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Three-Dimensional Control Charts for Regulating Processes Described by a Two-Dimensional Normal Distribution

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Abstract In the statistical management of processes in the initial phase, the stability of the technological process is determined based on the available samples. If the process is not stable, eliminating possible causes is brought into a statistically controlled position. At the same time, simple Shewhart control charts are used. In practice, some methods bring the process to a stable state (ISO standards, standards of various states). After the process has become stable, the boundaries of control charts are found for further management. Then, with the help of new samples, the process is managed. The article considers a process modeled by a two-dimensional normal distribution. New control charts have been found to check the normality and correlation of two-dimensional random variable components. The process is regulated using these charts, preserving the shape of the density of the individual components of the normal vector and linearity of these components. When constructing control charts, the Kolmogorov-Smirnov type agreement criterion and the Fisher criterion on the strength of the linear coupling of components were used. A concrete example shows the course of the introduction of these charts in production. The work results can be used in the initial phase of regulation and during the control check of the process under study. We used these control charts to assess product quality and quality control coming from the machine that produces the sleeves. It presents statistical methods for analyzing problems in factory practice and solutions for their elimination.

Keywords Statistical Control of Production, Control Charts, Statistical Hypothesis, Kolmogorov-Smirnov Type Consent Criterion, Fisher Criterion, Machine Operation Control

1. Introduction

Let's say that the company produces a large number of same-sex products. The working characteristic of the product is assigned to a random quantity of Z, let z_0 be its nominal value and $[z_0 - \Delta z,\ z_0 + \Delta z]$ assigned to the inventor-engineer. The product will have high quality if it fits the spec otherwise the product is considered as poor quality.

The $Z \neq z_0$ relationship is appropriate without much of an axiom that says "there will not be two identical objects", hence the concept of spec appeared. The engineering concept of "increasing the amount of quality products with control over the spec" has historically not justified itself. At the companies, the funds, losses have increased, and the requirements of consumers are not fully satisfied.

Based on American Scientist-Engineer U. Shewhart's conclusions of Mathematical Statistics in 1924, there was created a method of reducing the progress in production. His followers D. Juran, E. Deming and others have added a great contribution to the potential capabilities of this method [1]-[5]. This method is now known as the method of control charts for statistical management of processes.

The process of adoption of the new method by the science community and engineers, managers and managers of manufacturing enterprises has not been easy. After the 80-ies of the XX century, statistical instruments, known as control charts (CC), were introduced into the range of instruments that produced products in the manufacturing enterprises of all developed countries. From the nineties of the XX century, this method began to be used in manufacturing enterprises, service circles, medicine, education, public administration, small and medium-sized businesses and other industries. The number of scientific, scientific-practical articles and books has increased dramatically and is growing [6].

To continue our thoughts, we will briefly dwell on the concept of CC and the function of the CC method in statistical control of processes (SCP).

Z in unit times assigned from $\vec{Z} = (z_1, z_1, \dots, z_n)$ We take instant selections and determine the $g = g(z_1, z_1, \dots, z_n)$ statistics with which there is a correlation of product quality. In the simplest case of these statistics, Z can be a statistical estimate of the moments of a random quantity. g the mathematical expectation of a random quantity - $a = Mg$ and its average squared deviation with the help of - $\sigma = \sqrt{Dg}$ or with their statistical evaluation, we determine the following quantities: $LCL = a - 3\sigma$; $UCL = a + 3\sigma$; $CL = a$.

In the plane, we determine the moments of selection on the horizontal axis, the values of g on the vertical axis and the points of LCL, CL , UCL. After passing the parallel straight lines from the points LCL, CL, and UCL to the horizontal axis, we find the points corresponding to g in the plane and combine them with the cuts. Created diagram is called g-CC. The *lower control limit* of LCL g is called the *middle line* of CL g and the *upper control limit* of g. If $g \in (LCL, UCL)$ The process associated with the production of the product is called statistical control (SCP), otherwise it is not in the statistical control (SCP). In the case of SCP, the finite characteristics of Z will be almost unchanged, and the density function or the form of its evaluation will not change. In such a case, we will say that the process is in a stable condition, such determination of CC is called The Shewhart method. Currently, such CC are used in English-speaking production enterprises that produce products.

With the theory of probability and the methods of Mathematical Statistics, in most cases it will be possible to determine the distribution or limit distribution of g statistics and find suitable quantum tables for it. In such cases, the limits of CC can be found by the method of working intervals. In this case, 99% of g reliable interval limits are obtained as LCL and UCL. Sometimes, warning lines of g – CC are also inserted in order to determine the condition of the process. These lines are found as the boundaries of the 95% reliable interval for g [7]. Such types of g - CC are more commonly used in German-speaking production enterprises.

The mechanism for the use of CC is based on a sequential verification printout of statistical hypotheses. CC diagrams are considered the common language of various specialists and provide information about the state of the process. For this reason, in many enterprises, CCs are used as a legal document.

Any process of product coming up can be in one of the following four cases:

(a) SCP and 100% fits product specification;

(b) SCP and there are products with a certain amount that are of poor quality;

(v) Is not SCP and 100% fits product specification;

(g) Is not SCP and most of the products are of poor quality.

b) and v) cases with CCs in the SCP will be possible to bring a) case. In the case of g), it will be necessary to identify the important reasons that trace the process with CCs and eliminate them. Then it is gradually transferred to v), b) and a) cases. In identifying v) cases, the use of CCs, which have warning lines, is effective in detecting the condition. In production, the process of production of products varies. If we do not use CCs, then in the influence of entropy, the process will go toward g) depending on the situation. Depending on the problem in the SCP, different CCs are used. Of course, it takes time and expenditure of work, but the profits that come later will be much higher than the costs spent [8].

In order to accurately predict the state of the process with CCs, it is important to correctly determine the boundaries of the CCs. When determining boundaries, the process is required to be stable. In production, these works are carried out at the initial stage of learning the process state. [9], [10] in our work, CCs were found that determined the stability of the process when the working characteristic of the product was one or two-dimensional normal random quantity. In this article, opinions are made about how these results can be used in the production process.

We recall that multi-dimensional control charts for autocorrelation processes were analyzed, for example, in the following books and articles. [11]- [14] the new control charts that we have built are like Shewhart charts that do not take into account the autocorrelation.

2. Methods

If the result of a statistically controlled manufacturing process is described by a two-dimensional normal distribution and the source of variability is only the usual causes, then during technological operations, the shape of the distribution density does not show changes in position, spread, and shape. As a result, the process continues to produce products that meet the requirements. In this case, the distribution characteristics are used to evaluate the process. This all helps the enterprise to make economically correct decisions regarding the actions related to the process.

In connection with the above, the article proposes a method for maintaining the shape of the density of a distribution with certain parameters depending on the shape of the density of individual components of a two-dimensional normal vector and at the same time the linearity of these components. In this case, we use the CC technique.

To check the normality of individual components, we construct a CC using the Kolmogorov-Smirnov type of agreement criterion. CC checking the linearity of the components is based on the Fisher criterion (see, for example, $[15]-[17]$). As a result, we have a three-dimensional CC supporting the shape of a two-dimensional density.

Now, using the example of one CC, we will describe the methodology for building a new CC and the course of conducting these CC in production.

Let the random variable z determine the quality attribute of the manufactured products. Let $z_t = (z_{t1,} z_{t2,\dots,} z_{tn,})$ denote measurement results in time points $t = 1,2,\dots,m$, n-sample size. Let $g(z_t) = g(z_{t1,} z_{t2,\dots,} z_{tn,})$ denote sample characteristics correlated with product quality. $P(g(z_t) < x) = F(x)$ distribution (or limit distribution) of a quantity $g(z_t)$.

At the significance level $\propto$ CC characteristics: LCL (lower control limit) and, UCL $(upper\ control\ limit)\ are$ determined from the ratio

$P\big(g(z_t) \notin (LCL, UCL)\big) = \propto$ where P(.) represents the probability that event (.) will occur.

This is equivalent to the construction $(1-\propto)100\% = 99\%$ of the nth confidence interval, provided that the process is under statistical control.

CC's perform several functions in learning the process. One of the functions is to reexamine hypotheses.

H_0: The manufacturing process is statistically controlled,

H_1: The process is statistically uncontrolled.

In this case, if $g(z_t) \in (LCL, UCL)$ the hypothesis H_0 is accepted, otherwise,H_1.

The remaining functions of the CC are aimed at improving the process in the form of a Deming - Shewhart cycle (PDS(C)A): "Planning-Do-Study-Act" has the shape of a circle [4], [8].

3. Results

3.1. CCs that Check the Normality of the Components of a Two-dimensional Vector ($\rho_z - chart$)

Assume that the samples are $z_t = (z_{t1,} z_{t2,\dots,} z_{tn,})$, $t = 1,2,\dots,m$, are derived from the normal general population Z with respect to which there are two nonparametric hypotheses:

$$H_0 : F(z) = \Phi(z;\theta_0),$$
$$H_1 : F(z) \neq \Phi(z;\theta_0),$$

where $\theta_0 = (\mu_0, \sigma_0)$-actual value of the parameter.

To test the hypothesis of agreement, the Kolmogorov test is used, based on statistics

$$\sqrt{n}\ D_n(\theta_0) = \sqrt{n}\ sup|F_n(z) - \Phi(z;\theta_0)|,$$

where $F_n(z)$ empirical distribution function.

If θ_0 is unknown, use Kolmogorov-Smirnov type statistics

$$\sqrt{n}\ D_n(\theta_n^*) = \sqrt{n}\ |F_n(z) - \Phi(z;\theta_n^*)|\ ,$$

where $\theta_n^* = (\underline{z}, s^2), \underline{z}$ and s^2 ratings $\mu_0\ and\ \sigma_0^2$.

As indicated in [15], the distribution is statistically $\sqrt{n}\ D_n(\theta_n^*)$ different from the distribution of the statistics $\sqrt{n}\ D_n(\theta_0)$, with the quantiles differing by about 1.5 times.

In our work, we use statistics of the Kolmogorov-Smirnov type, so the simple main hypothesis H_0 here has the following form:

$$H_0 : F(z) = \Phi(z;\theta_n^*),\ \text{here}\ \theta_n^* = (\underline{z}_{tn}, s_{tn}^2),$$
$$H_1 : F(z) \neq \Phi(z;\theta_n^*),$$

at each $t = 1,2,\dots,m$.

Let $z_t^* = (z_{t1,}^* z_{t2,\dots,}^* z_{tn}^*)$ sample variation series $z_t = (z_{t1,} z_{t2,\dots,} z_{tn,})$. For practical implementation of Kolmogorov-Smirnov type statistics, $\sqrt{n}\ D_n(\theta_n^*)$ replace with the equivalent expression $\sqrt{n}\ \rho_t(\theta_n^*)$, where $\rho_{\ t}(\theta_n^*) = \left|\Phi\big(z_{tk\ ,}^*; \underline{z}_{tn}, s_{tn}^2\big) - \frac{2k-1}{2n}\right| + \frac{1}{2n}$, here $\Phi\big(z_{tk\ ,}^*; \underline{z}_{tn}, s_{tn}^2\big) = \Phi(Y_{tk\ ,}^*)$, $Y_{tk\ ,}^* = \frac{z_{tk}^* - \ ,\underline{z}_{tn,}^*}{s_{tn}}$, $\Phi\big(Y_{tk\ ,}^*\big) \sim N(0,1)$-normal distribution.

Now we turn to the construction of the CC based on the Kolmogorov-Smirnov type criterion.

As a reference value $g(z_t)$ we take $\rho_t(\theta_n^*) = \rho_t$. In the sense of the problem, we construct a CC with an upper bound.

Theorem 1[9]. If the hypothesis H_0 is valid at a given level of significance, the equality takes place

$$UCL_\rho = \frac{k_{1-\alpha}}{\sqrt{n}},$$

where k_p order quantile $p = 1 - \alpha$=0.99 distribution of statistics $\sqrt{n}\ \rho_t(\theta_n^*)$.

From $(1-\alpha)100\%$ =99% confidence interval for statistics$\sqrt{n}\ \rho_t(\theta_n^*)$ we get this equality.

Next, the constructed CC with the reference value$\rho_t$and the bound

$$UCL_\rho = \frac{k_{1-\alpha}}{\sqrt{n}}\ \ \text{denote, by "}\rho_z - chart\text{"}.$$

Corollary 1 (**Realization** $\rho_z - charts$). At time t, for the validity of the hypothesis H_0 at the significance level, it is sufficient to perform the inequality

$$\rho_t < \frac{k_{1-\alpha}}{\sqrt{n}}.$$

3.2. CC Checking the Linearity of the Components of a Two-dimensional Vector (r-chart)

Let n pairs of observations $(x,y)_t = \{(x_{t1}, y_{t1}), (x_{t2}, y_{t2}), \dots, (x_{tn}, y_{tn})\}$, $t = 1,2,\dots,m$ form a selection from (X,Y) for which the random vector is (X,Y) distributed normally.

It is of interest to us to construct a CC that determines the linear relationship between the measured features X and Y.

Denote by $K(x,y)$ correlation coefficient between X and Y, $R(x,y)$- its theoretical assessment and $r = r(x,y)$ realization X and Y.

To construct CC (r −chart) determining the strength of the linear connection we rely on the Fisher criterion testing the statistical hypothesis:

$$H_0 : K(x,y)=k_0,$$

$$H_1 : K(x,y) \neq k_0$$

where k_0–a value that expresses the strength of a linear connection $(k_0 \neq 0)$.

For the reference value we take $r_t = r_t(x,y)$. With respect to the control boundaries and the power function of the CC, the following statement holds.

Theorem 2 [10]

If the hypothesis H_0 is valid at a given level of significance, there are equalities $LCL_r = th(z_1)$, $UCL_r = th(z_2)$, where, $z_1 = argth(k_0) - \frac{\psi(1-\alpha)}{\sqrt{n-3}}$,

$z_2 = argth(k_0) + \frac{\psi(1-\alpha)}{\sqrt{n-3}}$, $th(z) = \frac{e^{2z}-1}{e^{2z}+1}$, $argth(z) = \frac{1}{2} ln\, ln \frac{1+z}{1-z}$, $\psi(p)$ -p-the quantile of the normal distribution$\sim N(0,1)$.

The CC power function is determined by the following relation: $G(r) = 1 - \Phi\left(\frac{argth(UCL_r)-a(r_t)}{\sigma}\right) + \Phi\left(\frac{argth(LCL_r)-a(r_t)}{\sigma}\right)$, where, $a(r_t) \approx argth(r_t) + \frac{r_t}{2(n-3)}, \sigma = \frac{1}{\sqrt{n-3}}$, $n \geq 20$, $\Phi(\cdot) - N(0,1) -$ normal distribution.

Corollary 2 (Realization $r-$ charts)

At time t, for the validity of the hypothesis H_0 at the significance level, it is sufficient to perform the inequality

$$LCL_r < r_t < UCL_r.$$

In the proof of Theorem 2, to determine the control bounds of the r –chart, we use $(1-2\propto)100\%$ –the confidence interval obtained by R. A. Fisher. $th(z_1) < r_t < th(z_2)$

By definition, we find the power function by calculating the probability

$$G(r) = P(r < LCL_r) + P(r > UCL_r).$$

In this case, we use the inequality

$$\frac{1+r}{1-r} \leq \frac{1+x}{1-x}, \quad -1 \leq x < 1, r < x, when\ x = LCL_r \text{ or } x = UCL_r$$

Next, we use the fact that for the normalizing Fisher transform, $z = argth(r)$ value $\frac{z-M(z)}{\sqrt{D(z)}}$ distributed system with parameters (0,1).

4. Conclusions

To illustrate our theoretical results, let us consider an example of a machine that produces casings (gun casings, cylinder liners) with a landing nominal diameter of d=30mm (see [16]). The control of the fitting diameter of the sleeve is carried out in two mutually perpendicular planes. Errors of deviation from the nominal value in these planes are denoted by X and Y. As is stated in [16], the random vector (X,Y) is normally distributed and the components are linearly connected:

$$(X,Y) \sim N(\mu_{0x,}\mu_{0y,}\sigma^2_{0x,}\sigma^2_{0y,}\rho^0_{xy}).$$

Moreover, the parameter estimates are such that $\mu_{0x} \approx 7.82$ mk, $\mu_{0y} \approx 8.86, \sigma^2_{0x} \approx 4.86\, mk, \sigma^2_{0y} \approx 5.5 mk, \rho^0_{xy} \approx 0.5$. We take these estimates as the true values of the parameters and continue to study this process using the CC technique.

Within the standard we take samples of (X,Y) and check the hypotheses:

$$H_0 : F(x,y) = \Phi(x,y\colon 7.82; 8.86; 4.86; 5.5; 0.5),$$

$$H_1 : F(x,y) \neq \Phi(x,y\colon 7.82; 8.86; 4.86; 5.5; 0.5)$$

This task is equivalent to testing hypotheses:

$$A)\ H_0 : F(x,+\infty) = \Phi(x\colon 7.82; 4.86),$$

$$H_1 : F(x,+\infty) \neq \Phi(x\colon 7.82; 4.86)$$

$$B)\ H_0 : F(+\infty,y) = \Phi(y\colon 8.86; 5.5),$$

$$H_1 : F(+\infty,y) \neq \Phi(y\colon 8.86; 5.5)$$

$$C)\ H_0 : \rho_{xy} \geq 0.5,$$

$$H_1 : \rho_{xy} < 0.5.$$

Remark. In problem C), we assumed $H_0 : \rho_{xy} \geq 0.5$, $H_1 : \rho_{xy} < 0.5$. Instead of H_0: $\rho^0_{xy} = 0.5, H_1$: $\rho^0_{xy} \neq 0.5$, since the process under study has a strong linear dependence and does not affect the quality of the produced sleeves [16]. This means there is a need to construct $r-$ chart with LCL_r.

Using the statements of theorems 1 and 2 we construct a three dimensional CC: $\rho_x - \rho_y - r_{xy}$. The results of the calculations are provided in the following table.

Table 1. Potential indicator values

t	1	2	3	4	5	6	7
n	150	150	150	150	150	50	60
UCL_ρ $\propto= 0.01$	0.08	0.08	0.08	0.08	0.08	0.02	0.02
ρ_x	0.049	0.046	0.054	0.062	0.059	0.14	0.13
ρ_y	0.049	0.046	0.035	0.062	0.059	0.18	0.23
LCL_r $\propto= 0.05$	0.35	0.35	0.35	0.35	0.35	0.21	0.24
r	0.60	0.99	0.47	0.99	0.56	0.02	0.18
G(r)%	0.1	1.2	9.7	1.2	0.5	91.3	71.6

For the stability of the machine that releases the sleeves, it is necessary to perform inequalities $\rho_x < UCL_\rho$, $\rho_y < UCL_\rho$ and $r > LCL_r$. In this case, the value G(r)% within the standard. From the table, we can see that for $t = 6$ and $t = 7$, these inequalities are not satisfied, that the process is unstable and needs correction. In this case, this is due to a smaller sample size.

In summary, as can be seen from Table 1, the production process when t=1,2,4 and 5 unit times is described in the introduction b), and when t=3 is in v), and when t=5,6 in g).

To further improve the process in the PDS(C)A cycle, there is still a need to use the Shewhart CC to determine the proportion of unusable products and find out the causes of abnormal phenomena [4], [8].

For the successful implementation of a three-dimensional $\rho_x - \rho_y - r_{xy}$ charts, a new software product is needed, which we are currently doing.

There, instead of a table, there will be diagrams of these CCs.

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