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Pricing of A European Call Option in Stochastic Volatility Models

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Abstract Volatility occupies a strategic place in the financial markets. In this context of crisis, and with the great movements of the markets, traders have been forced to turn to volatility trading for the potential gain it provides. The Black-Scholes formula for the value of a European option to purchase the underlying depends on a few parameters which are more or less easy to calculate, except for the realized volatility at maturity which makes a problem, because there is no single value, nor an established way to calculate it.
In this article, we exploit the Martingale pricing method to find the expected present value of a given asset relative to a risk-neutral probability measure. We consider a bond-stock market that evolves according to the dynamics of the Black-Scholes model, with a risk-free interest rate varying with time. Our methodology has effectively directed us towards interesting formulas that we have derived from the exact calculation, giving the present value of the volatility realized over a period of maturity for a European option in a stochastic volatility model.

Keywords Stochastic Model, Volatility, SABR Model, Pricing

AMS Classification Numbers: 11B39, 33C05

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1 Introduction

Nobody denies that the development of financial markets is mainly due to the availability of financial instruments such as stocks and bonds, and the flexibility of their barter, sale and purchase, as well as the availability of information that allows owners and investors to make reasonable decisions about their transactions in the financial market, without being emotionally drawn by feelings of greed or fear.
Among the financial products which are strongly involved in these operations and which are very well known in Europe and America, we find the standard call or put option: This is a contract by which its owner can, without obligation, buy or sell a quantity of the asset at a time to be determined in advance or later, provided that this time does not exceed a date which we call the expiration date of the contract. To reduce the risks, the owner must base himself on a solid reason which allows him to determine this date before or choose it well when it arrives, by calculating the expected pricing of the financial products on a fixed date.
Calculating option pricing has always been a subject of stochastic modeling. The first was that of Black-Scholes [2] , which was introduced in 1973, and it is the basic form for the underlying models of price dynamics that will come after. It is composed of a predictable part and a fluctuation part that cannot be controlled, the latter representing a Brownian motion

amplified by a constant which is called in this field the volatility of the model.
This constant volatility did not help explain the findings that the volatility extracted from the B and S formula which was in fact not constant at all. Stochastic volatility models will effectively overcome this need. Among these models, we cite that of Barndorff-Nielsen and Shephard (BNS) which was created in 2001 [5]. The volatility of the BNS is a stochastic process given as a Levy process with positive jumps.
One of the financial instruments created to manage the risk of changes in the volatility of different assets is the volatility swap [9] and [10], it is a forward contract on the realized future volatility of an asset, which allows investors to directly trade the volatility of an asset in a manner similar to price index trading and obtain a gain given by a clear formula when the expiration date arrives.
The problem lies in the fact that the gain depends on the realized volatility of the asset at maturity, and is not easy to calculate since there is no exact formula that gives its value.
An approximation of the volatility strike is given by Carr and Lee in [15]. Rujivan and Rakwongwan assessed in [14] the value of this strike through discrete sampling, assuming that realized volatility is observed only discretely over time.
We have retained the essential, that until now, there is no analytical pricing formula for volatility swaps that answers our question already explained.
The structure of the article is as follows: section 2, we describe the studied model. Thereafter, we derive the formulas in section 3. Section 4 will be devoted to conclusions and future research.

2 Notations and organization of the article

We adopt the notations and context of ([9]).
Let $(\Omega, \mathcal{F}, P)$ be a probability space, we define on this space two Brownian motions W^1 and W^2 and the natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$ adapted of these two Brownian motions.
We place ourselves in the context of a Bond-Stock market modeled by a couple (B, S) stochastic processes, that $B = (B_t)_{t\in[0,T]}$ and $S = (S_t)_{t\in[0,T]}$.
The process B being predictable and S adapted to $(\mathcal{F}_t)_{t\in[0,T]}$.
We consider a stochastic volatility model that evolves according to the following dynamics:

$$\begin{cases} dB_t = r_t B_t dt \\ dS_t = \mu_t S_t dt + \sigma_t a(S_t) dZ_t \\ d\sigma_t = b(\sigma_t)dt + \nu(\sigma_t)dW_t \\ Cov(dZ_t, dW_t) = \rho dt \end{cases}$$

that r_t, is a predictable process that represents the interest rate (variable), μ_t is an adapted process that represents the yield on the risky asset S, σ_t is an adapted process that represents the implied volatility associated with asset S, and finally ρ is the supposed constant correlation coefficient.

We also make a hypothesis of stationarity on realized volatility, more precisely, we suppose that $Y_t = \int_0^t \sigma_R^2(s)ds$ is in independent increments. An interesting example of such a model is introduced by Barnadoff Nielson and Shephard in [5].
The realized volatility over period $[0, t]$ at any moment $t \in]0, T]$ is given by $\sigma_R^2(t) = \frac{1}{t}\int_0^t \sigma_s^2 ds$.
For more details on realized volatility and its significance, see ([9],[10]).
A volatility swap on this model is a forward contract that pays to the holder on the maturity date T the amount:

$$c(\sigma_R(T) - \Sigma)$$

where Σ is a fixed level of volatility and the contract period is $[0, T]$ (see [11]). The constant c is a factor converting volatility surplus or deficit into money. For simplicity, we choose $c = 1$ in this paper.

We want to give the discounted value at the moment t of the swap thus defined, knowing the history of the volatility until the moment t and the volatility on the valuation date t. In other words, we evaluate first

$$\gamma_1(r, t, T) = E_Q(\sigma_R(T) - \Sigma | \sigma_R(t) = r)$$

where $\mathcal{Q}$ is a risk-neutral probability and r is a real positive and then

$$\gamma_2(x, r, t, T) = E_Q(\gamma_1(\sigma_t, t, T) | \sigma_t = x)$$

3 Integral form of the swaps value

In this section we will establish an integral form of $\gamma_1(r, t, T)$ et $\gamma_2(x, r, t, T)$

Proposition 1. *We have for any* $r > 0$,

$$\gamma_1(r,t,T) = r\sqrt{\frac{t}{T}} + \frac{1}{\sqrt{\pi T}}\int_0^\infty \frac{exp(-tr^2u^2)(1-v(u,t))}{u^2}du - \Sigma$$

where v *is defined by:* $v(u,t) = E_Q\left(exp(-u^2\int_t^T \sigma_s^2 ds)\right)$.

Proof. since

$$\sigma_R(T) = \frac{1}{\sqrt{\pi T}}\int_0^\infty \frac{1-exp(-T\sigma_R^2(T)u^2)}{u^2}du$$

we obtain

$$\gamma_1(r,t,T) = \frac{1}{\sqrt{\pi T}}\int_0^\infty \frac{1-E_Q\left(exp(-T\sigma_R^2(T)u^2)|\sigma_R(t)=r\right)}{u^2}du - \Sigma.$$

Furthermore

$$E_Q\left(exp(-T\sigma_R^2(T)u^2)|\sigma_R(t)\right) = E_Q\left(exp(-t\sigma_R^2(t)u^2)exp(-u^2\int_t^T \sigma_s^2 ds)|\sigma_R(t)\right)$$

$$= exp(-t\sigma_R^2(t)u^2)E_Q\left(exp(-u^2\int_t^T \sigma_s^2 ds)|\sigma_R(t)\right)$$

The stationarity hypothesis of $(\sigma_t)_{t\in[0,T]}$ implies that

$$E_Q\left(exp(-u^2\int_t^T \sigma_s^2 ds)|\sigma_R(t)\right) = E_Q\left(exp(-u^2\int_t^T \sigma_s^2 ds)\right)$$

thus

$$E_Q\left(exp(-T\sigma_R^2(T)u^2)|\sigma_R(t)=r\right) = exp(-tr^2u^2)E_Q\left(exp(-u^2\int_t^T \sigma_s^2 ds)\right)$$

we can write,

$$\gamma_1(r,t,T) = \frac{1}{\sqrt{\pi T}}\int_0^\infty \frac{1-exp(-tr^2u^2)v(u,t)}{u^2}du - \Sigma$$

$$= \frac{1}{\sqrt{\pi T}}\int_0^\infty \frac{1-exp(-tr^2u^2)+exp(-tr^2u^2)(1-v(u,t))}{u^2}du - \Sigma$$

$$= r\sqrt{\frac{t}{T}} + \frac{1}{\sqrt{\pi T}}\int_0^\infty \frac{exp(-tr^2u^2)(1-v(u,t))}{u^2}du - \Sigma.$$

□

We can rewrite the formula above using an integral over a bounded interval, by introducing the residual value on the observation at a time t pretty close to the date of maturity T defined by $X_t = \frac{1}{T}\int_t^T \sigma_s^2 ds$.

Proposition 2. *With the notations above*

$$\gamma_1(r,t,T) = r\sqrt{\frac{t}{T}} - \Sigma + \frac{t^{1/2}r}{2\sqrt{\pi T}}\int_0^1 u^{1/2}(1-u)^{3/2}E_Q\left(\frac{X_t}{\frac{t}{T}r^2+uX_t}\right)du$$

Proof. We can write $v(u,t) = \sum_{n=0}^{+\infty}(-1)^n T^n \frac{u^{2n}E_Q(X_t^n)}{n!}$
therefore,

$$\int_0^\infty \frac{exp(-tr^2u^2)(1-v(u,t))}{u^2}du = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}T^nE_Q(X_t^n)}{n!}\int_0^{+\infty} u^{2(n-1)}e^{-tr^2u^2}du$$

$$= \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}T^nE_Q(X_t^n)\Gamma(n-1/2)}{2n!r^{2n-1}t^{n-1/2}}$$

where Γ is the Eulerian function defined for any complex $z, Re(z) > 0$ by

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} \exp(-t)dt$$

using $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 (1-u)^{x-1} u^{y-1} du$, we can write

$$\int_0^{\infty} \frac{exp(-tr^2u^2)(1-v(u,t))}{u^2} du = \int_0^1 E_Q \left(\sum_{n=1}^{+\infty} \frac{(-1)^{n+1} T^n X_t^n (1-u)^{3/2} u^{n-1/2}}{2t^{n-1/2} r^{2n-1}} \right) du$$

$$= \int_0^1 E_Q \left(\frac{u^{1/2}(1-u)^{3/2} t^{1/2} r X_t}{2(\frac{t}{T} r^2 + u X_t)} \right) du$$

□

4 conclusion and perspectives

We have established in this paper a closed pricing formula for the volatility swap in a stochastic volatility model. In the future, we would like to expand this formula in some specific cases of the classical models, such as the SABR [6] or BNS model or similar models.

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