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Introduction to Applied Algebra: Book Review of Chapter 8-Linear Equations (System of Linear Equations)

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Abstract This chapter review presents two ideas and techniques in solving Systems of Linear Equations in the most simple minded straightforward manner to enable the student as well as the instructor to follow it independently with very little guidance. The focus is on using simpler and easier approaches such as Determinants; and Elementary Row Operations to solve Systems of Linear Equations. We found the solution set of a few systems of linear equations by a successive ratio of the determinant of all the matrices formed from replacing each column of the coefficient matrix by the right hand side vector and the determinant of the coefficient matrix repeatedly giving the values of the variables in the system in the order in which they appeared. Similarly, we also used the three types of elementary row operations namely; Row Swap; Scalar Multiplication; and Row Sum to find the solution set of systems of linear equations through row echelon form to reduced row echelon form to find the solution set of some systems of linear equations. Technical forms of systems of linear equations were used to illustrate the two approaches in finding their solution sets. In each approach we started by finding the coefficient matrices from the systems of linear equations.

Keywords System of Linear Equations, Determinants, Elementary Row Operations, Row Echelon Form, Reduced Row Echelon Form, Augmented Matrix

1. Introduction

The set of linear equations namely, $\mu_i' = \sum_{i=1}^{n} a_{ij}\mu_i$ $(i, j = 1, 2, \dots, n)$ (*) define the vector $y' = (\mu_1', \mu_2', \dots, \mu_n')$ as a linear homogeneous function of the coordinates of $y = (\mu_1, \mu_2, \dots, \mu_n)$ and in accordance with the usual functional notation it is natural to write $y' = A(y)$; it is usual to omit the brackets and we hence set in place (*) as $y' = Ay$.

The function or operator A when regarded as a single entity is called a *Matrix*; it is completely determined, relatively to the fundamental basis, when the An^2 numbers a_{ij} are known, in much the same way as the vector y is determined by its coordinates. We call the a_{ij} the coordinates of A [6].

The essential information of a system of linear equations can be written succinctly in a rectangular array called a matrix.

As the number of equations and unknowns in a linear system increases, so does the complexity of the algebra in finding solutions. Two matrices A and B are said to be row equivalent if each can be obtained from the other by a sequence of Elementary Row Operations [5].

To each elementary row operation e there corresponds an elementary row operation e_1 of the same type as e such that $e_1\big(e(A)\big) = e\big(e_1(A)\big) = A$ for each coefficient matrix A. In other words, the inverse operation (function) of an elementary row operation exists and is an elementary row operation of the same type [10].

The new matrix obtained by Gaussian elimination algorithm is called Reduced Row Echelon Form (RREF). Using an augmented matrix we can easily find the solution of systems of linear equations by the RREF obtained from the augmented matrix. To write an augmented matrix in an RREF, it first has to be written in the Row Echelon Form (REF) in which case all nonzero rows are above any rows with all zeros and all leading ones of any row are in a column to the right of the leading one of the row above it and thus, all entries in any column below a leading one are zero. In addition if each leading one is the only nonzero entry in its column then we say the matrix is in its RREF.

In solving a system of linear equations, there are two fundamental questions you have to ask: Is there a solution to the system of linear equations?; If so, is it a unique solution? [2]

The solution set of systems of linear equations can be found by using matrix determinant approach as illustrated in the next section.

2. Materials and Methods

A System of Linear Equations is a set of two or more linear equations with two or more variables. In short, a finite set of linear equations is called a System of Linear Equations. The variables are called unknowns. Systems of Linear Equations arise in all forms of applications in many different fields of study [4].

A solution or a particular solution of systems of linear equations is a list of values for the unknowns which is a solution of each of the equations in the system. The set of all solutions of the system is called the solution set or the general solution of the system [7].

The method of Determinants and Elementary Row Operations are used to solve the System of Linear Equations of the form:

$$\begin{matrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{matrix}$$

of any size. The system above in matrix form is:

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}} = \underset{b}{\begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}}$$

We write the system in a single augmented matrix form as:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

Thus, consider a system of m equations in n variable, then the $m \times n$ matrix C formed by setting C_{ij} to be the coefficient of x_j in the ith equation is called the coefficient matrix of this system and the augmented matrix A is an $m \times (n+1)$ matrix formed just as C but whose last column contains the constants of each equation [1].

From the following system of linear equations, we obtain the coefficient and augmented matrices as follows:

A system of linear equations:

$$\begin{aligned} x_1 + x_2 - x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= -13 \\ -x_1 + 2x_2 - x_3 &= 8 \end{aligned}$$

Coefficient matrix:

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ 2 & -1 & 3 & -13 \\ -1 & 2 & -1 & 8 \end{array}\right]$$

2.1. Theorem

If A is an $m \times n$ matrix, with columns $a_1, a_2, \dots, a_n$ and if b is in R^m, then the matrix equation:

$$Ax = b$$

has same solution set as the vector equation**,**

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

which, in turn has the same solution set as the system of linear equations whose augmented matrix is:

$$[a_1 \quad a_2 \quad \dots \quad a_n \quad |b]$$

The $m \times n$ matrix A is called the coefficient matrix, the m-vector b is called the right hand side and the n-vector x is called a solution of the linear equations if $Ax = b$ holds [3].

2.2. Corollary

Any system $\sum_{j=1}^{n} a_{ij}x_j = b_i$; a_{ij} and b_i are constants and a_{ij} is the coefficient of the unknown x_j for $= 1, \dots, m$; all of m simultaneous linear equations in n unknowns can be reduced to an equivalent system whose

ith equation has the form:

$$x_i + c_{i.i+1}x_{i+1} + c_{i.i+2}x_{i+2} + \cdots + c_{in}x_n = d_i \quad (1)$$

for some subsets of r of the integers
$i = 1, \ldots, m, plus\ m - r\ equations$ of the form

$$0 = d_k$$

Proof

If by interchanging two equations we get a compatible equivalent system of linear equations and if the system is a smaller system of m equations in the $n-1$ unknowns then we may get a degenerate equations of the form $0 = d_k$. If all $d_k = 0$, these can be ignored; if $d_k \neq 0$, the original system is incompatible (has no solutions).

The system in (1) above looks like the following echelon form below:

$$x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n = d_1$$
$$x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2$$
$$x_3 + \cdots + c_{3n}x_n = d_3$$
$$x_r + \cdots + c_{rn}x_n = d_r$$

Solutions of any system of the echelon form are easily described.

Consider $x_n, x_{n-1}, x_{n-2}, \ldots, x_1$ in succession. If a given x_i in this sequence is the first variable in a system of linear equations then it is determined by
$x_n, \ldots, x_{i+1}$ from the relation

$$x_i = d_i - c_{i,i+1} - c_{i,i+2} - \cdots - c_{in}x_n$$

If it is not, then this x_i can be chosen arbitrarily. Proved.

2.3. Theorem

Let

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ \cdot \qquad\qquad & \cdot \\ \cdot \qquad\qquad & \cdot \\ \cdot \qquad\qquad & \cdot \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0 \end{aligned} \quad (2)$$

be a system of m linear equations in n unknowns and assume that $n > m$. Then the system has a non-trivial solution.

Proof

By induction considering that one equation in n unknowns, $n > 1$;

$$a_1x_1 + \cdots + a_nx_n = 0$$

If all the coefficients $a_1, \ldots, a_n$ are equal to 0, then any value of the variables will be a solution set and will give a non-trivial solution set certainly. Assume there exists some coefficient $a_1 = 0$. After we renumber the variables and the coefficients, we may suppose that it is a_1. Then we give $x_1, \ldots, x_n$ arbitrary values for example we make $x_2 = \cdots = x_n = 1$, and solve for x_1, we let $x_1 = \frac{-1}{a_1}(a_1 + \cdots + a_n)$

In that sense, we have a non-trivial solution set for the system of linear equations.

Further, consider the fact that the theorem is true of any $m-1$ equations in more than $m-1$ unknowns. It can be proved that it is true for m equations in n unknowns whenever $n > m$. If all the coefficients of eqn (2) are equal to 0, then after renumbering the equations and variables, we can assume that it is a_{11}. Namely, we shall consider the system of equations

$$\left(A_1 - \frac{a_{21}}{a_{11}}A_1\right).X = 0$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\left(A_m - \frac{a_{m1}}{a_{11}}A_1\right).X = 0$$

which can also be written in the form

$$\begin{aligned} &A_2.X - \frac{a_{21}}{a_{11}}A_1.X = 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ &A_m.X - \frac{a_{m1}}{a_{11}}A_1.X = 0 \end{aligned} \quad (3)$$

Notice that in this system the coefficient of x_1 is equal to 0. Thus, eqn (3) can be seen as a system of $m-1$ equations in $n-1$ unknowns and then have $n-1 > m-1$. By the initial assumption, it is possible to find a non-trivial solution set $(x_2, \ldots, x_n)$ for this system. We shall solve for x_1 in the first equation as $x_1 = \frac{-1}{a_{11}}(a_{12}x_2 + \cdots + a_{1n}x_n)$

In this way, we find a solution set of $A_1.X = 0$.

But by eqn (3) we must have $A_i.X = \frac{a_{i1}}{a_{11}}A_1.X$ for $i = 2, \ldots, m$. Hence, $A_i.X = 0$ for $i = 2, \ldots, m$ and hence, we have gotten a non-trivial solutions set to our original system in eqn (2).

This argument we gave allows us to proceed stepwise from one equation to two equations then to three equations and so on. Proved.

3. Results and Discussions

3.1. Using Determinant to Solve Systems of Linear Equations

We can find the solution set of any system of linear equations by the method of determinants. By this approach after writing out the matrix from the given system as matrix A, we first find it's determinant. After that we replace the first column by B and find the determinant after that we divide the result by the determinant of the original matrix A and the result gives the value of the first variable in the given system. Similarly, we replace second column by B and find the determinant after that divide the result by the determinant of the original matrix A to get the value of the second variable in the system. If there are more variables and columns in the original matrix we continue to repeat the procedure to find the remaining variables.

Illustrative Example I

Solve the following system of linear equations by the determinant method.

$$-b + 3c = -1$$

$$a + b + c = -2$$

$$4a + 2b + 2c = 2$$

First, we write the system in matrix form as below:

$$\begin{bmatrix} 0 & -1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Next we find the determinant of matrix A as below;

$$det\ A = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 1 & 1 \\ 4 & 2 & 2 \end{vmatrix}$$

$$= 0\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= 0 - 2 - 6 = -8$$

Next, we find the first variable a by replacing the first column of the coefficient matrix A by B and find the determinant and then divide the result by the determinant of original matrix A found. Hence we have;

$$det\ A_1 = \begin{vmatrix} -1 & -1 & 3 \\ -2 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} + 3\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix}$$

$$= 0 - 6 - 18 = -24$$

Therefore, $a = \frac{detA_1}{detA} = \frac{-24}{-8} = 3$

Similarly, we have for the second variable b by replacing the second column of matrix A by B and find the determinant and then divide the result by the determinant of the original coefficient matrix A found. Hence;

$$det\ A_2 = \begin{vmatrix} 0 & -1 & 3 \\ 1 & -2 & 1 \\ 4 & 2 & 2 \end{vmatrix}$$

$$= 0\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & -2 \\ 4 & 2 \end{vmatrix}$$

$$= 0 - 2 + 30 = 28$$

$$b = \frac{\det A_2}{\det A} = \frac{28}{-8} = -\frac{7}{2}$$

Further, we have for the third variable c by replacing the third column of the coefficient matrix A by B and find the determinant and then divide the result by the determinant of the original coefficient matrix A found. Hence we have;

$$det\ A_3 = \begin{vmatrix} 0 & -1 & -1 \\ 1 & 1 & -2 \\ 4 & 2 & 2 \end{vmatrix}$$

$$= 0\begin{vmatrix} 0 & -2 \\ 2 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & -2 \\ 4 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix}$$

$$= 0 + 10 + 2 = 12$$

$$c = \frac{\det A_3}{\det A} = \frac{12}{-8} = -\frac{3}{2}$$

Therefore, from the above results, the solution set of the system is; $a = 3, b = -\frac{7}{2}, c = -\frac{3}{2}$

Illustrative Example II

Solve the following system of linear equations by the determinant method:

$$x - 2y + z = 0$$

$$2y - 8z = 8$$

$$-4x + 5y + 9z = -9$$

First, we write the system in matrix form as below:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

Next we find the determinant of matrix $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$ as $detA\ below$:

$$det\ A = \begin{vmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & -8 \\ 5 & 9 \end{vmatrix} - (-2)\begin{vmatrix} 0 & -8 \\ -4 & 9 \end{vmatrix} + (1)\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix}$$

$$= 58 - 64 + 8 = 2$$

Next, we find the first variable x by replacing the first column of matrix A by B and find the determinant and then divide the result by the determinant of the original coefficient matrix A found. Hence we have;

$$det\ A_1 = \begin{vmatrix} 0 & -2 & 1 \\ 8 & 2 & -8 \\ -9 & 5 & 9 \end{vmatrix}$$

$$= (0)\begin{vmatrix} 2 & -8 \\ 5 & 9 \end{vmatrix} - (-2)\begin{vmatrix} 8 & -8 \\ -9 & 9 \end{vmatrix} + 1\begin{vmatrix} 8 & 2 \\ -9 & 5 \end{vmatrix}$$

$$= 0 - 0 + 58 = 58$$

Therefore, $x = \frac{detA_1}{detA} = \frac{58}{2} = 29$

Similarly, we have for the second variable y by replacing the second column of the coefficient matrix A by B and find the determinant and then divide the result by the determinant of the original coefficient matrix A found. Hence we have;

$$det\ A_2 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 8 & -8 \\ -4 & -9 & 9 \end{vmatrix}$$

$$= 1\begin{vmatrix} 8 & -8 \\ -9 & 9 \end{vmatrix} - (0)\begin{vmatrix} 0 & -8 \\ -4 & 9 \end{vmatrix} + 1\begin{vmatrix} 0 & 8 \\ -4 & -9 \end{vmatrix}$$

$$= 0 - 0 + 32 = 32$$

Therefore, $y = \frac{detA_2}{detA} = \frac{32}{2} = 16$

Further, we find the value of variable z by replacing the third column of the coefficient matrix A by B and find the determinant and then divide the result by the determinant of the original coefficient matrix A found. Hence we have;

$$det\ A_3 = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 2 & 8 \\ -4 & 5 & -9 \end{vmatrix}$$

$$= 1\begin{vmatrix} 2 & 8 \\ 5 & -9 \end{vmatrix} - (-2)\begin{vmatrix} 0 & 8 \\ -4 & -9 \end{vmatrix} + 0\begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix}$$

$$= -58 + 64 + 0 = 6$$

Therefore, $z = \frac{detA_3}{detA} = \frac{6}{2} = 3$

Therefore, from the above results, the solution set of the system is; $x = 29, y = 16, z = 3$

3.2. Using Elementary Row Operations to Solve Systems of Linear Equations

The elementary row operations also called the row reduction algorithm leads directly to an explicit description of the solution set of a system of linear equations when the algorithm is applied to the augmented matrix of the given system. Row operations are reversible so if the augmented matrices of two systems of linear equations are row equivalent, then the two systems of linear equations have the same solution set. We perform Elementary Row Operations by; swapping or interchanging any two rows in their original forms, multiplying any row by some nonzero constant, and adding one row to another row. When Elementary Row Operations are performed on any given matrix in such a way that the first nonzero entry of every row is a 1 and called the leading one (leading 1); the. leading 1 in each row is to the right of all other leading ones (leading 1s) in the row above it; every entry in the column below any leading 1 is zero; and finally rows consisting entirely of zeros are made the bottom rows, then we get the matrix in Row Echelon Form (REF). If in an REF, all entries above and/or below all leading 1s are zeros, we say the system is in the Reduced Row Echelon Form (RREF). Reduced Row Echelon Form (RREF) is unique but Row Echelon form (REF) is not unique [9]. In performing Elementary Row Operations for REF, we first make the first nonzero entry a 1 and move on to make all entries below it zero then move to do same to all other nonzero first entries on all other rows. Similarly, we make all entries above the column of the last leading 1 zeros in an REF and move to do same to all other leading 1s to the left to obtain the RREF. The last additional column of any RREF matrix gives the solution set of the given system of linear equations in ascending order in which the variables appeared in the system. Every matrix is row equivalent to a matrix in row echelon form [8]. We can describe the rank of any matrix as either the number of leading ones in a Reduced Row Echelon Form (RREF) or the number of nonzero rows in a Row Echelon Form (REF) [3].

Illustrative Example I

Solve the following system of linear equations by Elementary Row Operations:

$$4x - 2y - 5z = 11$$
$$x + y + z = 2$$
$$-2x + 3y - 2z = -14$$

Putting the system in an augmented matrix form we get the result below;

$$\left[\begin{array}{ccc|c} 4 & -2 & -5 & 11 \\ 1 & 1 & 1 & 2 \\ -2 & 3 & -2 & -14 \end{array}\right]$$

From the above augmented matrix, interchange the row 1 with the row 2 to get the result below;

$$\left[\begin{array}{ccc|c} 4 & -2 & -5 & 11 \\ 1 & 1 & 1 & 2 \\ -2 & 3 & -2 & -14 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & -2 & -5 & 11 \\ -2 & 3 & -2 & -14 \end{array}\right]$$

From the above result, perform $R_2 \rightarrow -4R_1 + R_2$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 4 & -2 & -5 & 11 \\ -2 & 3 & -2 & -14 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -6 & -9 & 3 \\ -2 & 3 & -2 & -14 \end{array}\right]$$

From the above result, perform $R_3 \rightarrow 2R_1 + R_3$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -6 & -9 & 3 \\ -2 & 3 & -2 & -14 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -6 & -9 & 3 \\ 0 & 5 & 0 & -10 \end{array}\right]$$

From the above result, perform $R_2 \rightarrow -\frac{1}{6}R_1$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -6 & -9 & 3 \\ 0 & 5 & 0 & -10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 5 & 0 & -10 \end{array}\right]$$

From the above result, perform $R_3 \to -5R_2 + R_3$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 5 & 0 & -10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{15}{2} & -\frac{15}{2} \end{array}\right]$$

From the above result, perform $R_3 \to -\frac{2}{15}R_3$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{15}{2} & -\frac{15}{2} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \text{ REF}$$

From the above result, perform $R_2 \to -\frac{3}{2}R_3 + R_2$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

From the above result, perform $R_1 \to (-1)R_3 + R_1$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

From the above result, perform $R_1 \to (-1)R_2 + R_1$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{ RREF}$$

Therefore, from the above result, the solution set of the system is; $x = 3,\ y = -2\ and\ \ z = 1.$

Illustrative Example II

Solve the following system of linear equations by Elementary Row Operations:

$$x - 2y + z = 0$$

$$2y - 8z = 8$$

$$-4x + 5y + 9z = -9$$

Augmenting the system in matrix form gives;

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

From the above augmented matrix, perform $R_3 \to 4R_1 + R_3$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

From the above result, perform $R_2 \to \frac{1}{2}R_2$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right]$$

From the above result, perform $R_3 \to 3R_2 + R_3$ to get the REF below;

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From the above result, perform $R_3 \to 4R_1 + R_3$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From the above result, perform $R_2 \to 4R_3 + R_2$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From the above result, perform $R_1 \to R_1 + (-1)R_3$ to get the result below;

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From the above result, perform $R_1 \to 2R_2 + R_1$ to get the RREF below;

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore, from the above result, the solution set of the system is;

$$x = 29, y = 16\ and\ z = 3$$

4. Conclusions

In conclusion, we gave a detailed solution as to how to solve systems of linear equations using two distinct approaches; by Determinant and Elementary Row Operations, which both proved to be simpler and easy to follow by every student and instructor since every step was outlined systematically in all the illustrative examples solved to show the two approaches.

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