < 1 - |\sqrt{2} - c|\} \subseteq \{w : |w^2 - 1| < 2|w|\}.$$

Theorem 2.2. *The $S^*_{\rightmoon}$ radius is characterized as $R_{S^*_{\rightmoon}} \approx 0.486401$. Here, $R_{S^*_{\rightmoon}}$ refers to the smallest positive root of the polynomial $(2+\sqrt{2})R^2 - (5+2\sqrt{2})R + 3$. Consequently, the result is sharp.*

Proof. Since $C(r)$ resembles an increasing function of r, therefore for $R = R_{S^*_{\rightmoon}}$, the centre of (2.1) yields $C(R) = \sqrt{2}$. Here, we have $0 \le r < R, 1 \le C(r) < \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{9}{2} + 4\sqrt{2}}\right)$ or $0 \le r < R, \frac{1}{2}\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{9}{2} + 4\sqrt{2}}\right) - C(r) > 0$.

Thus, equation $(2+\sqrt{2})R^2 - (5+2\sqrt{2})R + 3 = 0$ has the smallest positive root when $R = R_{S^*_{\rightmoon}}$.

We indicate that $S^*_{\rightmoon}$ is a subclass of the parabolic starlike class S^*. For $0 \le r < R$, we hold

$$-\frac{r}{(2r-3)(1-r)} < 1 + \sqrt{2} - \frac{1}{1-r} = 1 - \left|\sqrt{2} - \frac{1}{1-r}\right|, \quad (2.3)$$

where the function

$$h(r) = \frac{1}{1-r} - \frac{r}{(2r-3)(1-r)} = \frac{r-3}{(2r-3)(1-r)},$$

refers to an increasing function of r for $0 < r < 1 = R_{S^*}$. Here, equation $h(r) = \sqrt{2}+1, R = R_{S^*_{\rightmoon}}$ expresses the smallest positive root.

Therefore, (2.1) as well as (2.3) yield

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-r}\right| < 1 - \left|\sqrt{2} - \frac{1}{1-r}\right| : |z| \le r.$$

We can now deduce from (1.2) that R is the correct radius. For $r \in [0,R), C(r) \in [1, C(R)) \subseteq [1, c(0.5)) \approx [1,2) \subseteq (\sqrt{2}-1, \sqrt{2}+1)$, the centre $C(r)$ of (2.1) is an increasing function of r.
Consider

$$f(z) = \frac{2}{3}\left(-1 + \frac{1-\dfrac{z}{2}}{(1-z)^2}\right).$$

For this given function

$$\frac{zf'(z)}{f(z)} = \frac{z-3}{(2z-3)(1-z)}.$$

Using the equation for R, we have

$$r-3 = (\sqrt{2}+1)(2r-3)(1-r),$$

and we may efficiently observe that for $z = \frac{1}{2}\left(3 - \frac{1}{\sqrt{2}} - \sqrt{3\sqrt{2}-\frac{5}{2}}\right)$,

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| = 2\left(\frac{zf'(z)}{f(z)}\right) = 2(\sqrt{2}+1).$$

Thus, the result is sharp.

□

2.3 S^*_C-radius of $Co(\alpha)$

Sharma et al. [22] examined the class $S^*_c = S^*(\phi_c) = S^*\left(1 + \left(\frac{4}{3}\right)z + \left(\frac{2}{3}\right)z^2\right)$ and established Lemma (1.3).

Theorem 2.3. *The S^*_c radius is defined as $R_{S^*_c} \approx 0.565741$. Here, $R_{S^*_c}$ resembles the smallest positive root of the polynomial $6R^2 - 14R + 6$. The findings here is definite.*

Proof. $R = R_{S^*_c}$ denotes the smallest positive root that can be acquired from the equation

$$6R^2 - 14R + 6 = 0.$$

The class S^*_c is a subclass of the parabolic starlike class S^*. For $0 < r < 1 = R_{S^*_c}$, the function

$$h(r) = \frac{1}{1-r} - \frac{r}{(2r-3)(1-r)} = \frac{r-3}{(2r-3)(1-r)},$$

denotes an increasing function of r with $R = R_{S^*_c}$ as the smallest positive root of equation $h(r) = 3$. Subsequently, we get

$$-\frac{r}{(2r-3)(1-r)} < 3 - \frac{1}{1-r}, \tag{2.4}$$

for $0 < r < R$. Therefore, (2.1) as well as (2.4) generate

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-r}\right| < 3 - \frac{1}{1-r}, |z| = r < R.$$

Thus, for

$$r \in (0,R), C(r) \in (C(0.4), C(0.6)) \approx (1.666667, 2.5) \subseteq \left(\frac{1}{3}, \frac{5}{3}\right),$$

the centre $C(r)$ of (2.1) refers to an increasing function of r. Currently, by applying Lemma (1.3), we manage to signify that the disc $\{w : |w-c| < 3-c\} \subseteq \Omega_c$.
For verifying sharpness, we analyze the following function

$$f(z) = \frac{2}{3}\left(-1 + \frac{1-\frac{z}{2}}{(1-z)^2}\right).$$

For this function,

$$\frac{zf'(z)}{f(z)} = \frac{z-3}{(2z-3)(1-z)},$$

and using the equation for R, we get $R-3 = 3(2R-3)(1-R)$. Hence, for $z = R$, we obtain

$$\frac{zf'(z)}{f(z)} = 3 = \phi_c(1).$$

□

2.4 S^*_e-radius of $Co(\alpha)$

Mendiratta et al. [15] proposed the S^*_e subclass of starlike functions pertinent to the exponential equation. All functions $f \in A$ are consequent to the subordination $\frac{zf'(z)}{f(z)} \prec e^z$ are in this class S^*_e. Here, the inequality $\left|\log\left(\frac{zf'(z)}{f(z)}\right)\right| < 1$ as well as $\frac{zf'(z)}{f(z)} \prec e^z$ equals to each other.

Theorem 2.4. *The S^*_e radius is described as $R_{S^*_e} \approx 0.531250$. Then, $R_{S^*_e}$ denotes the smallest positive root of the polynomial $2eR^2 + (1-5e)R - (3-3e)$. Here, the result is sharp.*

Proof. For $0 < r < 1 = R_{S^*}(Co(\alpha))$, the function

$$h(r) = \frac{1}{1-r} - \frac{r}{(2r-3)(1-r)} = \frac{r-3}{(2r-3)(1-r)},$$

is an increasing function of r. The number $R = R_{S^*_e}(Co(\alpha)) < 1 = R_{S^*}(Co(\alpha))$ is the polynomial's smallest positive root as given below

$$2eR^2 + (1-5e)R - (3-3e) = 0, \tag{2.5}$$

or $h(R) = e$. Thus, it shows that $0 < r < R$, $h(r) < h(R) = e$. Therefore

$$-\frac{r}{(2r-3)(1-r)} < e - \frac{1}{1-r}. \tag{2.6}$$

Therefore, Equations (2.1) as well as (2.6) yield

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1-r}\right| < e - \frac{1}{1-r}, |z| = r < R. \tag{2.7}$$

For $r \in (0, R)$, the function $C(r) = \dfrac{1}{(1-r)}$ is an increasing function of r. Hence, it ensures that $C(r) \in (C(0.4), C(0.6)) \approx (1.666666, 2.5) \subseteq (1.543081, 2.718281) \approx \left(\dfrac{e+e^{-1}}{2}, e\right)$. Through Lemma (1.4) for $\dfrac{e+e^{-1}}{2} < c < e$, it fulfils $\{w : |w-c| < r_c\} \subseteq \{w : |\log(w) < 1|\}$ when r_c is provided by (1.4).

From (2.7), we identify that $w = \dfrac{zf'(z)}{f(z)}, |z| < R$ fulfills $|w-c| < e-c$ and, hence, it pursues that $|\log(w) < 1|$. This indicates that the class $Co(\alpha)$ has a S_e^*-radius of at least R. To demonstrate the sharpness of the result, we refer to the function

$$f(z) = \frac{2}{3}\left(-1 + \frac{1-\frac{z}{2}}{(1-z)^2}\right).$$

Using Equation (2.5) in the following function

$$\frac{zf'(z)}{f(z)} = \frac{z-3}{(2z-3)(1-z)},$$

We get $R - 3 = e(2R-3)(1-R)$; thus $z = R$ yields

$$\left|\log\left(\frac{zf'(z)}{f(z)}\right)\right| = \left|\log\left(\frac{z-3}{(2z-3)(1-z)}\right)\right| = |\log(e)| = 1.$$

Consequently, this establishes that R is determined to be the precise S_e^*-radius of the class $Co(\alpha)$.

□

3 Conclusion

For the class of concave univalent functions $Co(\alpha)$, the radii of different subclasses of starlike functions, involving starlikeness connected with a particular rational function $R_{S_R^*}$, lune $R_{S_{\complement}^*}$, cardioid $R_{S_c^*}$, and the exponential function $R_{S_e^*}$, are obtained in this paper.

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