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Some Results on Number Theory and Analysis

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In memory of Emiliana Maguiña Cabana.

Abstract In this work we obtain bounds for the sum of the integer solutions of quadratic polynomials of two variables of the form $P = (10x + 9)(10y + 9)$ or $P = (10x + 1)(10y + 1)$ or $P = (10x + 7)(10y + 3)$ where P is a given natural number that ends in one. This allows us to decide the primality of a natural number P that ends in one. Also we get some results on twin prime numbers. In addition, we use special linear functionals defined on a real Hilbert space of dimension n, $n \geq 2$, in which the relation is obtained: $a_1 + a_2 + \cdots + a_n = \lambda[a_1^2 + \cdots + a_n^2]$, where a_i is a real number for $i = 1, ..., n$. When $n = 3$ or $n = 2$ we manage to address Fermat's Last Theorem and the equation $x^4 + y^4 = z^4$, proving that both equations do not have positive integer solutions. For $n = 2$, the Cauchy-Schwartz Theorem and Young's inequality were proved in an original way.

Keywords Diophantine Equation, Prime Numbers, Twin Prime Numbers, Cauchy-Shwarz Inequality, Fermat's Last Theorem

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1 Introduction

We know that for a Hilbert space H and M a closed subspace of H we have that $H = M \oplus M^{\perp}$, following the ideas given in the article [5] and [6] we create linear functionals $f : \mathbb{R}^n \to \mathbb{R}$ obtaining the relation:

$$a_1 + a_2 + \cdots + a_n = \lambda(a_1^2 + a_2^2 + \cdots + a_n^2)$$

where the $a_i \in \mathbb{R}$. We have only used the cases $n = 2$ or $n = 3$, or the combination of both cases.

In the study of quadratic polynomials in two variables that represent a natural number, better bounds are obtained for the sum $A + B$, where (A, B) is the integer solution of one of the equations $P = (10x + 9)(10y + 9)$ or $P = (10x + 1)(10y + 1)$ or $P = (10x + 7)(10y + 3)$, where P is a natural number that ends in one.

By studying Fermat's equation, $x^n + y^n = z^n$ and assuming that there is an integer solution for $n \geq 3$, we obtained the equation of a sphere, that is :

$$\left(a_1 - \frac{1}{2\lambda}\right)^2 + \left(a_2 - \frac{1}{2\lambda}\right)^2 + \left(a_3 - \frac{1}{2\lambda}\right)^2 = \frac{3}{4\lambda^2}.$$

Which was parameterized in spherical coordinates. When n is odd, the solution is independent of the choice of coordinate axes and no positive integer solution other than the trivial one was found.

By studying the equation $x^4 + y^4 = z^2$, see [3], in this book the author shows that this equation does not have a positive integer solution using the method of infinite descent due to Fermat. We show that this equation has no positive integer solution using the cases $n = 2$ and $n = 3$.

Finally, using the case $n = 2$ we prove Schwarz's inequality, see [4] Theorem 3.11. Similarly, young's inequality is proved differently from the classical proofs. This last result is very important to prove Hölder's inequality.

2 Bounds for integer solutions of quadratic polynomials in two variables.

Determining when a natural number is prime is a computationally difficult problem. There are classical methods that allow this process to be carried out, but in this work we attack the problem using simple methods.

In [1] B.M. Cerna proved that any natural number P ending in 1 has one of the following forms:

$$\begin{aligned} P &= (10A+9)(10B+9) \text{ or} \\ P &= (10A+1)(10B+1) \text{ or} \\ P &= (10A+7)(10B+3) \end{aligned}$$

where A and B are natural numbers.

In the following theorem we establish bounds for the sum of the possible values of A and B, which will allow us to determine the primality of P.

Theorem 2.1. *Let P be a natural number ending in one. If there is $(A, B) \in \mathbb{N} \times \mathbb{N}$ such that:*

(i) $P = (10A + 9)(10B + 9)$ *or*

(ii) $P = (10A + 1)(10B + 1)$ *or*

(iii) $P = (10A + 7)(10B + 3)$.

Then we have

for (i) $\dfrac{\sqrt{P}-9}{5} \leq A + B \leq 2\dfrac{\sqrt{P}-9}{5},\ A < B.$

for (ii) $\dfrac{\sqrt{P}-1}{5} \leq A + B \leq 2\dfrac{\sqrt{P}-1}{5},\ A < B.$

for (iii) $\dfrac{\sqrt{P+28}-7}{5} \leq A + B \leq \dfrac{2}{5}\left(\sqrt{P+2}-3\right),\ A < B.$

Proof. For equations $P = (10x + 9)(10y + 9)$ and $P = (10x + 1)(10y + 1)$ the proof process is the same.

If $(A, B) \in \mathbb{N} \times \mathbb{N}$ is a solution of the equation (i), then (B, A) is also a solution of (i). The line L through the points (A, B) and (B, A) is $L:\ x + y = A + B$. The line L_T through the point $x_0 = y_0 = \frac{\sqrt{P}-9}{10}$ and furthermore, as this line being tangent to the curve (i) is given by $L_T : x + y = \frac{\sqrt{P}-9}{5}$ (see Figure 1).

The proyections of the vector $(x, y) \in L, L_T$ when $x = y$ and the vector (x_0, y_0) on the x-axis and intersection of the line L_T with the x-axis given the results.

For the equation (iii) we have

$$P = (10A + 3)(10B + 3) + 4(10B + 3),$$

from this last relationship we have

$$q = P - 4(10B + 3) = (10A + 3)(10B + 3) \tag{1}$$

As it was done in (i) we apply it in (1) so

$$\frac{\sqrt{q}-3}{5} \leq A + B \leq \frac{2}{5}[\sqrt{q}-3] \tag{2}$$

By (1), $P = q + 40B + 12$, from this relationship and (2) we get

$$B \leq \frac{P-21}{50} \text{ and } \frac{2[\sqrt{q}-3]}{5} \leq \frac{2}{5}[\sqrt{P-12}-3] \tag{3}$$

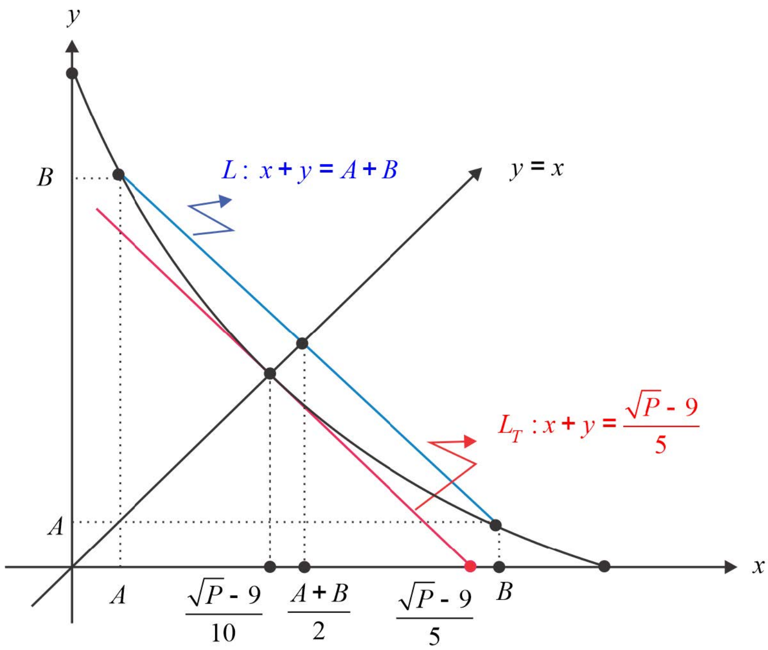

Figure 1: The curve L_T.

Then by (1) and (3) we have

$$\frac{\sqrt{q}-3}{5} \geq \frac{\sqrt{\frac{P+24}{5}}-3}{5}. \tag{4}$$

Also of the relations (2), (3) and (4) we have:

$$\frac{\sqrt{\frac{P+24}{5}}-3}{5} \leq A+B \leq \frac{2}{5}\left[\sqrt{P-12}-3\right]. \tag{5}$$

Also the equation (1) can be written as

$$P = (10A+7)(10B+7) - 4(10A+7)$$

from this relationship we have

$$P + 4(10A+7) = (10A+7)(10B+7). \tag{6}$$

Using relation (i) in (6)

$$\frac{\sqrt{P+28+40A}-7}{5} \leq A+B \leq \frac{2}{5}\left[\sqrt{P+28+40A}-7\right] \tag{7}$$

of relationships (5) and (7) with $A < B$ we get

$$\frac{\sqrt{P+28}-7}{5} \leq A+B \leq \frac{2}{5}\left[\sqrt{P-12}-3\right]. \tag{8}$$

□

The previous theorem establishes bounds for the sum of the possible values of A and B. If the possible values of A and B not satisfie the conditions (i), (ii) and (iii) of the theorem, then P is a prime number.

Example 2.1. *In the theorem 2.1 of [2] we describe an example, where*

a) $AB = 4500 + 4500\frac{N}{M} - \frac{1}{20} - \frac{N}{20M}$

b) $A+B=5000-5000\frac{N}{M}-\frac{1}{18}+\frac{N}{18M}$

c) $AB+A+B=9500-500\frac{N}{M}-\frac{19}{180}+\frac{N}{180M}$

with $G.C.D(M,N)=1$, $\tau=500\frac{N}{M}+\frac{19}{180}-\frac{1}{180}\frac{N}{M}$ *and* τ *asume* 26 *posibles valores para* $A\geq 11$. *From (b) and using (i) of the theorem* 2.1 *we get*

$$0,92482\leq\frac{N}{M}\leq 0,96241. \tag{9}$$

Hence replacing τ *in (9), we have* $462\leq\tau\leq 481$. *In addition, the following expressions are obtained*

$$(A-1)(B-1)=-501+19\tau;\;\;(A+1)(B+1)=9501-\tau;\;\;AB=9\tau+4499 \tag{10}$$

and from there, for any value of $A=\mathring{3}+2$ *or* $\mathring{3}+1$ *we get* $\tau=\mathring{3}$. *So* $\tau\in\{462,468,471,474,477,480\}$, τ *assumes six possible values.*

As

$$\begin{aligned} P &= \mathring{4}+3=100AB+90(A+B)+81=(10A+9)(10B+9) \\ & \quad \mathring{4}+3=2(A+B)+1, \end{aligned} \tag{11}$$

thus

$$A+B=\mathring{4}+1 \text{ or } A+B=\mathring{4}+3.$$

Also by (11) we have

$$2A=\mathring{4}\;\text{ and } 2B=\mathring{4}+2\;\text{ or }\;2A=\mathring{4}+2 \text{ and }\;\;2B=\mathring{4} \tag{12}$$

and by (12) and (10)

$$\begin{aligned} (2A-2)(B-1) &= -501\times 2+38\tau \\ (\mathring{4}-2)(B-1) &= 2\tau-(\mathring{4}+2) \\ \mathring{4}-2B+2 &= 2\tau \end{aligned} \tag{13}$$

where the result is equal when $2A=\mathring{4}+2$ *y* $2B=\mathring{4}$.
By (13) it is clear that $\tau=\mathring{4}+1$ *or* $\tau=\mathring{4}+3$. *Therefore* $\tau\in\{465,471,477\}$.

If $A+B=4k_2+1$, *then from (10) we know that* $\tau=\mathring{3}$ *and by (b)* $A+B=5001-10\tau$, *so* $A+B=\mathring{3}$, *that is,* $A+B=3k_1=4k_2+1=5001-10\tau$, *which forces us to have* $\tau=\mathring{6}$ *which is false. Therefore*

$$A+B=4k+3 \tag{14}$$

and by (14) the following possibilities are obtained

$$\tau=12\lambda+9\quad\text{or}\quad\tau=12\lambda+3 \tag{15}$$

but $A=3a+1$, $B=3b+2$ *or* $A=3a+2$, $B=3a+1$, *replacing these relations in (10) and (15) we get*

$$3a(3b+1)=-501+3(4\lambda+3)\cdot 19 \text{ and } (3a+2)(3b+3)=9501-(12\lambda+9).$$

For $A=3a+1$, $B=3b+2$, $2A=\mathring{4}$ *and* $2B=\mathring{4}+2$ *we get* $a=\mathring{4}$ *and* $b=\mathring{4}$ *or* $a=\mathring{4}+1$ *and* $b=\mathring{4}+3$, *by replacing these values in (2.1) we get a contradiction.*

The same results are obtained for the other cases. Therefore, the only possibility that guarantees a solution is when $\tau=12\lambda+3$ *that is* $\tau\in\{471\}$.

Using the ideas of the Theorem 2.1 of [2] we will demonstrate a very important lemma which serves to demonstrate the Hölder inequality and consequently the Minkowsky inequality.

Lemma 2.1. *If* a,b,λ *and* u *are non-negative numbers and* $u+\lambda=1$, *then* $a^\lambda b^u\leq\lambda a+ub$.

Proof. Let $F(x,y)=(a^\lambda b^u)x+(\lambda a+ub)y$ be so it is clear that F is a continuous linear functional, then $\text{Ker}F=(-(\lambda a+ub),a^\lambda b^u)$, $\{\text{ Ker}F\}^\perp=(a^\lambda b^u,\lambda a+ub)$. Thus

$$F(1,1)=a^\lambda b^u+\lambda a+ub. \tag{16}$$

As $\mathbb{R}^2 = \text{Ker}F \oplus \{ \text{ Ker}F\}^{\perp}$ we have

$$(1,1) = \lambda_1 \left(-(\lambda a + ub), a^{\lambda} b^{u}\right) + \lambda_2 \left(a^{\lambda} b^{u}, \lambda a + ub\right) \tag{17}$$

and relationships (16) and (17) we get

$$a^{\lambda} b^{u} + \lambda a + ub = \lambda_2 \left(a^{2\lambda} b^{2u} + (\lambda a + ub)^2\right) \tag{18}$$

and

$$a^{\lambda} b^{u}[1 - \lambda_2 a^{\lambda} b^{u}] = (\lambda a + ub)\left[\lambda_2(\lambda a + ub) - 1\right] \tag{19}$$

which implies

$$1 - \lambda_2 a^{\lambda} b^{u} \geq 0 \quad \text{and} \quad \lambda_2(\lambda a + ub) - 1 \geq 0 \tag{20}$$

or

$$1 - \lambda_2 a^{\lambda} b^{u} \leq 0 \quad \text{and} \quad \lambda_2(\lambda a + ub) - 1 \leq 0. \tag{21}$$

By (20) we get

$$\lambda a + \lambda b \geq a^{\lambda} b^{u} \tag{22}$$

and by (21)

$$\lambda a + ub \leq a^{\lambda} b^{u}. \tag{23}$$

We will show that (23) not happenes. By (23) and (19)

$$\lambda_2 \left[a^{\lambda} b^{u} + \lambda a + ub\right] \leq 2 \tag{24}$$

If $a, b \neq 0$. Thus by (24) and using $\lambda + u = 1$

$$\lambda_2 \leq \frac{2}{a} \text{ if } a \leq b \quad \text{or} \quad \lambda_2 \leq \frac{2}{b} \text{ if } b \leq a \tag{25}$$

making a or b big enough, we get the only chance $\lambda_2 = 0$.
Therefore (23) is impossible, so only the relationship happens (22). □

Corollary 2.1. *Let H be a Pre-Hilbert space over $\mathbb{C}$, then*

$$| < x, y > | \leq \sqrt{< x, x >}\sqrt{< y, y >}.$$

Now we will use the Theorem 2.1 to study the twin prime numbers. First we have a general result.

Theorem 2.2. *Let P be a natural number ending in 1 and $P = \mathring{3}+2$. If **(i)** $P = (10x+9)(10y+9)$ or **(ii)** $P = (10x+1)(10y+1)$ or **(iii)** $P = (10x+7)(10y+3)$ and **(iv)** $P+2 = (10x+9)(10y+7)$ or **(v)** $P+2 = (10x+1)(10y+3)$. If there are integer solutions $(A, B) \in \mathbb{N} \times \mathbb{N}$ and $(C, D) \in \mathbb{N} \times \mathbb{N}$ of the quadratic equations representing P and $P+2$, then*

- ***(i)*** $\dfrac{\sqrt{P} - 9}{5} \leq A + B \leq \dfrac{2(\sqrt{P} - 9)}{5}$ *or*
- ***(ii)*** $\dfrac{\sqrt{P} - 1}{5} \leq A + B \leq \dfrac{2(\sqrt{P} - 1)}{5}$ *or*
- ***(iii)*** $\dfrac{\sqrt{P+28} - 7}{5} \leq A + B \leq \frac{2}{5}\left(\sqrt{P - 12} - 3\right)$ *with $A < B$ and*
- ***(iv)*** $\dfrac{\sqrt{P+20} - 9}{5} \leq C + D \leq \frac{2}{5}\left(\sqrt{P - 12} - 7\right)$ *with $C < D$. or*
- ***(v)*** $\dfrac{\sqrt{P+8} - 3}{5} \leq C + D \leq \frac{2}{5}\left(\sqrt{P} - 1\right)$ *with $C < D$.*

Proof. The results **(i)**, **(ii)**, **(iii)**, **(iv)** and **(v)** are obtained in a similar way to what was done in the Theorem 2.1. □

Remark 2.1. *From the Theorem 2.2 is easy to obtain bounds for A, B, C, D, AM and CD. To find the integer solutions it is necessary to use the Theorem 2.1 of [2], where*

$$AB = \frac{(P-81)}{200}\frac{(M+N)}{M}, \qquad A+B = \frac{(P-81)}{180}\frac{(M-N)}{M}$$
$$CD = \frac{(P-61)}{200}\frac{(M_1+N_1)}{M_1}, \qquad 7C+9D = \frac{(P-61)}{20}\frac{(M_1-N_1)}{N_1}.$$

with $A < B$, $C < D$, $N < M$, $M_1 < N_1$, N and M are relative primes, N_1 and M_1 are relative primes.

Also for any $A, B, C, D \in \mathbb{N}$ we have $A + B = \mathring{3}$ and $C + D = \mathring{3}$, and we can use the technique of the Example 2.1.

Theorem 2.3. *Let P be a natural number with $P = \mathring{3} + 2$. If P and $P + 2$ are prime numbers, then exist m and n relatively prime such that $p + 1 = m = \sqrt{n-1}$ and if $n - 2 = p_1p_2$ where p_1 and p_2 are prime numbers, then P and $P + 2$ are prime numbers.*

Proof. Consider the aplication $F(x, y) = Px + (P + 2)y$, similar to the Lemma 2.1 we have $\text{Ker}\,F = \{(-(P + 2), P)\}$, $\{\text{Ker}\,F\}^{\perp} = \{(P, P + 2)\}$. Then

$$F(1, 1) = P + P + 2. \tag{26}$$

Also

$$(1, 1) = \lambda_1 (-(P + 2), P) + \lambda_2 (P, P + 2) \tag{27}$$

and by (26) and (27) we have

$$P + 2 + P = \lambda_2 \left(P^2 + (P + 2)^2\right). \tag{28}$$

Hence $\lambda_2 \in \mathbb{Q}$. Let $\lambda_2 = \frac{m}{n}$ be with m and n relatively prime. By (28) we get

$$(P + 2)(m(P + 2) - n) = P(n - mP) \tag{29}$$

and using the fact that $P + 2$ and P are primes, then $P + 1 = m = \sqrt{n-1}$.

If $m = p + 1 = \sqrt{n-1}$, then $P^2 + 2p + 1 = n - 1$ and so $P(P + 2) = n - 2 = p_1p_2$. Therefore P and $P + 2$ are prime numbers. □

In the following theorem we prove the Fermat's last theorem using basic tools developed in this article.

Theorem 2.4. *If $n \in \mathbb{N}$ odd, $n \geq 3$, the equation $x^n + y^n = z^n$ has no solution in positive integers $x, y, z \geq 1$.*

Proof. Suppose that equation $x^n + y^n = z^n$ has an integer solution for $n \in \mathbb{N}$ odd with $n \geq 3$. Let A, B and C be natural numbers that are relatively prime numbers to each other such that

$$A^n + B^n = C^n. \tag{30}$$

Let $F(x, y, z) = A^n x + B^n y + C^n z$ be a real function of several variables, then consider

$$KerF = \{(-C^n, 0, A^n), (-B^n, A^n, 0)\}, \{KerF\}^{\perp} = \{(A^n, B^n, C^n)\}.$$

Then

$$F(1, 1, 1) = 2C^n. \tag{31}$$

Also $(1, 1, 1) = \lambda_1(-C^n, 0, A^n) + \lambda_2(-B^n, A^n, 0) + \lambda_3(A^n, B^n, C^n)$. Hence, applying F we obtain

$$2C^n = \lambda_3(A^{2n} + B^{2n} + C^{2n}) \tag{32}$$

From (32)

$$\begin{cases} A^n = \frac{1}{\lambda_3}\sin\phi\cos\theta \\ B^n = \frac{1}{\lambda_3}\sin\phi\sin\theta \\ C^n = \frac{1}{\lambda_3}(\cos\phi + 1) \end{cases} \tag{33}$$

So by (33) and using (30) we get

$$(\cos\theta + \sin\theta) = \frac{(\cos\phi + 1)}{\sin\phi} = K \tag{34}$$

It is clear that $\theta = \theta(A,B,C,n)$, $\phi = \phi(A,B,C,n)$, $\lambda_3 = \lambda_3(A,B,C,n)$ and $K = K(A,B,C,n)$, $\theta \in \langle 0, \frac{\pi}{2}\rangle$, $\phi \in \langle 0, \pi\rangle$.

From (34) $\sin\theta = \sqrt{\dfrac{1-K\sqrt{2-K^2}}{2}}$, $\cos\theta = \sqrt{\dfrac{1+K\sqrt{2-K^2}}{2}}$, $\cos\phi = \dfrac{K^2-1}{K^2+1}$ and $\sin\phi = \dfrac{2K}{K^2+1}$. Also from the third equation of (33) it follows that $\cos\phi \in \mathbb{Q}$ because λ_3 is rational number, so $K^2 \in \mathbb{Q}$, $1 \le K \le \sqrt{2}$. As $\sin^2\theta \in \mathbb{Q}$ making $\lambda K = \pm\sqrt{2-K^2}$ with $\lambda \in \mathbb{Q}$, then $K^2 = \dfrac{2}{\lambda^2+1}$ and $0 \le \lambda^2 \le 1$ imply that $-1 \le \lambda \le 1$.

As λ_3 and λ are rational numbers, consider $\lambda_3 = \dfrac{M}{N}$ and $\lambda = \frac{r}{t}$ with $G.C.D(M,N) = 1$ and $G.C.D(r,t) = 1$. As $-1 \le \lambda \le 1$, using the relations of (33), if we consider $0 \le \lambda \le 1$, we obtain

$$C^n = \frac{4Nt^2}{M\left(3t^2+r^2\right)}, \quad B^n = \frac{2\,t\mathrm{N}(t-r)}{M\left(3t^2+r^2\right)}, \quad A^n = \frac{2Nt(t+r)}{M\left(3t^2+r^2\right)}. \tag{35}$$

Also, if we consider the case $-1 \le \lambda \le 0$, we obtain analogous expressions. Without loss of generality, which will be supported later, from the equation (30), if we assume that C is even, then B and A are odd.

We observe from (35) that if the factor $\dfrac{2Nt}{M\left(3t^2+r^2\right)}$ is a natural number greater than one, A, B and C have common prime factors, which is a contradiction. Then consider

$$\frac{1}{\beta} = \frac{2Nt}{M\left(3t^2+r^2\right)}, \ \beta \in \mathbb{N}, \ \beta \ge 1. \tag{36}$$

From (36) and (35) we have

$$C^n = \frac{1}{\beta}2t, \quad B^n = \frac{1}{\beta}(t-r), \quad A^n = \frac{1}{\beta}(t+r). \tag{37}$$

If $p \neq 2$ was a prime divisor of the numbers β, $t-r$ y $t+r$, then $t-r = p\theta_1$, $t+r = p\theta_2$ and so $2t = p(\theta_1+\theta_2)$, and as $p \neq 2$ implies that p divide to t, which implies that p divide to r, which is a contradiction, since $G.C.D(r,t) = 1$. Therefore, the unique prime divisor of β is 2, then

$$C^n = t, \quad B^n = \frac{t-r}{2}, \quad A^n = \frac{t+r}{2} \tag{38}$$

As C is even, t is even, so r is odd. Consequently, of (36)

$$1 = \frac{4Nt}{M\left(3t^2+r^2\right)} \tag{39}$$

which implies that (38) y (39) are situations that will not happen. Therefore, $\beta = 1$ and so

$$C^n = 2t, \ B^n = t-r, \ A^n = t+r. \tag{40}$$

If t was odd, (40) would be a contradiction, since 2 does not have an exact nth root and $C \in \mathbb{I}$, which is a contradiction. Therefore, t is even and r is odd. Then t has the following form

$$t = 2^{n-1}p_1^{n\alpha_1}\dots p_k^{n\alpha_k}, \ n \ge 2. \tag{41}$$

In addition from (40) we have $B^n - A^n = 2r$. Now if n was 2, we would have $(B-A)(B+A) = 2r$, but $B-A$ and $B+A$ are pairs, which is absurd. Also, is easy to see that for the other even values of n the same contradiction is reached. Therefore n has to be odd.

In addition of (36) we get

$$\frac{2Nt}{M(3t^2+r^2)} = 1, \ \text{ for all } \ N, M, t, r. \tag{42}$$

Now, we will justify why C was supposed to be even. If C is odd, then A is even and B is odd. Define $-A = \widehat{A}$, $-C = \widehat{C}$, $B = \widehat{B}$. Thus, we consider

$$G(x,y,z) = \widehat{C}^n x + \widehat{B}^n y + \widehat{A}^n z, \quad \widehat{C}^n + \widehat{B}^n = \widehat{A}^n.$$

Now, doing the same process done with the linear functional F, we get

$$\widehat{A}^n = \frac{1}{\widehat{\lambda}_3}\left(\cos\widehat{\phi}+1\right), \ \widehat{B}^n = \frac{1}{\widehat{\lambda}_3}\sin\widehat{\theta}\sin\widehat{\phi}, \ \widehat{C}^n = \frac{1}{\widehat{\lambda}_3}\sin\widehat{\phi}\cos\widehat{\theta}, \tag{43}$$

where $\widehat{\theta} \in \langle \frac{3\pi}{2}, 2\pi \rangle$ y $\widehat{\phi} \in \langle 0, \pi \rangle$. We also get

$$\frac{\cos\widehat{\phi}+1}{\sin\widehat{\phi}} = \cos\widehat{\theta} + \sin\widehat{\theta} = K_1,\ K_1 > 0$$

doing similar operations as in the first case, we obtain

$$A^n = 2t_1,\ B^n = r_1 - t_1,\ C^n = r_1 + t_1 \tag{44}$$

or

$$A^n = t_1,\ \ B^n = \frac{r_1 - t_1}{2},\ \ C^n = \frac{r_1 + t_1}{2} \tag{45}$$

where $K_1^2 = \dfrac{2}{\lambda_1^2 + 1}$ and $\lambda_1 = \dfrac{r_1}{t_1} > 1$, with $G.C.D(r_1, t_1) = 1$. Therefore, (45) is a contradiction, because t_1 is even and r_1 is odd. Then it is justified that this process is only valid if n is odd.

From the relation (42) and the fact $G.C.D(M, N) = G.C.D(t, r) = 1$, we conclude that

$$N = 3t^2 + r^2 \ \text{ and } \ M = 2t. \tag{46}$$

Thus, from the third equation of (33)

$$C^n = 2\frac{N}{M}\cos^2\frac{\phi}{2}. \tag{47}$$

Then by (47) and using the fact that $C^n = 2t$ and by (46) we get

$$t = \frac{\sqrt{N}}{\sqrt{2}}\cos\frac{\phi}{2} = \frac{\sqrt{N}}{2}\sqrt{\cos\phi + 1}. \tag{48}$$

So by (48) and (34)

$$t = \frac{\sqrt{N}}{2}\frac{\sqrt{2}K}{\sqrt{K^2+1}},\ \text{ with } K^2 = \frac{2}{\lambda^2+1} \text{ and } \lambda = \frac{r}{t}. \tag{49}$$

At once, by replacing the value of K we have

$$t = \frac{\sqrt{2}}{2}K\sqrt{r^2+t^2}. \tag{50}$$

If $r^2 + t^2$ was a perfect square, that is, if exist $z \in \mathbb{N}$ such that $r^2 + t^2 = z^2$, then K must be of the form $K = \frac{\sqrt{2}\alpha}{m}$ such that $G.C.D(\alpha, m) = 1$, with $\frac{1}{\sqrt{2}} < \frac{\alpha}{m} \leq 1$, becuase $(1 \leq K \leq \sqrt{2})$. Then in (50) we get

$$t = \frac{\alpha}{m}z \tag{51}$$

Therefore, it is clear that $z = \tau m$ for some $\tau \in \mathbb{N}$, so in $r^2 + t^2 = z^2$

$$r^2 + \alpha^2\tau^2 = \tau^2 m^2, \tag{52}$$

This implies that r and t have the common factor τ, which is a contradiction, because $M.D.C(t; r) = 1$. Therefore, $r^2 + t^2$ cannot be a perfect square. Hence by (50), the expression of $r^2 + t^2$ should be as follows

$$r^2 + t^2 = 2K^{2a}w^2. \tag{53}$$

for some $a, w \in \mathbb{N}$. Then replacing (53) in (50), we have

$$t = \frac{\sqrt{2}}{2}K^{a+1}w\sqrt{2} = wK^{a+1} \tag{54}$$

from there $a = 2b + 1$ because K^2 is a rational number, for some $b \in \mathbb{N}$. So, we get $t = wK^{2(b+1)}$, with $K^2 = \dfrac{2}{\lambda^2+1}$ and $\lambda = \frac{r}{t}$, the

$$t = \frac{w.2^{b+1}.t^{2b+2}}{(r^2+t^2)^{b+1}},$$

hence

$$(r^2+t^2)^{b+1} = w \cdot 2^{b+1} \cdot t^{2b+1}.$$

This last relationship is a contradiction, since $G.C.D(r,t)=1$ and t is even and r is odd.

Therefore, this proves that the Fermat's last theorem $x^n + y^n = z^n$ has no solution if n is odd. □

The next theorem we prove that the Fermat's last theorem $x^n + y^n = z^n$ has no solution if n is even, this completes the proof of Fermat's last theorem.

Theorem 2.5. *If $n \in \mathbb{N}$ even, the equation $x^n + y^n = z^n$ has no solution in positive integers $x, y, z \geq 1$.*

Proof. Suppose that equation $x^n + y^n = z^n$ has an integer solution for $n \in \mathbb{N}$ even. Let A, B and C be natural numbers that are relatively prime numbers to each other such that

$$A^n + B^n = C^n.$$

Let $F(x,y) = A^n x + B^n y$ be a linear functional, then $Ker\, F = (-B^n, A^n)$ and $\{Ker\, F\}^{\perp} = \{(A^n, B^n)\}$. So

$$F(1,1) = A^n + B^n = C^n. \tag{55}$$

Also

$$(1,1) = \lambda_1(-B^n, A^n) + \lambda_2(A^n, B^n) \tag{56}$$

The relationships (55) and (56) imply

$$C^n = \lambda_2\left(A^{2n} + B^{2n}\right). \tag{57}$$

By (57) we get

$$\begin{aligned} C^n &= \lambda_2\left((A^n+B^n)^2 - 2A^nB^n\right) = \lambda_2\left(C^{2n} - 2A^n(C^n - A^n)\right) \\ \frac{C^n}{\lambda_2} &= C^{2n} - 2A^nC^n + 2A^{2n} \Rightarrow \frac{C^n}{2\lambda_2} = \frac{C^{2n}}{2} - A^nC^n + A^{2n} \\ \left(A^n - \frac{C^n}{2}\right)^2 &= \frac{C^n}{4} + \frac{C^n}{2x_2} - \frac{C^{2n}}{2} = \frac{C^n}{2x_2} - \frac{C^{2n}}{4}, \end{aligned}$$

then

$$A^n = \frac{C^n}{2} \pm \sqrt{\frac{C^n}{2\lambda_2} - \frac{C^{2n}}{4}}.$$

Hence

$$2A^n = C^n \pm \sqrt{\frac{2C^n}{\lambda_2} - C^{2n}} \tag{58}$$

Consider

$$\frac{2C^n}{\lambda_2} - C^{2n} = T^2,\ T \in \mathbb{N} \tag{59}$$

Let's suppose that $A^n > B^n$ then from the relations (58) and (59)

$$2A^n = C^n + T \ \text{ and } \ 2B^n = C^n - T. \tag{60}$$

We have the following possibilities:

Case a) In (60) if C is even, then A and B are odd and T is even with $T = 2\lambda$, $\lambda \in \mathbb{N}$, λ is odd. By (60)

$$2A^n - 2B^n = 2T,\ A^n - B^n = T = 2\lambda. \tag{61}$$

We observed that (61) is false because $n = 2k$ and the following relationship cannot happen

$$(A^k - B^k)(A^k + B^k) = 2\lambda.$$

Case b) In (60) if C is odd, then suppose A is even and B is odd. By (59) for $\lambda_2 = \frac{r}{t}$, with r and t relatively prime numbers and by (60) T is odd, so

$$\frac{2C^n t}{r} - C^{2n} = T^2. \tag{62}$$

By (62) we have

$$(Tr)^2 + (C^n r - t)^2 = t^2. \tag{63}$$

Using Fermat's theorem for the case 2 there are M and N relatively prime numbers such that

$$t = M^2 + N^2,\ Tr = 2MN,\ C^n r - t = M^2 - N^2 \tag{64}$$

or

$$t = M^2 + N^2,\ Tr = M^2 - N^2,\ C^n r - t = 2MN \tag{65}$$

Again by (62) if t and r are odd, then r divide C^n. Consider

$$C^n = kr,\ k \in \mathbb{N}\ , k \text{ odd.} \tag{66}$$

Now from (66) and (64)

$$kr^2 = 2M^2. \tag{67}$$

Then (67)is a contradiction because k and r are odd. Also by (62) if t is even, then r is odd and r divide C^n. Consider

$$C^n = kr, k \in \mathbb{N},\ k \text{ odd} \tag{68}$$

Replacing (68) in (64) we get

$$kr^2 = 2M^2. \tag{69}$$

Hence (69) is a contradiction.

Finally from (62) if t is odd and r is even, then $r = 2\tilde{k}$ where $\tilde{k}$ divide C^n, but (62) would be a contradiction because the sum of two odd numbers would be equal to another odd.

Consider the second possibility (65) $C^n r - t = 2MN$, then $C^n r = (M+N)^2$ and by (66) if t and r are odd

$$kr^2 = (M+N)^2. \tag{70}$$

By (70) we have $k = \alpha^2$, $\alpha \in \mathbb{N}$, α odd, then in (70)

$$\alpha r = M + N.$$

Therefore $Tr = (M+N)(M-N) = \alpha r(M-N)$ and so

$$T = \alpha(M-N). \tag{71}$$

Also by (60)

$$\begin{aligned} 4A^n B^n &= C^{2n} - T^2 = k^2 r^2 - \alpha^2 (M-N)^2 \\ &= \alpha^4 r^2 - \alpha^2 (M-N)^2,\ n = 2\overline{k},\ \overline{k} \text{ odd} \\ 4A^n B^n &= \alpha^2(\alpha^2 r^2 - (M-N)^2) \end{aligned}$$

As α is odd, then α/A or α/B, in addition of (66) α/C, this a contradiction. If t is even and r is odd this is impossible in the equation (65) $C^n r - t = 2MN$. Therefore there is no solution for $n \geq 4$ even.

□

In [3] the book "Algebraic Theory of Numbers" in the Theorem 2 (page 16), shows that the equation $x^4 + y^4 = z^2$ has no integer solution $x, y, z \geq 1$, which implies that $x^4 + y^4 = z^4$ has no integer solution $x, y, z \geq 1$. In the proof used by the author he mentions that a slight variant leads to the infinite descent method due to Fermat. We use the technique described in the previous theorems and we will give a simple and subtle proof of this result.

Theorem 2.6. *The equation $x^4 + y^4 = z^2$ has no solution in positive integers $x, y, z \geq 1$.*

Proof. Suppose there are $A, B, C \geq 1$ integers such that $A^4 + B^4 = C^2$ where A, B and C that are relatively prime numbers to each other. Consider $F(x, y) = A^4x + B^4y$, then $Ker\, F = \{(-B^4, A^4)\}$, $\{Ker\, F\}^{\perp} = \{(A^4, B^4)\}$. So

$$F(1,1) = A^4 + B^4 = C^2 \tag{72}$$

in addition

$$(1,1) = \lambda_1(-B^4, A^4) + \lambda_2(A^4, B^4). \tag{73}$$

From the relations (72) and (73) we get

$$\begin{aligned} C^2 &= \lambda_2(A^8 + B^8) = \lambda_2\left((A^4 + B^4)^2 - 2A^4B^4\right), \\ C^2 &= \lambda_2\left[C^4 - 2A^4(C^2 - A^4)\right], \\ C^2 &= \lambda_2\left[C^4 - 2A^4C^2 + 2A^8\right]. \end{aligned} \tag{74}$$

Is clear that $\lambda_2 = \lambda_2(A, B, C)$, $\lambda_2 \in \mathbb{Q}^+$, so by (74)

$$\frac{C^2}{2\lambda_2} - \frac{C^4}{4} = \left(A^4 - \frac{C^2}{2}\right)^4. \tag{75}$$

Hence in (75)

(I) If C is even, $C = 2k$ for some $k \in \mathbb{N}$, then

$$\frac{4k^2}{2\lambda_2} - \frac{16k^4}{4} = \left(A^4 - 2k^2\right)^2 \tag{76}$$

For $\lambda_2 = \frac{r}{t}$ with $G.C.D(r, t) = 1$, replacing in (76)

$$\frac{2k^2t}{r} - 4k^4 = \left(A^4 - 2k^2\right)^2 \tag{77}$$

and making $r = k^2$ we have

$$2t - 4k^4 = \left(A^4 - 2k^2\right)^2$$

the last equality is a contradiction to the fact that A is odd. By (77) if $r = 2k^2$, we get

$$t - 4k^4 = (A^4 - 2k^2)^2. \tag{78}$$

Also by the relation (74)

$$4k^2t = 2k^2(A^8 + B^8). \tag{79}$$

Hence by (78) and (79)

$$2t = (A^8 + B^8) = 8k^4 + 2(A^4 - 2k^2)^2. \tag{80}$$

Then by (80)

$$(B - A)(B + A)(B^2 + A^2) = 2(2k^2 - A^4). \tag{81}$$

As A and B are odd, the relation (81) is a contradiction.

(II) If C is odd, then A is even and B is odd or (A is odd and B is even).

Define

$$G(x, y, z) = A^4x + B^4y + C^2z \tag{82}$$

By the proof of the theorem 2.4 we have

$$A^4 = \frac{1}{\lambda_3}\cos\theta\sin\Phi, B^4 = \frac{1}{\lambda_3}\sin\theta\sin\Phi, C^2 = \frac{1}{\lambda_3}(\cos\Phi + 1). \tag{83}$$

$$C^2 = t,\ B^4 = \frac{t-r}{2},\ A^4 = \frac{t+r}{2} \tag{84}$$

and also

$$1 = \frac{4Nt}{M(3t^2+r^2)} \tag{85}$$

By (84) C is odd, we have that r and t are odd and $G.C.D(r,t) = 1$. From the relations (83) and (85)

$$A^8 + B^8 = \frac{1}{\lambda_3^2}\sin^2\Phi = \frac{(r^2+t^2)t^2}{8(3t^2+r^2)}, \tag{86}$$

where $\lambda_3 = \dfrac{M}{N} = \dfrac{4t}{3t^2+r^2}$, $\sin\Phi = \dfrac{k^2-1}{k^2+1}$, $k^2 = \dfrac{2}{\lambda^2+1}$, $\lambda \in \mathbb{Q}$, $\lambda = \dfrac{r}{t}$. The relations (86) is a contradiction because $A^8 + B^8 \in \mathbb{N}$ but $\dfrac{(r^2+t^2)t^2}{8(3t^2+r^2)} \notin \mathbb{N}$ since $G.C.D(r,t) = 1$ and the only common prime factor of r^2+t^2 and $3t^2+r^2$ is 2. □

3 Conclusions

When X is a Hilbert space and $f : X \to \mathbb{R}$ is a continuous linear functional, then $X = \ker f \oplus M$, where M is the subspace orthogonal to $\ker f$. Using this result with the choice of a suitable linear functional, we make links with the problems that were our object of study, this technique allowed us to solve and obtain important results in number theory and real analysis, these results are proved in a peculiar way.

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