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Weighted Least Squares Estimation for AR(1) Model With Incomplete Data

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Abstract Time series forecasting is the main objective in many life applications such as weather prediction, natural phenomena analysis, financial or economic analysis, etc. In real-life data analysis, missing data can be considered as a feature that the researcher faces because of human error, technical damage, or catastrophic natural phenomena, etc. When one or more observations are missing, it might be urgent to estimate the model as well as to estimate the missing values which lead to a better understanding of the data, and more accurate prediction. Different time series require different effective techniques to have better estimates for those missing values. Traditionally, the missing values are simply replaced by mean and mode imputation, deleted or handled using other methods, which are not convenient enough to address missing values, as those methods can cause bias. One of the most popular models used in estimating time-series data is autoregressive models. Autoregressive models forecast the future values in terms of the previous ones. The first-order autoregressive AR (1) model is the one which the current value is based on the immediately preceding value, then estimating parameters of AR (1) with missing observations is an urgent topic in time series analysis. Many approaches have been developed to address the estimation problems in time series such as ordinary least square (OLS), Yule Walker (YW). Therefore, a suggested method will be introduced to estimate the parameter of the model by using weighted least squares. The properties of the (WLS) estimator are investigated. Moreover, a comparison between those methods using AR (1) model with missing observations is conducted through a Monte Carlo simulation at various sample sizes and different proportions of missing observations, this comparison is conducted in terms of mean square error (MSE) and mean absolute error (MAE). The results of the simulation study state that (WLS) estimator can be considered as the preferable method of estimation. Also, time series real data with missing observations were estimated.

Keywords AR(1) Model with Missing Observation**,** Estimation, Least Squares Estimator, Weighted Least Squares

1. Introduction

In the presence of missing observations, the parameters of AR models are difficult to be estimated as the standard estimation algorithms cannot be applied to such AR models with missing observations. Moreover, the missing observations problem affects the statistical inference associated with this model. The literatures have several approaches to estimating AR model with missing observations.

Dunsmuir and Robinson [3] suggested the estimator of AR(1) model without constant with missing observations be using Yule-Walker method when $|\rho| < 1$ by:

$$\hat{\rho}_{YW} = \frac{\sum_{k=2}^{n} y_k y_{k-1} / \sum_{k=2}^{n} a_k a_{k-1}}{\sum_{k=1}^{n-1} y_{k-1}^2 / \sum_{k=1}^{n-1} a_k} \tag{1}$$

They also stated the advantages of this method over alternatives. They assessed the performance of the method used in estimating simple models by using simulations, and then applied it to a time series of pollution levels containing some missing observations. Shively [13] constructed some tests for autoregressive disturbances in a time series regression with missing observations, where the disturbance terms are generated by AR(1) process and AR(p) process with a possible seasonal component. Takeuchi [14] suggested another estimator of AR(1) model without constant with missing observations by using OLS method when $|\rho| < 1$ by:

$$\hat{\rho}_{ols} = \frac{\sum_{k=2}^{n} a_k y_k y_{k-1}}{\sum_{k=2}^{n} a_k y_{k-1}^2}, \quad (2)$$

Hamaz and Ibazizen [7] used Pitman-Closeness Criterion to compare between two estimation methods of missing values in AR (1). Jie, et al. [10] employed the polynomial transformation and an extension of stochastic gradient algorithm is proposed to fit AR models with missing observations. El-sayed et al. [5] derived OLS estimation for AR (1) with (without) constant term in case of missing values. In addition, the properties of the estimator's linearity and unbiased are discussed. Abdel Wahab [1] introduced three methods of estimation for AR (1) and AR (2) models with (without) constant with missing values. These methods were: Yule Walker method, OLS method and weighted symmetric method. Saadatmand, et al [12] considered the estimation of a missing value for AR (1) with exponential innovations and compared two methods of estimation of the missing value, with respect to Pitman's measure of closeness. Abdel Wahab and Issa [2] derived the general form of the mean and variance of AR (P) Model with Missing Observations, and a special case also introduced. In [8] they derived OLS estimator for AR (1) panel data model with missing data, which can be considered as an extension of El-Sayed, et al [5] In addition. Enany, et al. [6] introduced a closed form estimator for ρ in case of missing observations using maximum likelihood (ML). Issa [9] derived (OLS) estimators for AR(1) with constant terms and missing observations, Also, studied the properties of this estimator, moreover, suggested modification ordinary least square (MOLS) estimator which is an extension of the work of Youssef [16]. A comparative study between (OLS), Yule-Walker (YW) and (MOLS) is considered in case of stationary and near unit root time series, using Monte Carlo simulation.

The objectives of research to present a new method of estimation are to minimize the sum of the squares of the random variables of the estimated residuals. If we have equal weights then ordinary least squares (OLS) regression has been obtained. However, if the structure of the data suggests unequal weights, then it would be inappropriate to ignore the regression weights. Accordingly, estimator of AR(1) model with missing data is derived by using weighted least squares method (WLS). In addition, some properties of the estimator have been discussed. A Monte Carlo simulation study is conducted to compare between the suggested estimator with other estimation methods at different sample sizes as well as different percentages of missing. Moreover, time series real data with missing data was estimated. The rest of the paper is organized as follows: in section (2) the model and its assumptions have been introduced. In Section (3), WLS estimator for AR(1) model with missing observations has been obtained. Moreover, the properties of the estimator are investigated. In section (4), simulation studies are carried out to compare between different estimators OLS estimator, YW estimator, and the proposed estimators (WLSI and WLSII). A real data set is analyzed. Finally, in section (5), a conclusion of the theory and simulation study has been presented.

2. The Model and Assumptions

The first order autoregressive model takes the following form;

$$x_k = \rho x_{k-1} + \varepsilon_k, \quad k = 1,2,\dots,n \quad (3)$$

Where x_{k-1} is an explanatory variable (regressors) and ρ is a coefficient. Parzen [11] formulated a time series model with missing observation as a specific case of amplitude modulated stationary process. So, we express observed data $(y_k), k = 1,2,3,\dots,n$ by

$$y_k = a_k x_k, \quad k = 1,2,\dots,n \quad (4)$$

Where a_k represented the state of observation.

$$a_k = \begin{cases} 1 & if\ x_k\ is\ observed, \\ 0 & if\ x_k\ is\ missing \end{cases}$$

To specify the model in (4), the following assumptions are imposed:

a) The unknown parameter ρ is constrained to $|\rho| < 1$ for stationarity.
b) y_0 is fixed and when $n \to \infty$ the effect of ε_1 will be negligible and so it will tend to zero
c) ε_k is (i.i.d) with a Gaussian distribution with mean 0 and variances σ_ε^2 and the fourth moment of ε_k exists.
d) $E(y_{k-1}, \varepsilon_k) = 0$, the independent variables are predetermined in the sense that they are orthogonal to the contemporaneous error term for every $k = 2,3,\dots,n$.
e) $E(\varepsilon_i, \varepsilon_k) = 0$
f) $a_{k-1}a_k = 1$ (This assumption stated by Takeuchi [14])

3. WLS Estimator for AR(1) Model With Missing Observations

In this section, the estimator of AR(1) model with missing observations and its properties will be derived using WLS method.

Lemma

Suggested weighted least squares estimator for AR (1) with missing observations is obtained by

$$\hat{\rho}_{WLS} = \frac{\sum_{k=2}^{n} w_k a_k y_k y_{k-1}}{\sum_{k=2}^{n} w_k a_k y_{k-1}^2}$$

$$var(\hat{\rho}_{WLS}) = \sigma_\varepsilon^2 \sum_{k=2}^{n} a_k \left(a_k w_k y_{k-1} \left(\sum_{k=2}^{n} a_k w_k y_{k-1}^2 \right)^{-1} \right)^2$$

Where

$$w_k = |y_{k-1}|^{-2\gamma}$$
$$w_k = |y_{k-1}|^{\gamma-1}$$

Where, γ is the coefficient of heteroscedasticity according to Brewer [3] which is used in regression models. In our article γ will be revised in case of AR(1) model with missing observations to minimize the residual sum of squares.

Proof

Pre multiply a_k of both sides of equation (3), to get,

$$a_k x_k = \rho a_k x_{k-1} + a_k \varepsilon_k \quad (5)$$

Equation (4) can be rewritten as:

$$y_{k-1} = a_{k-1} x_{k-1} \quad (6)$$

Pre multiply a_k of both sides of equation (6) and by using assumption (f) then:

$x_{k-1} = a_k y_{k-1}$ and also, $x_k = a_{k-1} y_k$. So, equation (5) will be

$$y_k = \rho a_k y_{k-1} + a_k \varepsilon_k \quad (7)$$

Let, Q be the weighted sum of squares of the random factors of model (7) of the estimated residuals:

$$Q = \sum_{k=2}^{n} w_k a_k \varepsilon_k^2 = \sum_{k=2}^{n} w_k [y_k - \rho a_k y_{k-1}]^2 \quad (8)$$

By differentiating equation (8) with respect to ρ and set to zero then,

$$\hat{\rho}_{WLS} = \frac{\sum_{k=2}^{n} w_k a_k y_k y_{k-1}}{\sum_{k=2}^{n} w_k a_k y_{k-1}^2} \quad (9)$$

Substitute the value of weight number (I) in equation (9), to get;

$$\hat{\rho}_{WLS.I} = \frac{\sum_{k=2}^{n} a_k |y_{k-1}|^{-2\gamma} y_k y_{k-1}}{\sum_{k=2}^{n} a_k |y_{k-1}|^{-2\gamma} y_{k-1}^2} \quad (10)$$

When $\gamma = 0$, then $\hat{\rho}_{WLS.I} = \hat{\rho}_{OLS}$

Substitute the value of weight number (II) in equation (9), then;

$$\hat{\rho}_{WLS.II} = \frac{\sum_{k=2}^{n} a_k |y_{k-1}|^{\gamma-1} y_k y_{k-1}}{\sum_{k=2}^{n} a_k |y_{k-1}|^{\gamma-1} y_{k-1}^2} \quad (11)$$

When $\gamma = 1$, then $\hat{\rho}_{WLS.II} = \hat{\rho}_{OLS}$

From equation (7), it is easy to verify that $\hat{\rho}_{wls}$ can be rewritten as liner form

$$\hat{\rho}_{WLS} = \sum_{k=2}^{n} z_k y_k \quad (12)$$

Where,

$$z_k = a_k w_k y_{k-1} \left(\sum_{k=2}^{n} a_k w_k y_{k-1}^2\right)^{-1} \quad (13)$$

Also, by substituting y_k of equation (7) in equation (12), to get

$$\hat{\rho}_{WLS} = \rho + \sum_{k=2}^{n} a_k z_k \varepsilon_k \quad (14)$$

By taking the expection of equation (14) and by using assumation (c) , then,

$$E(\hat{\rho}_{WLS}) = \rho$$

Since,

$$var(\hat{\rho}_{WLS}) = E[\hat{\rho}_{WLS} - \rho]^2$$

Equation (14) can be rewritten as:

$$E[\hat{\rho}_{WLS} - \rho]^2 = \boldsymbol{E}\left[\sum_{k=2}^{n} a_k z_k \varepsilon_k\right]^2$$

$$E[\hat{\rho}_{WLS} - \rho]^2 = E\left(\left(\sum_{k=2}^{n} a_k z_k \varepsilon_k\right)^2 + 2\sum_{k=2}^{n}\sum_{\substack{i=2 \\ i\neq k}} \alpha_k \alpha_i z_k z_i \varepsilon_k \varepsilon_i\right)$$

Using assumptions (c and e), to get:

$$E[\hat{\rho}_{WLS} - \rho]^2 = E\left(\sum_{k=2}^{n} a_k z_k \varepsilon_k\right)^2 = \sigma_\varepsilon^2 \sum_{k=2}^{n} a_k z_k^2$$

By substituting value of z_k in equation (13), we get ouer proof.

Alternatives format of the variance of WLS Estimator

1) Variance of weight (I):

Variance of WLS estimator by using weight (I) is:

$$var(\hat{\rho}_{WLS.II}) = \sigma_\varepsilon^2 \sum_{k=2}^{n} a_k \left(a_k \left|y_{k-1}\right|^{-2\gamma} y_{k-1} \left(\sum_{k=2}^{n} a_k \left|y_{k-1}\right|^{-2\gamma} y_{k-1}^2 \right)^{-1} \right)^2$$

2) Variance of weight (II):

Variance of WLS estimator by using weight (II) is:

$$var(\hat{\rho}_{WLS.III}) = \sigma_\varepsilon^2 \sum_{k=2}^{n} a_k \left(a_k \left|y_{k-1}\right|^{\gamma-1} y_{k-1} \left(\sum_{k=2}^{n} a_k \left|y_{k-1}\right|^{\gamma-1} y_{k-1}^2 \right)^{-1} \right)^2$$

4. Simulation Study

This section aims to investigate the properties of the proposed estimation through the simulation study in small, moderate, and large samples. The model is generated as follows:

1) AR (1) model without constant is generated. The errors are generated ~ IIDN(0,1), and the autoregressive parameter ρ is chosen to be 0.3,0.5 and 0.9.
2) Different sample sizes have been used: n =15, 50, 100, and 200.
3) Different values of γ has been used as: $\gamma = 0.1, \dots, 0.9$.
4) To investigate the resistance of the proposed estimators, we randomly generate different percentages of missing values (p) equals to [(10 to 20) and (30 to 50)].
5) All Monte Carlo experiments involved 10000 replications.

We compare performance of the proposed weighted least squares estimators (WLSI and WLSII) defined in equations (10 and 11) with the Yule Walker estimator (YW) which is defined in equation (1), and the ordinary least squares estimator (OLS) in equation (2). the mean squared error (MSE) and mean absolute error (MAE) have been computed as a measure of comparison. To compare between the different methods; we firstly use the three methods to estimate the autoregressive parameter in the presence of the missing observations. Then we use the estimated parameter to provide estimates of observed observations. The results of simulation study will be explained in different cases: (a) $0.1 \leq \gamma < 0.4$ and (b) $0.4 \leq \gamma \leq 0.5$, (c) $0.5 < \gamma < 1$.

a) When $\mathbf{0.1 \leq \gamma < 0.4}$

The results of the simulation study are summarized in tables (1 to 3), using different sample sizes, different percentages of missing observations and different values of $\hat{\rho}$, then for comparison purpose we obtained the average MSE and MAE for different estimation methods. Using the results contained in tables (2 to 4), whatever the values of the simple size, the percentages of missing observations and different values of $\hat{\rho}$, the (WLSI) was found to be less MSE and MAE than any other estimator of AR (1) with missing observations, followed by (WLSII) estimator.

b) When $0.4 \leq \gamma \leq 0.5$

The results of the simulation study are summarized in tables (5 to 6). In different sample sizes, different percentages of missing observations and different values of $\hat{\rho}$, the (WLSII) was found to be less MSE and MAE than any other estimator of AR (1) with missing observations, followed by (WLSI) estimator.

c) When $0.5 < \gamma \leq 0.1$

Firstly, using the results contained in tables (7 to 10), we will compare the MSE of these estimators as follows:

In case of different sample sizes, different percentages of missing observations and different values of $\hat{\rho}$, the (WLSII) estimator has the best MSE, followed by (OLS and YW) estimators and (WLSI) has the biggest MSE and RMSE of any other estimators in case of $\rho = 0.3$ *and* 0.5, but when $\rho = 0.9$ *and* $n = 15$, the (WLSI) it is second only to the (WLSI) estimator.

Secondly, these estimators will be compared using the (MAE) criterion. The (WLSII) estimator has the best MAE, followed by (YW) estimator and the (OLS) estimator has the biggest MAE of any other estimators.

5. Real Data Application

The feasibility of the proposed estimator is illustrated using a daily average number of defects per truck time series data found in final inspection at the end of assembly line of a truck manufacturing plant Wei [15]. Figure1 shows this data that consist of 45 daily observations of consecutive business.

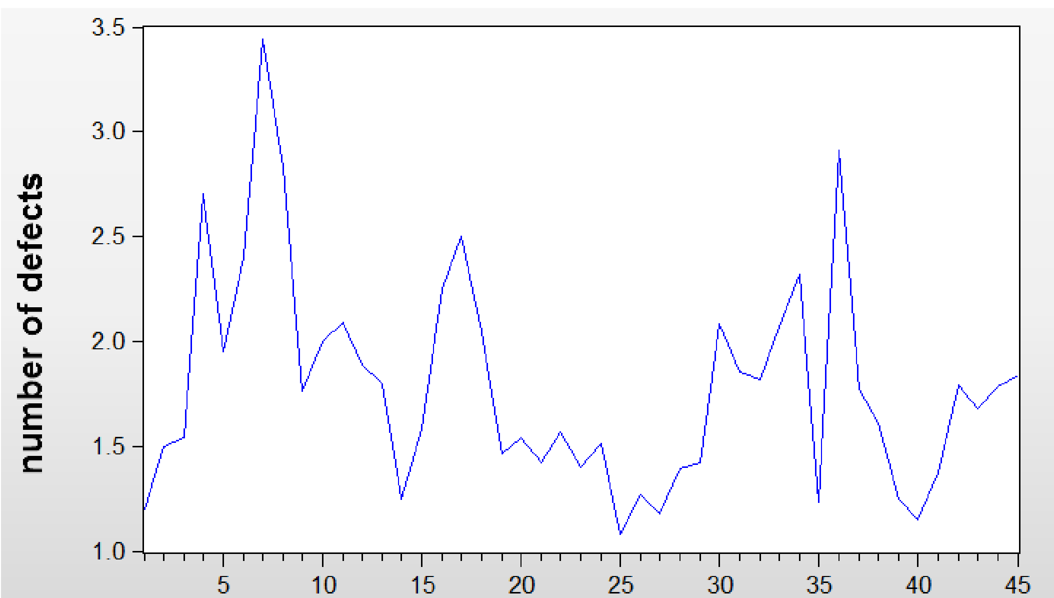
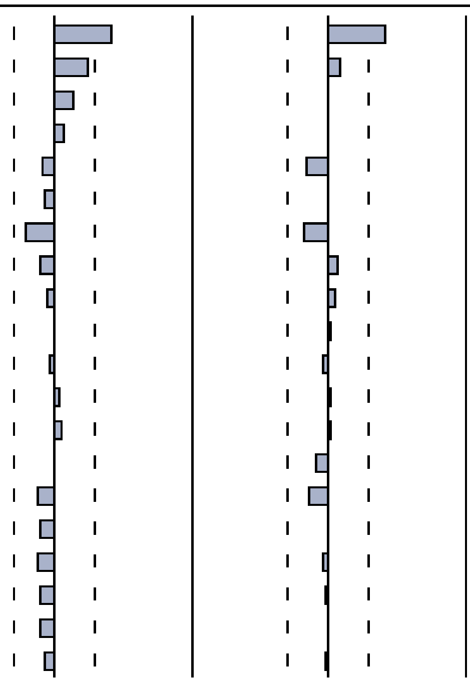


Figure 1. Daily average number of defects

By using Phillips-Perron and Augmented Dickey-Fuller tests for stationary assumption of the time series. The null hypothesis of the unit root test is rejected for different lags. The next step is to determine the best model to fit the data. The sample ACF and PACF are shown in Figure 2 which noted that the AR (1) model is more appropriate to fit the data.

To compare between (OLS), (YW), (WLSI) and (WLSII) methods of estimation for the parameter of AR (1) model in the presence of the missing observations and different values of γ. To measure the accuracy, mean squared error (MSE) and mean absolute error (MAE) are computed. The results of table (1) indicated that the new method (WLSI and WLSII) gives a good performance for the values of MAE and MSE for most percentages of missing observations and different values of γ with respect to other methods.



	AC	PAC	Q-Stat	Prob
1	0.429	0.429	8.8369	0.003
2	0.261	0.094	12.176	0.002
3	0.141	-0.001	13.176	0.004
4	0.079	0.000	13.500	0.009
5	-0.085	-0.160	13.886	0.016
6	-0.073	0.003	14.179	0.028
7	-0.207	-0.177	16.565	0.020
8	-0.110	0.066	17.253	0.028
9	-0.052	0.052	17.409	0.043
10	-0.007	0.013	17.412	0.066
11	-0.037	-0.034	17.496	0.094
12	0.038	0.021	17.589	0.129
13	0.045	0.022	17.723	0.168
14	-0.003	-0.086	17.724	0.220
15	-0.125	-0.145	18.826	0.222
16	-0.103	-0.008	19.602	0.239
17	-0.130	-0.040	20.872	0.232
18	-0.108	-0.013	21.781	0.242
19	-0.101	-0.008	22.608	0.255
20	-0.072	-0.014	23.046	0.287

Figure 2. Sample ACF and PACF for daily average number of defects

Table 1. MSE and MAE for different (methods, % of missing observations and γ)

γ	Estimators		percentages	
		5 to 10	10 to 15	20 to 25
0.1	OLS	0.4911 (0.4831)	0.6274 (0.5766)	1.3279 (1.0753)
	YW	0.4911 (0.4867)	0.6301 (0.5448)	1.3348 (0.8636)
	WLSI	0.4914 (0.4801)	0.6277 (0.5417)	1.3280 (0.8618)
	WLSII	0.4972 (0.4828)	0.6333 (0.5432)	1.3298 (0.8622)
0.2	OLS	0.4112 (0.4347)	0.6871 (0.6270)	1.0287 (0.8438)
	YW	0.4134 (0.4692)	0.7126 (0.5994)	1.0289 (0.6869)
	WLSI	0.4124 (0.4286)	0.6875 (0.5842)	1.0296 (0.6362)
	WLSII	0.4160 (0.4308)	0.6887 (0.5847)	1.0325 (0.6863)
0.3	OLS	0.5175 (0.4777)	0.6102 (0.5569)	1.3183 (1.0731)
	YW	0.5238 (0.4544)	0.6118 (0.5206)	1.3307 (0.8710)
	WLSI	0.5187 (0.4500)	0.6114 (0.5180)	1.3233 (0.8680)
	WLSII	0.5191 (0.4501)	0.6118 (0.5181)	1.3251 (0.8689)
0.4	OLS	0.4532 (0.4491)	0.6476 (0.5816)	1.2139 (0.9533)
	YW	0.4618 (0.4655)	0.6504 (0.5335)	1.2408 (0.7569)
	WLSI	0.4555 (0.4429)	0.6521 (0.5332)	1.2181 (0.7347)
	WLSII	0.4545 (0.4436)	0.6501 (0.5325)	1.2163 (0.7323)
0.5	OLS	0.4293 (0.4474)	0.6206 (0.5732)	1.2708 (1.0211)
	YW	0.4302 (0.4748)	0.6405 (0.5579)	1.2708 (0.8077)
	WLSI	0.4369 (0.4762)	0.6249 (0.5420)	1.2758 (0.8132)
	WLSII	0.4312 (0.4750)	0.6217 (0.5401)	1.2720 (0.8101)
0.6	OLS	0.6142 (0.5492)	0.5257 (0.5103)	1.0206 (0.8687)
	YW	0.6201 (0.5055)	0.5274 (0.5060)	1.0394 (0.7544)
	WLSI	0.6237 (0.5001)	0.5376 (0.5063)	1.0326 (0.7524)
	WLSII	0.6153 (0.4983)	0.5270 (0.5061)	1.0218 (0.7471)
0.7	OLS	0.5173 (0.4945)	0.7912 (0.6679)	1.0470 (0.8730)
	YW	0.5189 (0.4841)	0.7912 (0.5643)	1.0691 (0.7417)
	WLSI	0.5306 (0.4872)	0.8108 (0.5740)	1.0583 (0.7363)
	WLSII	0.5179 (0.4840)	0.7921 (0.5658)	1.0475 (0.7301)
0.8	OLS	0.5135 (0.4905)	0.6364 (0.5595)	1.3740 (1.0812)
	YW	0.5153 (0.4798)	0.6370 (0.4986)	1.3745 (0.8298)
	WLSI	0.5285 (0.4837)	0.6626 (0.5146)	1.3928 (0.8297)
	WLSII	0.5138 (0.4793)	0.6368 (0.4985)	1.3742 (0.8267)
0.9	OLS	0.5540 (0.5297)	0.8018 (0.7004)	1.0552 (0.8638)
	YW	0.5541 (0.5184)	0.8105 (0.6259)	1.0580 (0.7065)
	WLSI	0.5802 (0.5245)	0.8208 (0.6326)	1.0744 (0.7128)
	WLSII	0.5539 (0.5182)	0.8018 (0.6191)	1.0552 (0.7018)

Note: MAE values are written in parentheses

6. Conclusion

In this article, the problem of estimating the parameters of AR(1) with missing observations has been derived by using (WLS) and so two proposed estimators with different weights have been introduced. The variance of the estimated parameters has been proved with different cases. Also, the properties of (WLS) estimator have been studied. Monte Carlo simulation has been constructed using different methods of estimation (OLS, YW, WLSI and WLSII). Based on MSE and MAE criteria, the results of simulation are divided to the interval of γ: when $(0.1 \leq \gamma < 0.4)$ the (WLSI) can be considered the preferable method of estimation, followed by (WLSII) estimator and when $(0.4 \leq \gamma \leq 0.5)$ and when $(0.5 < \gamma < 1)$ the (WLSII) is the better one, followed by (WLSI) and (YW and OLS) estimators respectively. Therefore, the best interval of γ is$(0 < \gamma < 1)$.

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Appendix

Table 2. Results of AR (1) Model Estimation When γ = 0.1

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.0952	1.0952	1.0245	1.0518	1.1359	1.1363	1.0614	1.0882	1.8702	1.8742	1.7644	1.7850
		MAE	1.0138	0.8033	0.7749	0.7876	1.0515	0.8172	0.7875	0.8000	1.7308	0.9854	0.9433	0.9521
	30 to 50	MSE	1.1399	1.1402	1.0440	1.0799	1.1851	1.1855	1.1521	1.1643	3.1204	3.1371	2.9861	3.0139
		MAE	1.0251	0.8073	0.7678	0.7855	1.1450	0.8518	0.8383	0.8433	2.8037	1.2447	1.1856	1.1982
50	10 to 20	MSE	1.0384	1.0384	1.0153	1.0241	1.0807	1.0808	1.0575	1.0662	1.7913	1.7927	1.7627	1.7705
		MAE	1.0139	0.8028	0.7939	0.7978	1.0552	0.8180	0.8086	0.8122	1.7488	0.9786	0.9669	0.9700
	30 to 50	MSE	1.0832	1.0833	1.0502	1.0627	1.2755	1.2772	1.1723	1.2068	2.9229	2.9273	2.8845	2.8946
		MAE	1.0467	0.8163	0.8025	0.8083	1.1468	0.8537	0.8110	0.8273	2.8230	1.2420	1.2245	1.2291
100	10 to 20	MSE	1.0261	1.0261	1.0144	1.0189	1.0700	1.0700	1.0577	1.0621	1.7067	1.7073	1.6937	1.6979
		MAE	1.0140	0.8036	0.7988	0.8006	1.0573	0.8191	0.8142	0.8161	1.6864	0.9651	0.9598	0.9615
	30 to 50	MSE	1.0563	1.0563	1.0396	1.0460	1.1576	1.1577	1.1409	1.1471	2.8053	2.8068	2.7873	2.7931
		MAE	1.0385	0.8127	0.8061	0.8088	1.1380	0.8488	0.8421	0.8446	2.7573	1.2245	1.2168	1.2193
200	10 to 20	MSE	1.0221	1.0221	1.0160	1.0185	1.0565	1.0565	1.0505	1.0528	1.6667	1.6669	1.6606	1.6628
		MAE	1.0160	0.8043	0.8020	0.8028	1.0502	0.8162	0.8138	0.8147	1.6568	0.9579	0.9554	0.9563
	30 to 50	MSE	1.0463	1.0463	1.0384	1.0415	1.1455	1.1455	1.1375	1.1405	2.7686	2.7689	2.7601	2.7631
		MAE	1.0374	0.8126	0.8094	0.8107	1.1358	0.8478	0.8446	0.8458	2.7448	1.2202	1.2168	1.2180

Table 3. Results of AR (1) Model Estimation When γ = 0.2

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.0989	1.0990	1.0319	1.0486	1.1447	1.1451	1.0742	1.0900	1.9036	1.9080	1.7966	1.8087
		MAE	1.0173	0.8032	0.7768	0.7847	1.0598	0.8206	0.7927	0.8000	1.7616	0.9894	0.9476	0.9529
	30 to 50	MSE	1.1791	1.1793	1.0816	1.1037	1.2779	1.2802	1.1766	1.1968	3.1411	3.1580	3.0096	3.0249
		MAE	1.0603	0.8207	0.7827	0.7933	1.1489	0.8546	0.8137	0.8238	2.8217	1.2470	1.1894	1.1965
50	10 to 20	MSE	1.0421	1.0421	1.0197	1.0250	1.0845	1.0846	1.0626	1.0679	1.7913	1.7924	1.7635	1.7682
		MAE	1.0176	0.8058	0.7968	0.7992	1.0589	0.8202	0.8115	0.8138	1.7488	0.9817	0.9699	0.9718
	30 to 50	MSE	1.0827	1.0828	1.0496	1.0573	1.1836	1.1839	1.1518	1.1589	2.9445	2.9470	2.9069	2.9131
		MAE	1.0461	0.8148	0.8018	0.8052	1.1436	0.8509	0.8380	0.8410	2.8440	1.2424	1.2267	1.2295
100	10 to 20	MSE	1.0312	1.0312	1.0201	1.0226	1.0655	1.0655	1.0542	1.0568	1.7159	1.7163	1.7035	1.7059
		MAE	1.0190	0.8049	0.8004	0.8015	1.0529	0.8173	0.8127	0.8137	1.6954	0.9669	0.9619	0.9628
	30 to 50	MSE	1.0657	1.0657	1.0496	1.0532	1.1590	1.1590	1.1432	1.1468	2.8114	2.8122	2.7948	2.7984
		MAE	1.0477	0.8173	0.8109	0.8124	1.1394	0.8492	0.8429	0.8444	2.7633	1.2258	1.2187	1.2202
200	10 to 20	MSE	1.0209	1.0209	1.0154	1.0167	1.0566	1.0566	1.0510	1.0523	1.6730	1.6731	1.6670	1.6682
		MAE	1.0149	0.8035	0.8012	0.8017	1.0503	0.8166	0.8143	0.8148	1.6629	0.9585	0.9560	0.9565
	30 to 50	MSE	1.0480	1.0480	1.0401	1.0420	1.1460	1.1460	1.1380	1.1398	2.7573	2.7576	2.7489	2.7507
		MAE	1.0391	0.8127	0.8095	0.8103	1.1363	0.8485	0.8453	0.8460	2.7336	1.2181	1.2147	1.2154

Table 4. Results of AR (1) Model Estimation When γ = 0.3

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.0997	1.0997	1.0359	1.0404	1.1315	1.1320	1.0636	1.0678	1.9035	1.9106	1.7989	1.8025
		MAE	1.0181	0.8049	0.7807	0.7827	1.0474	0.8158	0.7886	0.7905	1.7617	0.9920	0.9490	0.9505
	30 to 50	MSE	1.1553	1.1558	1.0625	1.0684	1.2840	1.2853	1.1876	1.1931	3.1945	3.2127	3.0702	3.0745
		MAE	1.0387	0.8131	0.7770	0.7798	1.1544	0.8558	0.8181	0.8207	2.8705	1.2585	1.2052	1.2071
50	10 to 20	MSE	1.0416	1.0416	1.0204	1.0218	1.0803	1.0803	1.0599	1.0613	1.7780	1.7792	1.7520	1.7533
		MAE	1.0170	0.8046	0.7960	0.7966	1.0548	0.8182	0.8101	0.8108	1.7359	0.9780	0.9676	0.9681
	30 to 50	MSE	1.0828	1.0829	1.0532	1.0552	1.1898	1.1900	1.1597	1.1617	2.9077	2.9113	2.8716	2.8733
		MAE	1.0460	0.8163	0.8046	0.8054	1.1495	0.8529	0.8415	0.8424	2.8081	1.2376	1.2215	1.2222
100	10 to 20	MSE	1.0286	1.0286	1.0184	1.0191	1.0680	1.0680	1.0576	1.0583	1.7069	1.7072	1.6951	1.6958
		MAE	1.0164	0.8046	0.8003	0.8006	1.0553	0.8183	0.8141	0.8144	1.6866	0.9641	0.9593	0.9596
	30 to 50	MSE	1.062	1.062	1.047	1.048	1.1599	1.1599	1.1451	1.1461	2.8156	2.8166	2.8000	2.8009
		MAE	1.0440	0.8159	0.8100	0.8105	1.1402	0.8496	0.8436	0.8440	2.7673	1.2266	1.2201	1.2204
200	10 to 20	MSE	1.0225	1.0225	1.0174	1.0178	1.0579	1.0579	1.0528	1.0532	1.6830	1.6831	1.6774	1.6777
		MAE	1.0164	0.8041	0.8021	0.8022	1.0516	0.8171	0.8151	0.8152	1.6730	0.9605	0.9583	0.9585
	30 to 50	MSE	1.0530	1.0530	1.0457	1.0462	1.1462	1.1462	1.1386	1.1391	2.7578	2.7582	2.7502	2.7507
		MAE	1.0441	0.8151	0.8122	0.8124	1.1365	0.8488	0.8457	0.8459	2.7340	1.2179	1.2149	1.2151

Table 5. Results of AR (1) Model Estimation When γ = 0.4

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.1011	1.1012	1.0468	1.0361	1.1381	1.1386	1.0812	1.0708	1.9146	1.9180	1.8180	1.8098
		MAE	1.0194	0.8065	0.7864	0.7818	1.0536	0.8181	0.7962	0.7917	1.7719	0.9910	0.9540	0.9504
	30 to 50	MSE	1.1569	1.1578	1.0789	1.0656	1.2764	1.2782	1.1973	1.1836	3.1620	3.1779	3.0473	3.0369
		MAE	1.0401	0.8150	0.7851	0.7791	1.1476	0.8530	0.8223	0.8164	2.8402	1.2541	1.2058	1.2010
50	10 to 20	MSE	1.0422	1.0423	1.0250	1.0215	1.0837	1.0837	1.0661	1.0628	1.7781	1.7791	1.7549	1.7519
		MAE	1.0177	0.8046	0.7980	0.7965	1.0580	0.8195	0.8124	0.8110	1.7359	0.9782	0.9689	0.9676
	30 to 50	MSE	1.0782	1.0783	1.0527	1.0478	1.1872	1.1874	1.1617	1.1569	2.9310	2.9343	2.9021	2.8979
		MAE	1.0417	0.8134	0.8035	0.8014	1.1470	0.8540	0.8440	0.8421	2.8309	1.2428	1.2303	1.2286
100	10 to 20	MSE	1.0296	1.0296	1.0214	1.0196	1.0693	1.0693	1.0607	1.0589	1.6988	1.6993	1.6885	1.6870
		MAE	1.0174	0.8037	0.8005	0.7997	1.0566	0.8190	0.8156	0.8150	1.6786	0.9638	0.9595	0.9588
	30 to 50	MSE	1.0579	1.0579	1.0463	1.0438	1.1589	1.1590	1.1459	1.1435	2.8375	2.8390	2.8234	2.8212
		MAE	1.0400	0.8133	0.8088	0.8078	1.1393	0.8496	0.8444	0.8434	2.7890	1.2316	1.2255	1.2246
200	10 to 20	MSE	1.0189	1.0189	1.0145	1.0137	1.0577	1.0577	1.0535	1.0526	1.6797	1.6798	1.6749	1.6741
		MAE	1.0128	0.8027	0.8009	0.8006	1.0514	0.8168	0.8152	0.8148	1.6697	0.9597	0.9577	0.9574
	30 to 50	MSE	1.0465	1.0465	1.0405	1.0393	1.1454	1.1454	1.1393	1.1381	2.7603	2.7606	2.7536	2.7525
		MAE	1.0376	0.8128	0.8104	0.8100	1.1357	0.8485	0.8460	0.8455	2.7366	1.2190	1.2163	1.2159

Table 6. Results of AR (1) Model Estimation When γ = 0.5

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.0995	1.0996	1.0704	1.0326	1.1378	1.1383	1.1007	1.0655	1.9049	1.9108	1.8279	1.8003
		MAE	1.0179	0.8059	0.7964	0.7804	1.0532	0.8179	0.8045	0.7897	1.7630	0.9876	0.9561	0.9451
	30 to 50	MSE	1.1604	1.1613	1.1154	1.0670	1.2488	1.2506	1.2002	1.1540	3.1351	3.1493	3.0449	3.0085
		MAE	1.0433	0.8142	0.7985	0.7785	1.1229	0.8462	0.8274	0.8080	2.8165	1.2474	1.2092	1.1944
50	10 to 20	MSE	1.0392	1.0393	1.0292	1.0178	1.0850	1.0850	1.0748	1.0633	1.7912	1.7917	1.7739	1.7639
		MAE	1.0146	0.8037	0.7997	0.7950	1.0593	0.8193	0.8156	0.8109	1.7487	0.9812	0.9740	0.9699
	30 to 50	MSE	1.0800	1.0800	1.0657	1.0489	1.1872	1.1876	1.1720	1.1557	2.9435	2.9478	2.9231	2.9089
		MAE	1.0435	0.8149	0.8093	0.8023	1.1469	0.8522	0.8461	0.8396	2.8429	1.2459	1.2362	1.2306
100	10 to 20	MSE	1.0251	1.0251	1.0199	1.0142	1.0619	1.0619	1.0569	1.0513	1.7110	1.7114	1.7043	1.6989
		MAE	1.0129	0.8033	0.8012	0.7989	1.0493	0.8162	0.8142	0.8120	1.6907	0.9664	0.9636	0.9614
	30 to 50	MSE	1.0532	1.0532	1.0457	1.0375	1.1573	1.1573	1.1505	1.1421	2.7781	2.7792	2.7682	2.7608
		MAE	1.0355	0.8119	0.8088	0.8054	1.1377	0.8486	0.8460	0.8426	2.7304	1.2211	1.2168	1.2138
200	10 to 20	MSE	1.0212	1.0212	1.0188	1.0158	1.0575	1.0575	1.0554	1.0524	1.6642	1.6642	1.6610	1.6583
		MAE	1.0152	0.8040	0.8030	0.8018	1.0513	0.8170	0.8162	0.8150	1.6542	0.9571	0.9557	0.9547
	30 to 50	MSE	1.0497	1.0497	1.0462	1.0420	1.1437	1.1437	1.1400	1.1360	2.7694	2.7696	2.7649	2.7611
		MAE	1.0408	0.8137	0.8123	0.8106	1.1340	0.8476	0.8461	0.8445	2.7456	1.2208	1.2190	1.2175

Table 7. Results of AR (1) Model Estimation When γ = 0.6

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.1037	1.1038	1.1389	1.0376	1.1374	1.1378	1.1637	1.0667	1.9339	1.9395	1.8983	1.8261
		MAE	1.0218	0.8068	0.8195	0.7813	1.0530	0.8169	0.8250	0.7885	1.7896	0.9958	0.9793	0.9518
	30 to 50	MSE	1.1534	1.1542	1.1849	1.0587	1.2743	1.2763	1.3036	1.1735	3.0956	3.1093	3.0618	2.9648
		MAE	1.0369	0.8153	0.8218	0.7763	1.1459	0.8522	0.8577	0.8126	2.7814	1.2412	1.2231	1.1873
50	10 to 20	MSE	1.0427	1.0427	1.0506	1.0203	1.0797	1.0797	1.0861	1.0571	1.7932	1.7942	1.7945	1.7681
		MAE	1.0181	0.8052	0.8088	0.7965	1.0542	0.8179	0.8199	0.8087	1.7506	0.9810	0.9805	0.9704
	30 to 50	MSE	1.0828	1.0829	1.0935	1.0501	1.1773	1.1777	1.1881	1.1461	2.9274	2.9319	2.9279	2.8908
		MAE	1.0462	0.8162	0.8198	0.8033	1.1375	0.8491	0.8527	0.8365	2.8271	1.2438	1.2415	1.2271
100	10 to 20	MSE	1.0280	1.0280	1.0323	1.0165	1.0613	1.0613	1.0658	1.0502	1.7055	1.7060	1.7068	1.6923
		MAE	1.0159	0.8039	0.8058	0.7995	1.0487	0.8159	0.8175	0.8113	1.6852	0.9634	0.9636	0.9578
	30 to 50	MSE	1.0587	1.0587	1.0645	1.0428	1.1582	1.1583	1.1636	1.1423	2.8097	2.8107	2.8128	2.7928
		MAE	1.0408	0.8139	0.8162	0.8075	1.1386	0.8493	0.8511	0.8427	2.7616	1.2263	1.2268	1.2193
200	10 to 20	MSE	1.0196	1.0196	1.0217	1.0141	1.0575	1.0575	1.0593	1.0518	1.6776	1.6778	1.6789	1.6715
		MAE	1.0136	0.8030	0.8039	0.8008	1.0512	0.8170	0.8177	0.8147	1.6675	0.9596	0.9601	0.9571
	30 to 50	MSE	1.0495	1.0495	1.0525	1.0418	1.1429	1.1429	1.1455	1.1347	2.7860	2.7862	2.7877	2.7775
		MAE	1.0407	0.8137	0.8150	0.8107	1.1332	0.8474	0.8483	0.8441	2.7620	1.2226	1.2232	1.2192

Table 8. Results of AR (1) Model Estimation When γ = 0.7

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.0950	1.0951	1.4764	1.0237	1.1601	1.1607	1.4485	1.0857	1.9561	1.9611	2.0919	1.8456
		MAE	1.0139	0.8029	0.8820	0.7744	1.0739	0.8257	0.8987	0.7960	1.8105	1.0000	1.0302	0.9562
	30 to 50	MSE	1.1592	1.1598	1.6370	1.0633	1.2671	1.2686	1.6653	1.1673	3.1400	3.1584	3.4506	3.0071
		MAE	1.0422	0.8145	0.8889	0.7752	1.1391	0.8499	0.9203	0.8096	2.8221	1.2521	1.2789	1.1953
50	10 to 20	MSE	1.0419	1.0419	1.1360	1.0190	1.0836	1.0837	1.1804	1.0601	1.7945	1.7955	1.9548	1.7669
		MAE	1.0173	0.8053	0.8354	0.7959	1.0579	0.8193	0.8493	0.8101	1.7518	0.9808	1.0026	0.9695
	30 to 50	MSE	1.0740	1.0741	1.2146	1.0416	1.1753	1.1755	1.3117	1.1429	2.8886	2.8919	3.0203	2.8508
		MAE	1.0378	0.8129	0.8502	0.7999	1.1355	0.8485	0.8830	0.8349	2.7899	1.2369	1.2639	1.2207
100	10 to 20	MSE	1.0262	1.0262	1.0739	1.0146	1.0679	1.0679	1.1210	1.0563	1.7099	1.7101	1.7698	1.6971
		MAE	1.0141	0.8031	0.8200	0.7986	1.0553	0.8180	0.8347	0.8135	1.6895	0.9654	0.9807	0.9602
	30 to 50	MSE	1.0627	1.0627	1.1338	1.0470	1.1565	1.1566	1.2155	1.1398	2.8162	2.8172	2.8698	2.7985
		MAE	1.0448	0.8155	0.8375	0.8094	1.1370	0.8481	0.8675	0.8414	2.7680	1.2263	1.2435	1.2189
200	10 to 20	MSE	1.0221	1.0221	1.0496	1.0164	1.0582	1.0582	1.0847	1.0524	1.6761	1.6761	1.6955	1.6701
		MAE	1.0160	0.8043	0.8135	0.8020	1.0519	0.8169	0.8259	0.8145	1.6660	0.9603	0.9675	0.9578
	30 to 50	MSE	1.0482	1.0482	1.0810	1.0401	1.1440	1.1441	1.1762	1.1356	2.7708	2.7709	2.8023	2.7623
		MAE	1.0394	0.8132	0.8251	0.8098	1.1344	0.8476	0.8588	0.8443	2.7469	1.2196	1.2306	1.2161

Table 9. Results of AR (1) Model Estimation When $\gamma = 0.8$

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.0896	1.0897	11.0620	1.0204	1.1539	1.1545	3.2212	1.0809	1.8983	1.9025	4.0031	1.7883
		MAE	1.0088	0.8006	1.0905	0.7727	1.0683	0.8224	1.0586	0.7933	1.7568	0.9869	1.1662	0.9433
	30 to 50	MSE	1.1669	1.1672	4.3905	1.0683	1.2710	1.2729	3.4736	1.1708	3.1605	3.1835	8.2840	3.0318
		MAE	1.0491	0.8161	1.0333	0.7751	1.1431	0.8500	1.0577	0.8080	2.8392	1.2511	1.4117	1.1910
50	10 to 20	MSE	1.0427	1.0427	2.2263	1.0198	1.0877	1.0878	2.8072	1.0647	1.7832	1.7839	2.4396	1.7546
		MAE	1.0180	0.8044	0.9589	0.7952	1.0620	0.8213	0.9749	0.8118	1.7408	0.9796	1.0981	0.9683
	30 to 50	MSE	1.0844	1.0844	3.8452	1.0512	1.1794	1.1796	2.6656	1.1455	2.9581	2.9614	5.6084	2.9211
		MAE	1.0477	0.8168	0.9806	0.8035	1.1397	0.8499	1.0102	0.8358	2.8568	1.2496	1.3900	1.2338
100	10 to 20	MSE	1.0334	1.0334	2.0241	1.0218	1.0645	1.0645	1.6390	1.0530	1.7105	1.7109	2.0887	1.6980
		MAE	1.0212	0.8053	0.9165	0.8006	1.0519	0.8170	0.9241	0.8124	1.6902	0.9666	1.0492	0.9614
	30 to 50	MSE	1.0625	1.0625	3.4579	1.0461	1.1648	1.1649	2.4481	1.1479	2.7982	2.7992	3.4661	2.7806
		MAE	1.0445	0.8144	0.9342	0.8079	1.1451	0.8513	0.9622	0.8445	2.7502	1.2246	1.3237	1.2171
200	10 to 20	MSE	1.0208	1.0208	1.4885	1.0150	1.0561	1.0561	1.4595	1.0502	1.6699	1.6701	2.0774	1.6638
		MAE	1.0148	0.8036	0.8752	0.8012	1.0499	0.8164	0.8889	0.8140	1.6599	0.9577	1.0265	0.9552
	30 to 50	MSE	1.0508	1.0508	2.2098	1.0422	1.1417	1.1417	1.7827	1.1337	2.7611	2.7615	3.3171	2.7522
		MAE	1.0419	0.8143	0.9030	0.8109	1.1320	0.8469	0.9297	0.8436	2.7373	1.2184	1.2888	1.2147

Table 10. Results of AR (1) Model Estimation When γ = 0.9

n	percentages	criteria	$\rho = 0.3$				$\rho = 0.5$				$\rho = 0.9$			
			Methods				Methods				Methods			
			OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII	OLS	YW	WLSI	WLSII
15	10 to 20	MSE	1.0848	1.0850	173.1895	1.0122	1.1457	1.1462	677.2799	1.0716	1.9333	1.9396	176.3661	1.8219
		MAE	1.0043	0.7980	1.8817	0.7680	1.0606	0.8215	1.9277	0.7914	1.7894	0.9920	1.6665	0.9473
	30 to 50	MSE	1.1557	1.1563	102.6896	1.0556	1.2683	1.2702	167.8470	1.1668	3.1571	3.1758	33.4943	3.0294
		MAE	1.0391	0.8136	1.4952	0.7717	1.1403	0.8503	1.6266	0.8075	2.8363	1.2478	1.6726	1.1906
50	10 to 20	MSE	1.0434	1.0434	259.8073	1.0201	1.0819	1.0819	118.6779	1.0576	1.7904	1.7915	23.2490	1.7624
		MAE	1.0188	0.8041	1.7580	0.7946	1.0564	0.8187	1.5456	0.8089	1.7478	0.9797	1.6017	0.9678
	30 to 50	MSE	1.0715	1.0716	272.7239	1.0395	1.1841	1.1843	148.1871	1.1509	2.9171	2.9208	44.1787	2.8784
		MAE	1.0354	0.8112	1.5740	0.7980	1.1440	0.8515	1.5749	0.8378	2.8174	1.2378	1.7456	1.2211
100	10 to 20	MSE	1.0274	1.0274	13.3454	1.0156	1.0665	1.0665	32.0124	1.0547	1.7113	1.7115	27.5995	1.6985
		MAE	1.0152	0.8037	1.3386	0.7990	1.0539	0.8175	1.4352	0.8126	1.6909	0.9653	1.5606	0.9601
	30 to 50	MSE	1.0544	1.0544	85.2205	1.0376	1.1552	1.1553	38.7818	1.1383	2.8024	2.8038	52.3149	2.7844
		MAE	1.0365	0.8118	1.3607	0.8052	1.1357	0.8485	1.3297	0.8418	2.7544	1.2251	1.7291	1.2173
200	10 to 20	MSE	1.0226	1.0226	10.9373	1.0167	1.0577	1.0577	27.7722	1.0517	1.6701	1.6702	75.0337	1.6638
		MAE	1.0165	0.8042	1.2614	0.8018	1.0514	0.8172	1.3249	0.8148	1.6600	0.9578	1.5222	0.9552
	30 to 50	MSE	1.0523	1.0523	31.0298	1.0436	1.1428	1.1428	18.6763	1.1344	2.7421	2.7425	64.0830	2.7333
		MAE	1.0434	0.8145	1.2492	0.8110	1.1330	0.8474	1.2561	0.8440	2.7185	1.2151	1.6317	1.2114