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Development of the Established Redlich-Kister Finite Difference Solution with MKSOR Iteration for Solving One Dimensional Diffusion Problems

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Abstract This paper contributes a new method known as the second-order Redlich-Kister Finite Difference (RKFD) solution to the partial differential equation, especially for one-dimensional (1D) diffusion problems. All derivative terms for the proposed method are needed for the discretization process of the second-order RKFD. Arranging the derivative terms will lead to the second-order RKFD approximation equations. Then, this approximation equation is applied to solve the system of the RKFD equation. As the large-scale and sparse coefficient matrix is obtained, it will be solved iteratively to regulate the high computational complexity by using the Gauss-Seidel (GS), Kaudd Successive Over Relaxation (KSOR) and Modified Kaudd Successive Over Relaxation (MKSOR) iterative methods. All of those iterative methods are developed according to the matrix structure of the system and are applied to three examples of the proposed problem. As a result, MKSOR iterative method showed significant improvement in terms of performance efficiency by contrast to GS and KSOR iterative methods. The performance efficiency is measured by the number of iterations, execution time and maximum norm.

Keywords MKSOR Iteration, Redlich-Kister Finite Difference, Finite Difference, Diffusion Problem, Iterative Methods

1. Introduction

A numerical solution is one of the frequent approaches used in mathematics and computer sciences for solving mathematical problems [1,2]. With the basis of developing, analyzing and simulating algorithms, this approach is widely used in solving partial differential equations (PDEs) particularly 1D diffusion problems. The mathematical theory of diffusion problem is well developed and described in many modelings in real-world including physical phenomena. Also, the diffusion problem is applied in science, physics, mathematics and engineering areas. For instance, it is used to describe quantities such as mass, energy, and heat [3,4], pollution in shallow lakes [5] and dispersion in porous media [6]. Real-world problems modeled by the diffusion problem lead to many researches attached to develop and present various computational methods to achieve the method with low computational complexity for solving one-dimensional diffusion problems. For instance, Chebyshev Wavelets Methods [7], Explicit Method [8] and Radial Basis Function [9] for solving the proposed problem in this investigation. Also, in [10], the authors have been presented the B-spline Finite Element Method for solving 1D diffusion problems.

Throughout various methods developed by focusing on the numerical method only, a few researchers have proposed the combination of the approximation function and the finite difference concept as the alternative methods. As a result, this alternative method rapidly develops to obtain the numerical solution. For instance, in [11], the author has proposed a new variant of this approach known as the Chebyshev finite difference method which is a combination of the Chebyshev approximation function and finite difference concept to solve the boundary value problems. Also, in [12], the author proposed the Rational finite-difference for solving the same problem as in [11]. Again two years later, the author [12] comes with a new combination called the Exponential finite difference discretization scheme [13].

From the listed methods in the second paragraph, this investigation was motivated to propose a new numerical method with the same concept as mentioned earlier and decided to consider one of the approximation functions called the Redlich-Kister (RK) polynomial function. Commonly, the application of this approximation function can be seen and found in physics and chemistry fields [14-16]. On the other side, the Redlich-Kister approximation function has not been explored and the least literature due to its application in numerical analysis. For the initial work of this function in the numerical area, the authors have investigated the two models of piecewise Redlich-Kister polynomial which are the first and third-order models [17]. These models have been constructed to analyze the correlation between the Gauss-Seidel iteration and the grid sizes considered. Thus, the performance of the piecewise third-order RK polynomial gives more accurate by contrast to the piecewise first-order RK polynomial. In recent years, this work is continued by the author [18] introduced the Redlich-Kister polynomial to find the solution to two-point boundary value problems. Also, the authors [19,20] proposed a new variant of Redlich-Kister as known RKFD in solving boundary value problems [19,20]. As a consequence, this investigation is interested to apply this combination as proposed in [19,20] over the one-dimensional diffusion problems.

In order to establish the interest combination in this investigation, the proposed problem must be thoroughly? the RKFD discretization scheme process to construct the RKFD approximation equation. The use of RKFD approximation equation generates the system of the RKFD equation which will be solved by two mainstream methods, which are direct and iterative methods as a linear solver. The factor of selection linear solver has been influenced by the characteristic of the generated linear system. The direct method is suitable for solving small-scale linear systems. In contrast to the direct method, the iterative method is preferable for the large and sparse linear system as mentioned in [21-23]. In the literature, several iterative methods have been developed and reviewed as solvers for the linear system, including Successive Over Relaxation (SOR) [24], Acceleration Over Relaxation (AOR) [25], KSOR method [26] and MKSOR method [27]. Among these methods listed, the MKSOR iterative method was chosen and considered as a linear solver in this investigation. Thus, for overall work in this paper will focus on the two newly established RKFD approximation equations extended together with the MKSOR method to find the solution to 1D diffusion problems.

As mentioned in earlier paragraph, this paper considers and deals with the general one-dimensional diffusion problem as follows

$$\frac{\partial U}{\partial t}(x,t) + V\frac{\partial U}{\partial x}(x,t) + ZU(x,t) = K\frac{\partial^2 U}{\partial x^2}(x,t) + r(x,t) \quad (1)$$

with the initial condition

$$U(x,0) = g_1(x),$$

and Dirichlet conditions,

$$U(x_0,0) = g_2(t),\ U(x_n,0) = g_3(t).$$

2. RKFD Approximation Equation

For establishing the RKFD approximation equation as proposed briefly in the first section, the (1) must thoroughly the RKFD discretization scheme process over the (1). The discretization process starts by defining the Redlich-Kister function of order n as below

$$U_n(x,t) = \sum_{k=0}^{n} a_k(t) \cdot T_k(x) \quad (2)$$

where $a_k, k = 0,1,2,\dots,n$ are to be calculated for the value of unknown parameters considered.

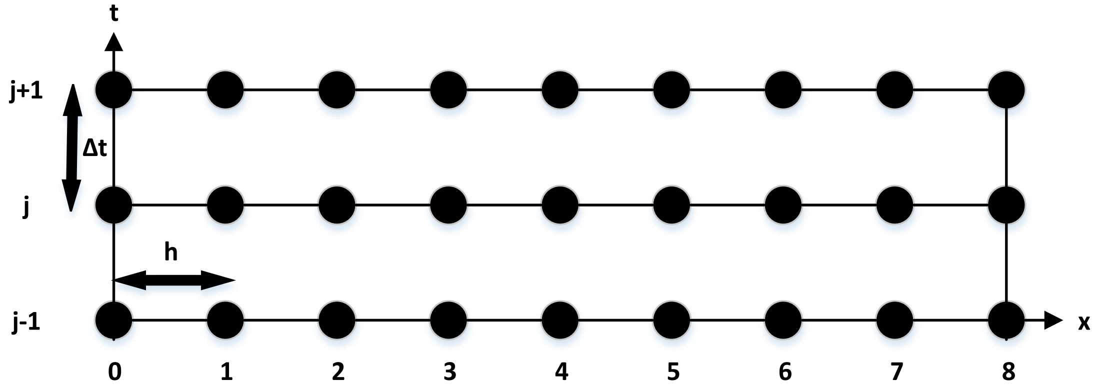

Figure 1. The grid network at three levels

According to Fig. 1, the distribution of the grid network at three levels considered is shown to elucidate the use of the RK function in this investigation. To be more clear on the application of the RK function, the construction of the first three RK functions based on all node points in Fig. 1 is needed as illustrated in Fig. 2.

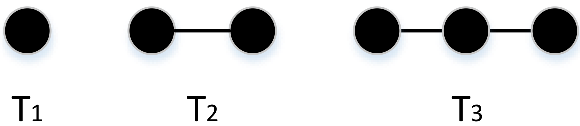

Figure 2. The path for T_1, T_2 and T_3.

By expanding the (2) by accord to Fig. 2, the second-order RK approximation function can be defined as

$$U(x,t) = a_0(t)T_0(x) + a_1(t)T_1(x) + a_2(t)T_2(x) \quad (3)$$

where the first three of (3) are defined as

$$T_0(x) = 1,$$
$$T_1(x) = x,$$
$$T_2(x) = x(1-x).$$

To set up the grid network in Fig. 1 as approximate domain of (3), let define and apply the node points, $x_c = x_0 + ch, c = 0,1,2,\ldots,n$ where the uniform step size is represented as $h = \frac{\emptyset-\mu}{n}, n = 2^p, p \geq 1$.Throughout this process, the $U(x,t)$ and $\mathrm{T}(x,t)$ at any three node points are represented as $U(x_k,t) = U_k(t), k = c-1, c, c+1$ and $\mathrm{T}(x_k) = T_k, k = c-1, c, c+1$ respectively.By substituting the approximation value functions obtained into (3), the following equations are obtained as

$$\mathrm{U}_{c-1}(x,t) = a_0(t)T_0(x_{c-1}) + \\ + a_1(t)T_1(x_{c-1}) + a_2(t)T_2(x_{c-1}) \quad (4)$$

$$\mathrm{U}_{c}(x,t) = a_0(t)T_0(x_c) + a_1(t)T_1(x_c) + a_2(t)T_2(x_c) \quad (5)$$

$$\mathrm{U}_{c+1}(x,t) = a_0(t)T_0(x_{c+1}) + \\ + a_1(t)T_1(x_{c+1}) + a_2(t)T_2(x_{c+1}) \quad (6)$$

Looking back to (3), there are the expressions of three parameters, $a_k, k = 0,1,2$ that need to be defined. The way to get these expressions is by solving all the (4), (5) and (6) through the matrix approach and substituting into the equation (3). Then, the (3) becomes

$$U(x,t) = N_0(x)U_{c-1}(t) + \\ + N_1(x)U_c(t) + N_2(x)U_{c+1}(t) \quad (7)$$

where the RK shapes functions of equation (7), $N_k, k = 0,1,2$ are defined respectively as

$$\begin{cases} N_0(x) = \frac{1}{2h^2}(x^2 - 2xhc - xh + h^2c^2 + h^2c), \\ N_1(x) = \frac{1}{h^2}(2xhc - x^2 - h^2c^2 + h^2), \\ N_2(x) = \frac{1}{2h^2}(x^2 - 2xhc + xh + h^2c^2 + h^2c), \end{cases} \quad (8)$$

and its derivatives of RK shape function (8) are as follows

$$\begin{cases} \mathrm{N}_0'(x) = \frac{1}{2h^2}(2x - h - 2hc), \\ \mathrm{N}_1'(x) = \frac{1}{h^2}(2hc - 2x), \\ \mathrm{N}_2'(x) = \frac{1}{2h^2}(2x + h - 2hc), \end{cases} \quad (9)$$

and

$$\begin{cases} \mathrm{N}_0''(x) = \frac{1}{h^2}, \\ \mathrm{N}_1''(x) = -\frac{2}{h^2}, \\ \mathrm{N}_2''(x) = \frac{1}{h^2}. \end{cases} \quad (10)$$

By taking the shape function (9) and (10) then applying them to (7), it can be written and defined respectively as

$$\left.\frac{\partial U}{\partial x}\right|_c = \mathrm{N}_0'(x_c)U_{c-1}(t) + \mathrm{N}_1'(x_c)U_c(t) + \mathrm{N}_2'(x_c)U_{c+1}(t) \quad (11)$$

and

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_c = \mathrm{N}_0''(x_c)U_{c-1}(t) + \mathrm{N}_1''(x_c)U_c(t) + \mathrm{N}_2''(x_c)U_{c+1}(t) \quad (12)$$

where $U(x_c,t) = U_c(t), c = 0,1,2,\ldots,n$ according to the approximation solution of function $U(x,t)$. As mentioned within the previous section on the most objective of this investigation, the (11) and (12) represented the two newly established RKFD discretization schemes. These established RKFD functions require the construction of the RKFD approximation equation over the proposed problem (1).

However, the (1) must be defined and transformed in the discrete form before constructing the RKFD approximation equation and becomes

$$\left.\frac{\partial U}{\partial t}\right|_{c,j} + V\left.\frac{\partial U}{\partial x}\right|_{c,j+1} + ZU_{c,j+1} = K\left.\frac{\partial^2 U}{\partial x^2}\right|_c + r_{c,j+1} \quad (13)$$

By recalling the two newly established RKFD in (11) and (12) and substituting them into (13), the general equation of the RKFD approximation equations for the one-dimensional diffusion problem is obtained. For the simplicity purpose, the two newly established RKFD approximation equations can be defined as

$$\alpha_c U_{c-1,j+1} + \beta_c U_{c,j+1} + \gamma_c U_{c+1,j+1} = R_{c,j}, \quad (14)$$

where

$$\alpha_c = -K\Delta t\mathrm{N}_0''(x_c) + V\Delta t\mathrm{N}_0'(x_c),$$
$$\beta_c = 1 - K\Delta t\mathrm{N}_1''(x_c) + V\Delta t\mathrm{N}_1'(x_c),$$
$$\gamma_c = -K\Delta t\mathrm{N}_2''(x_c) + V\Delta t\mathrm{N}_2'(x_c),$$
$$R_c = \Delta t r_{c,j+1} + (1 + \Delta t Z)U_{c,j}.$$

Based on the (14), a system of RKFD approximation equation can be generated by considering $c = 1,2,\ldots,n-1$. Thus, the generated system linear of RKFD equation in matrix form is expressed as

$$\mathrm{W}\cdot\underline{\mathrm{U}}_{\mathrm{j+1}} = R_{c,j+1}, j = 0,1,2,\ldots,n-1 \quad (15)$$

3. The MKSOR Iterative Method

As stated in the contribution of this investigation in the first section, the MKSOR iterative method needs to be established according to the characteristics of the generated system linear equation in (15). The basis development of the MKSOR iterative method by proposing the two weighted parameters, ω_1 and ω_2 or called the red-black approach [28]. Following that, the developed MKSOR method is used to solve (15). Therefore, before the implementation of the MKSOR method, the coefficient matrix in (15) needs to be decomposed into three summations of

$$(F + J + L) \cdot U = R \tag{16}$$

where J, F and L are represented diagonal, triangular lower and upper matrices respectively.

Then, the MKSOR iterative method can be established in point iteration form based on the (16) be expressed as [27,28]

$$U^{(q+1)} = [(1-\omega_1)J - \omega_1 F]^{-1}(J+L)U^{(q)} + \\ +[(1-\omega_1)J - \omega_1 F]^{-1}R \tag{17}$$

and

$$U^{(q+1)} = [(1-\omega_2)J - \omega_2 F]^{-1}(J+L)U^{(q)} + \\ +[(1-\omega_2)J - \omega_2 F]^{-1}R \tag{18}$$

where $U^{(q+1)}$ the represented the current value of U at the $(q+1)^{\text{th}}$ iteration.

By manipulating the (17) and (18), the general formulation of the MKSOR method that is used in solving the (15) is written as

$$U^{(q+1)} = \frac{1}{(1+\omega_1)}U_c^{(q)} + \frac{\omega_1}{(1+\omega_1)}\left(R_c - \alpha_c U_{c-1}^{(q+1)} - \gamma_c U_{c+1}^{(q)}\right) \tag{19}$$

and

$$U^{(q+1)} = \frac{1}{(1+\omega_2)}U_c^{(q)} + \frac{\omega_2}{(1+\omega_2)}\left(R_c - \alpha_c U_{c-1}^{(q+1)} - \gamma_c U_{c+1}^{(q)}\right) \tag{20}$$

where $c = 1,3,5,\ldots,n-1$ and $c = 2,4,6,\ldots,n-2$ are used to (19) and (20) respectively. Meanwhile, the weighted parameters of (19) and (20) have different values for each grid size considered.

4. Numerical Problem and Discussion

In order to verify the efficiency of the MKSOR iterative method formulated in the third section, three examples of one-dimensional diffusion problems were selected and solved iteratively. All experiments were conducted using a computer IdeaPad S340-15API with a processor AMD Ryzen 5 3500U with Radeon Vega Mobile Gfx 2.10 GHz with 12.0GB RAM. Also, the programming code has been written in C-language programming. Below are the following examples that are considered in this investigation.

Example 1

Consider the one-dimensional diffusion problem (1) as [7]

$$\frac{\partial U}{\partial t} = K\frac{\partial^2 U}{\partial x^2}, \tag{21}$$

with the initial and boundary conditions are given as

$$U(x,0) = sin(\pi x),$$
$$U(0,t) = 0,\ U(1,t) = 0,$$

and the analytical solution of problem (21) is

$$U(x,t) = sin(\pi x)e^{\pi^2 t}.$$

Example 2

Consider the one-dimensional diffusion problem (1) as [29]

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - U, \tag{22}$$

with the initial and boundary conditions are stated as

$$U(x,0) = x + e^{-x},$$
$$U(0,t) = 1,\ U(1,t) = e^{-1} + e^{-t}.$$

and the analytical solution of problem (22) is

$$U(x,t) = e^{-x} + xe^{-t}.$$

Example 3

Consider the one-dimensional diffusion problem (1) as [29]

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} - \frac{1}{4}U, \tag{23}$$

with the initial and boundary conditions are given as

$$U(x,0) = \frac{1}{2}x + e^{-\frac{x}{2}},$$
$$U(0,t) = 1,\ U(1,t) = e^{-\frac{1}{2}} + \frac{1}{2}e^{-\frac{t}{4}},$$

and the analytical solution of problem (23) is

$$U(x,t) = e^{-\frac{x}{2}} + \frac{x}{2}e^{-\frac{t}{4}}.$$

In executing these three examples, the efficiency of the MKSOR iterative method is compared to the other two iterative methods which are GS and KSOR methods. Furthermore, the implementation of these iterative methods will be considered. Three parameter measurements are the number of iterations (Iter), execution time (Time) and maximum norm (MaxNorm). For comparison purpose, all the parameters are recorded using five different sizes, $n = 256{,}512{,}1024{,}2048{,}4096$ for each selected example. The numerical results that through the iteration process completed are presented in Tables 1, 2 and 3. Note that, the GS method is benchmarking method and the use of tolerance error, $\varepsilon = 10^{-10}$ is in this investigation.

The collected results were obtained for problems (21), (22) and (23) which correspond to Table 1 to 3 respectively. As can be seen, the number of iterations and time by the combination of the RKFD approximation equation with the KSOR method is notably lower than the GS and KSOR methods for all different sizes considered. For n =256 as depicted in Table 1, the MKSOR method produces the 207 iterations compared to GS and KSOR methods with 7090 and 265 iterations respectively. In terms of time, the MKSOR method requires 0.26 seconds to complete the iteration for 256 sizes compared to GS and KSOR methods with 5.38 and 0.26 seconds respectively. Furthermore, the significant improvement when involving the grid size is increased. As tabulated in Table 1 for n=4096, the GS and KSOR methods produce 919902 in 11456.32 seconds and 4089 iterations in 50.78 seconds and the MKSOR method has minimized the numbers with 2860 iterations in 30.60 seconds. The accuracy for all the methods considered is comparable and has a good agreement over their analytical solution. Also, the study found the same pattern of results as Table 1 for another two examples that were considered in this investigation. Evidently, the use of two weighted parameters in the MKSOR method successfully improved the efficiency of testing the selected problems. In other words, the proposed combination RKFD approximation equation with the MKSOR method is the superior method for solving the 1D diffusion problems compared to the GS and KSOR methods.

Table 1. Numerical result based on comparison criteria considered for Example 1

	Method	**Grid Sizes, n**				
		256	512	1024	2048	4096
Iter	GS	7090	25030	86776	292338	919902
	KSOR	265	534	1064	2090	4089
	MKSOR	207	393	755	1469	2860
Time	GS	5.38	34.35	236.38	1591.56	11456.32
	KSOR	0.26	0.93	3.50	13.10	50.78
	MKSOR	0.26	0.72	2.46	8.34	30.60
MaxNorm	GS	2.9260e-05	2.7261e-05	1.9292e-05	1.2578e-05	5.1314e-05
	KSOR	2.9923e-05	2.9916e-05	2.9917e-05	2.9922e-05	2.9931e-05
	MKSOR	2.9965e-05	2.9995e-05	3.0066e-05	3.0135e-05	3.0225e-05

Table 2. Numerical result based on comparison criteria considered for Example 2.

	Method	**Grid Sizes, n**				
		256	512	1024	2048	4096
Iter	GS	9946	36485	132809	478769	1705251
	KSOR	315	629	1254	2497	4966
	MKSOR	284	570	1136	2267	4501
Time	GS	10.35	74.62	534.81	15.61	20395.43
	KSOR	0.42	1.85	6.48	8.13	62.02
	MKSOR	0.35	0.97	2.14	20395.43	31.17
MaxNorm	GS	1.1795e-04	1.1613e-04	1.0913e-04	8.1637e-05	4.3690e-05
	KSOR	1.1853e-04	1.1847e-04	1.1845e-04	1.1844e-04	1.1842e-04
	MKSOR	1.1853e-04	1.1846e-04	1.1844e-04	1.1840e-04	1.1835e-04

Table 3. Numerical result based on comparison criteria considered for Example 3.

	Method	Grid Sizes, n				
		256	512	1024	2048	4096
Iter	GS	10055	36897	134367	484644	1727314
	KSOR	318	635	1265	2517	5002
	MKSOR	284	570	1139	2269	4520
Time	GS	10.48	77.68	537.91	2623.41	18671.63
	KSOR	0.34	1.06	4.16	15.86	62.92
	MKSOR	0.18	0.57	2.32	12.58	49.44
MaxNorm	GS	7.1986e-06	5.3522e-06	3.0539e-06	3.3876e-05	1.5820e-04
	KSOR	7.8276e-06	7.8208e-06	7.8155e-06	7.8063e-06	7.7859e-06
	MKSOR	7.8266e-06	7.8165e-06	7.8040e-06	7.7892e-06	7.6985e-06

5. Conclusions & Future Work

This paper proposed the two newly established RKFD approximation equations with MKSOR method for 1D diffusion problems. The investigation started with the discretization process of the proposed problem to obtain the RKFD approximation equation. Then, it leads to generating a system of RKFD approximation equations and should be solved iteratively by using the GS, KSOR and MKSOR iterative methods. The validation of the efficiency of these iterative methods has been considered by testing three examples of the proposed problem at five different sizes. The collected results showed that the MKSOR method needed the least iterations and time, by contrast to the GS and KSOR methods. This statement has been supported by the comparison results in Tables 1 to 3. Overall, it is revealed and proved that the two newly established RKFD approximation equations with MKSOR iterative method are a better combination compared to the GS and KSOR iterations. As an extension, this work can continue with the same discretization scheme to solve other boundary cases like the multi-dimensional boundary value problem with the two-step iteration family [30,31], the half-sweep [32,33] and quarter-sweep [34,35] approaches.

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REFERENCES

[1] R. Bernatz, Fourier series and numerical methods for partial differential equations. John Wiley and Sons, 2010.

[2] S. S. Ray, Numerical analysis with algorithms and programming. Taylor & Francis Group, LLC, USA, 2016.

[3] M. Dehghan, Weighted finite difference techniques for one dimensional advection–diffusion equation, Applied Mathematics and Computation, Vol.147, 307-319, 2004.

[4] M. Dehghan, Numerical solution of the three-dimensional advection–diffusion equation, Applied Mathematics and Computation, Vol.150, 5-19, 2004.

[5] J. R. Salmon, L. A. Ligget, R. H. Gallager, Dispersion analysis in homogeneous lakes, International Journal for Numerical Methods Engineering, Vol.15, 1627-1642, 1980.

[6] Q. N. Fattah, J. A. Hoopes, Dispersion in anisotropic, homogeneous, porous media, Journal of Hydraulic Engineering, Vol.111, No.5, 810–827, 1985.

[7] M. R. Hooshmandasl, M. Haidari, F. M. MaalekGhaini, Numerical solution of one dimensional heat equation by using the Chebyshev Wavelets method, Journal of Applied and Computational Mathematics, Vol.1, No.6, 1-7, 2012.

[8] J. R. Li, L. Greengard, On the numerical solution of the heat equation I: Fast solvers in free space, Journal of Computational Physics, Vol.226, No.2, 1891-1901, 2007.

[9] M. Tatari, M. Dehghan, A method for solving partial differential equations via radial basis functions: Application to the heat equation, Engineering Analysis with Boundary Element, Vol.34, No.3: 206-212, 2010.

[10] V. Dabral, S. Kapoor, S. Dhawan, Numerical Simulation of one dimensional Heat Equation: B-Spline Finite Element Method, Indian Journal of Computer Science and Engineering, Vol.2, No.2, 222-235, 2011.

[11] E. M. Elbarbary, M. El-Kady, Chebyshev finite difference approximation for the boundary value problems, Applied Mathematics and Computation, Vol.139, No.2-3, 513-523, 2003.

[12] P. K. Pandey, Rational finite difference approximation of high order accuracy for nonlinear two point boundary value problems, Sains Malaysiana, Vol.43, No.7, 1105-1108, 2014.

[13] P. K. Pandey, Solving two point boundary value problems for ordinary differential equations using exponential finite difference method, Boletim da Sociedade Paranaense de Matemática, Vol.34, No.1, 45-52, 2016.

[14] S. Babu, R. Trabelsi, T. Srinivasa Krishna, N. Ouerfelli, A. Toumi, Reduced redlich–kister functions and interaction studies of dehpa+ petrofin binary mixtures at 298.15 k, Physics and Chemistry of Liquids, Vol.57, No.4, 536-546, 2019.

[15] Gayathri, T. Venugopal, K. Venkatramanan, Redlich-kister coefficients on the analysis of physico-chemical characteristics of functional polymers, Materials Today: Proceedings, Vol.17, 2083-2087, 2019.

[16] N. P. Komninos, E. D. Rogdakis, Geometric investigation of the three-coefficient redlich-kister expansion global phase diagram for binary mixtures, Fluid Phase Equilibria, 112728, 2020.

[17] M. K. Hasan, J. Sulaiman, S. Ahmad, M. Othman, S. A. Abdul Karim, Approximation of iteration number for gauss-seidel using redlich-kister polynomial, American Journal of Applied Sciences, Vol.7, 956-962, 2010.

[18] M. N. Suardi, J. Sulaiman, Solution of One-Dimensional Boundary Value Problem by Using Redlich-Kister Polynomial. Computational Science and Technology: 7th ICCST 2020, Pattaya, Thailand, 29–30 August, 2020, Vol.724, 487, 2021.

[19] M. N. Suardi, J. Sulaiman, Redlich-Kister Finite Difference Solution for Solving Two-Point Boundary Value Problems by using Ksor Iteration Family, Adv. Sci. Technol. Eng. Syst. J., Vol.6, No.1, 954-960, 2021.

[20] M. N. Suardi, J. Sulaiman, On redlich-kister finite difference solution of two-point boundary value problems using half-sweep kaudd successive over relaxation iteration, International Journal of Engineering Trends and Technology, Vol.69, No.2, 77–82, 2021.

[21] D. M. Young, Iterative Solution Of Large Linear Systems. London: Academic Press, 1971.

[22] W. Hackbusch, Iterative Solution of Large Sparse Systems of Equations. Springer-Verlag, 1995.

[23] Y. Saad, Iterative Methods for Sparse Linear Systems, International Thomas Publishing, 1996.

[24] R. Rahman, N. A. M. Ali, J. Sulaiman, F. A. Muhiddin, Caputo's finite difference solution of fractional two-point boundary value problems using SOR iteration, In AIP Conference Proceedings, 2013: 1, 2018.

[25] A. Sunarto, J. Sulaiman, A. Saudi, Implicit finite difference solution for time-fractional diffusion equations using AOR method, In Journal of Physics: Conference Series, 495: 1, 2014.

[26] N. Z. F. M. Radzuan, M. N. Suardi, J. Sulaiman, KSOR iterative method with quadrature scheme for solving system of Fredholm integral equations of second kind, Journal of Fundamental and Applied Sciences, Vol.9(5S), 609-623, 2017.

[27] M. N. Suardi, J. Sulaiman, Redlich-Kister finite difference solution for two-point boundary value problem by using MKSOR iteration. In AIP Conference Proceedings, Vol.2423, No.1: 020015, 2021.

[28] I. K. Youssef, A. A. Taha, On Modified Successive Overrelaxation Method. Applied Mathematics and Computation, Vol.219, 4601-4613, 2013.

[29] G. Summit, K. Devendra, S. Jagdev, Analytical Solutions of Convection Diffusion Problems by Combining Laplace Transform Method and Homotophy Perturbation Method, Alexandria Engineering Journal, Vol.54, 645-651, 2015.

[30] A. Dahalan, M. S. Muthuvalu, J. Sulaiman, Numerical solutions of two-point fuzzy boundary value problem using half-sweep alternating group explicit method, American Institute of Physics, Vol.1557, No.1, 103-107, 2013.

[31] A. Dahalan, J. Sulaiman, M. S. Muthuvalu, Performance of HSAGE method with Seikkala derivative for 2-D fuzzy Poisson equation, Applied Mathematical Sciences, Vol8, No.17-20, 885-899, 2014.

[32] M. K. Hasan, J. Sulaiman, S. A. Abdul Karim, M. Othman, Development of some numerical methods applying complexity reduction approach for solving scientific problem, 2010.

[33] Saudi, J. Sulaiman, Red-black strategy for mobile robot path planning, In World Congress on Engineering, Vol.2182, 2215-2219, 2010.

[34] N. I. M. Fauzi, J. Sulaiman, Quarter-Sweep Modified SOR iterative algorithm and cubic spline basis for the solution of second order two-point boundary value problems, Journal of Applied Sciences (Faisalabad), Vol.12, No.17, 1817-1824, 2012.

[35] J. V. Lung, J. Sulaiman, On quarter-sweep finite difference scheme for one-dimensional porous medium equations, International Journal of Applied Mathematics, Vol.33, No.3, 439, 2020.