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The Non-Trivial Zeros of The Riemann Zeta Function through Taylor Series Expansion and Incomplete Gamma Function

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Abstract The Riemann zeta (ζ) function $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ is valid for all complex number $s=x+iy : Re(s)>1$, for the line $x=1$. Euler-Riemann found that the function equals zero for all negative even integers: $-2,-4,-6,\cdots$ (commonly known as trivial zeros) has an infinite number of zeros in the critical strip of complex numbers between the lines $x=0$ and $x=1$. Moreover, it was well known to him that all non-trivial zeros are exhibiting symmetry with respect to the critical line $x=\frac{1}{2}$. As a result, Riemann conjectured that all of the non-trivial zeros are on the critical line, this hypothesis is known as the Riemann hypothesis. The Riemann zeta function plays a momentous part while analyzing the number theory and has applications in applied statistics, probability theory and Physics. The Riemann zeta function is closely related to one of the most challenging unsolved problems in mathematics (the Riemann hypothesis) which has been classified as the 8th of Hilbert's 23 problems. This function is useful in number theory for investigating the anomalous behavior of prime numbers. If this theory is proven to be correct, it means we will be able to know the sequential order of the prime numbers. Numerous approaches have been applied towards the solution of this problem, which includes both numerical and geometrical approaches, also the Taylor series of the Riemann zeta function, and the asymptotic properties of its coefficients. Despite the fact that there are around 10^{13}, non-trivial zeros on the critical line, we cannot assume that the Riemann Hypothesis (RH) is necessarily true unless a lucid proof is provided. Indeed, there are differing viewpoints not only on the Riemann Hypothesis's reliability, but also on certain basic conclusions see for example [16] in which the author justifies the location of non-trivial zero subject to the simultaneous occurrence of $\zeta(s)=\zeta(1-s)=0$, and omitting the impact of an indeterminate form $\infty.0$, that appears in Riemann's approach. In this study we also consider the simultaneous occurrence $\zeta(s)=\zeta(1-s)=0$ but we adopt an element-wise approach of the Taylor series by expanding n^{-x} for all $n=1,2,3,\cdots$ at the real parts of the non-trivial zeta zeros lying in the critical strip for $s=\alpha+iy$ is a non-trivial zero of $\zeta(s)$, we first expand each term n^{-x} at α then at $1-\alpha$. Then In this sequel, we evoke the simultaneous occurrence of the non-trivial zeta function zeros $\zeta(s)=\zeta(1-s)=0$, on the critical strip by the means of different representations of Zeta function. Consequently, proves that Riemann Hypothesis is likely to be true.

Keywords Riemann Zeta Function, Taylor Series Expansion, Incomplete Gamma Function, Meromorphic Functions

1 Introduction

The Riemann hypothesis (RH) is one of the millennium problem which is officially documented and briefly described on the website of Clay Mathematics Institute [1]. The German mathematician G.F.B Riemann (1826 – 1866) conjectured that "the sequential behavior of prime numbers is closely related to the function $\zeta(s)$" that is given by

$$\zeta(s)=\sum_{n=1}^{\infty}n^{-s}=\frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots. \tag{1}$$

The above functions is known as the Riemann zeta function, which is an absolutely convergent infinite series for every complex number $s=\alpha+iy$ satisfying $Re(s)>1$. Further he asserted that "all interesting solutions of the equation $\zeta(s)=0$ lie on a certain vertical straight line". Moreover Riemann numerically proved that the trivial zeros of the $\zeta(s)$ holds at the even negative integers $\{-2,-4,-6,\cdots\}$ but the non-trivial zeros exist in the set of complex numbers in which the $Re(s)=\frac{1}{2}$. In order to dive into the Riemann zeta function we refer readers and researchers to the official description of this problem by [1, 18] and the translated version of Riemann's 1859 manuscript by Alfred [7]. In addition a huge flood of papers prevails in the literature claiming the solution of RH, see for example [6, 9, 13, 17] and may more. After all the Clay Mathematics institute [1] displays the problem as still open to proof on their website which has been standing unsolved for a about 161 years.

The Riemann hypothesis evokes the existence and the location of zeros outside the region of convergence of this series and Euler product. Prior to further discussion with regards to such zeros, the one begins by obtaining a valid analytic continuation of the zeta function for all complex s which is guaranteed to be unique and of equivalent functional form over their domains due to the fact that $\zeta(s)$ is a meromorphic function. Initially, it is worth noting that the zeta function and the Dirichlet eta function satisfy the following relations, see for example [1, 3, 4, 5, 8, 10, 11, 12, 14] and [16].

$$\zeta(s)=\frac{1}{1-2^{1-s}}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}=\frac{1}{1^s}-\frac{1}{2^s}+\frac{1}{3^s}-\cdots \tag{2}$$

and

$$\zeta(1-s)=\frac{1}{1-2^{s}}\sum_{n=1}^{\infty}(-1)^{n+1}n^{s-1}=\frac{1}{1^{1-s}}-\frac{1}{2^{1-s}}+\frac{1}{3^{1-s}}-\cdots. \tag{3}$$

The two above alternative series in (2) and (3) extend the zeta function from $Re(s)>1$ to the larger domain $Re(s)>0$ because both right series converge in the same domain i.e. $Re(s)>0$. Moreover, it has been shown that $\zeta(s)$ is an entire function that is well defined for every s with positive real part except for the simple pole at $s=1$.

Particularly and among various functional equations that zeta function satisfies [2, 20] we consider the following functional equation in $0<Re(s)<1$, known as the critical strip. In the strip $0<Re(s)<1$, the zeta function satisfies the following functional expression

$$\zeta(s)=2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s). \tag{4}$$

2 Taylor Series Representation of Zeta Function

We consider an element wise approach to Taylor series representation of $\zeta(s)$ by expanding n^{-x} for every $n\in\mathbb{N}=\{1,2,3,\cdots\}$ at the real parts of the non-trivial zeta zeros lying in the critical strip; that is given that $s=\alpha+iy$ is a non-trivial zero of $\zeta(s)$, we first expand each term n^{-x} at α then at $1-\alpha$.

Let us recall that for $s=x+iy$, the zeta function is given by

$$\begin{aligned}\zeta(s)&=\left(\frac{1}{1-2^{1-s}}\right)\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s}\\&=\left(\frac{1}{1-2^{1-s}}\right)\left[\sum_{n=1}^{\infty}(-1)^{n-1}n^{-x}\cos(y\ln(n))-i\sum_{n=1}^{\infty}(-1)^{n-1}n^{-x}\sin(y\ln(n))\right]\end{aligned} \tag{5}$$

and

$$\begin{aligned}\zeta(1-s)&=\left(\frac{1}{1-2^{s}}\right)\sum_{n=1}^{\infty}(-1)^{n+1}n^{s-1}\\&=\left(\frac{1}{1-2^{s}}\right)\left[\sum_{n=1}^{\infty}(-1)^{n-1}n^{x-1}\cos(y\ln(n))+i\sum_{n=1}^{\infty}(-1)^{n-1}n^{x-1}\sin(y\ln(n))\right].\end{aligned} \tag{6}$$

Now, if $\zeta(s)=\zeta(1-s)=0$; then we have the following four possibilities of series

$$\left(\frac{1}{1-2^{1-s}}\right)\left[\sum_{n=1}^{\infty}(-1)^{n-1}n^{-x}\cos(y\ln(n))\right]=0, \tag{7}$$

$$\left(\frac{1}{1-2^{1-s}}\right)\left[\sum_{n=1}^{\infty}(-1)^{n-1}n^{-x}\sin(y\ln(n))\right]=0 \tag{8}$$

and

$$=\left(\frac{1}{1-2^{s}}\right)\left[\sum_{n=1}^{\infty}(-1)^{n-1}n^{x-1}\cos(y\ln(n))\right]=0, \tag{9}$$

$$=\left(\frac{1}{1-2^{s}}\right)\left[\sum_{n=1}^{\infty}(-1)^{n-1}n^{x-1}\sin(y\ln(n))\right]=0. \tag{10}$$

Since the factors $\frac{1}{1-2^{1-s}}$ and $\frac{1}{1-2^{s}}$, do not cause vanishing $\zeta(s)$ and $\zeta(1-s)$ respectively, and thus we consider only one among four of the above series defined by (7-10). Precisely, and without loss of generality; we consider only one series. For the sake of simplicity let $\zeta(x)$ refer to:

$$\zeta(x)=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-x}\cos(y\ln(n)). \tag{11}$$

Theorem 2.1. *If $s=\alpha+iy$, is a non-trivial zero of Zeta function, then $\zeta(x)$ of the form (11) can be written as*

$$\zeta(x)=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\cos(y\ln(n))\left[1+\sum_{n=1}^{\infty}\frac{(-1)^{k}(x-\alpha)^{k}(\ln(n))^{k}}{k!}\right]$$

consequently; the following results hold:

i. $Re(s)=\alpha=\frac{1}{2}$ for finitely many terms of the Taylor expansion of n^{-x} at α.

ii. $Re(s)=\alpha>\frac{1}{2}$ for infinitely many terms of the Taylor expansion of n^{-x} at α.

Proof. Let $s=\alpha+iy$ and $1-s=1-\alpha-iy$, be non-trivial zeta zeros in the critical strip: $0<\alpha<1$. For $n=1,2,3,\cdots$ and for $k=0,1,2,\cdots$ we deduce the Taylor series expansion of n^{-x} at α, likewise in the following manner

$$\begin{aligned} n^{-x}&=n^{-\alpha}\left(1-(x-\alpha)\ln(n)+\frac{1}{2!}(x-\alpha)^{2}(\ln(n))^{2}+\cdots+\frac{1}{k!}(x-\alpha)^{k}(\ln(n))^{k}\right)\\ &=n^{-\alpha}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^{k}(x-\alpha)^{k}(\ln(n))^{k}}{k!}\right). \end{aligned} \tag{12}$$

Thus the series in (11) can be expressed as a double series, which is possible by multiplying the both sides of (12) by $(-1)^{n-1}n^{-x}\cos(y\ln(n$ and then by the summation we obtain the following expression

$$\begin{aligned} \sum_{n=1}^{\infty}(-1)^{n-1}n^{-x}\cos(y\ln(n))=&\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\cos(y\ln(n))\\ &\times\left(1+\sum_{n=1}^{\infty}\frac{(-1)^{k}(x-\alpha)^{k}(\ln(n))^{k}}{k!}\right). \end{aligned} \tag{13}$$

Equivalently from (13), $\zeta(x)$ can be written as

$$\zeta(x)=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+n-1}\frac{(x-\alpha)^{k}(\ln(n))^{k}}{k!}n^{-\alpha}\cos(y\ln(n)), \tag{14}$$

this further yields

$$\zeta(x)=\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\cos(y\ln(n))\left(1+\sum_{n=1}^{\infty}\frac{(-1)^{k}(x-\alpha)^{k}(\ln(n))^{k}}{k!}\right). \tag{15}$$

Upon replacing $x=1-\alpha$ in (15) we receive,

$$\zeta(1-\alpha)=\zeta(\alpha)+\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\cos(y\ln(n))\left(\sum_{n=1}^{\infty}\frac{(-1)^{k}(1-2\alpha)^{k}(\ln(n))^{k}}{k!}\right). \tag{16}$$

Since $\zeta(1-\alpha)=\zeta(\alpha)=0$, then above expression reduces to

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\cos(y\ln(n))\left(\sum_{n=1}^{\infty}\frac{(-1)^k(1-2\alpha)^k(\ln(n))^k}{k!}\right)=0 \tag{17}$$

It is obvious that (17) vanishes at $\alpha=\frac{1}{2}$.
Moreover, for $k=\{0,1,2,\cdots,m\}$ the partial sum of equation (17), yields

$$\sum_{n=1}^{m}\frac{(-1)^k(1-2\alpha)^k(\ln(n))^k}{k!}=\frac{n^{2\alpha-1}\Gamma(m+1,(2\alpha-1)\ln(n))}{m!}-1. \tag{18}$$

Therefore, by putting (18) in (17) we have

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\cos(y\ln(n))\left(\frac{n^{2\alpha-1}\Gamma(m+1,(2\alpha-1)\ln(n))}{m!}-1\right)=0, \tag{19}$$

and by some simple calculation equation (19) leads to

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{\alpha-1}\cos(y\ln(n))\left(\frac{\Gamma(m+1,(2\alpha-1)\ln(n))}{m!}\right)=0. \tag{20}$$

Now there are two possibilities to solve the above Taylor series:

First, for the finitely n many terms, we can write

$$\frac{\Gamma(m+1,(2\alpha-1)\ln(n))}{m!}=1, \quad \text{for} \quad \alpha=\frac{1}{2}. \tag{21}$$

In easy words if we are assuming the accuracy of the Taylor series then $\alpha=\frac{1}{2}$, this is equivalent to RH (Riemann Hypothesis) see [1, 3, 4, 5, 8, 10, 11, 12, 14].

The second solution exists but for infinitely many terms is given by

$$\sum_{n=1}^{\infty}\frac{(-1)^k(1-2\alpha)^k(\ln(n))^k}{k!}=-n^{2\alpha-1}(n^{1-2\alpha}-1)=n^{2\alpha-1}(1-n^{1-2\alpha}) \quad \text{for} \quad \alpha>\frac{1}{2}. \tag{22}$$

Substituting (22) in (17), we obtain

$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{\alpha-1}\cos(y\ln(n))(1-n^{1-2\alpha})=0$$
$$\sum_{n=1}^{\infty}(-1)^{n-1}n^{\alpha-1}\cos(y\ln(n))-\sum_{n=1}^{\infty}(-1)^{n-1}n^{-\alpha}\cos(y\ln(n))=0.$$

The last expression leads us towards the following statement that

$$\zeta(1-\alpha)-\zeta(\alpha)=0.$$

But in return $\zeta(1-\alpha)=\zeta(\alpha)=0$ means that RH remains unproved. □

In view of Theorem 2.1, we deduce a corollary stated as under:

Corollary 2.1. *If $s=\alpha+iy$ is a non-trivial Zeta zero; then either of the following solutions exist.*

i. $Re(s)=\alpha=\frac{1}{2}$ for finitely many terms of the Taylor expansion of n^{-x} at $1-\alpha$,

ii. $Re(s)=\alpha<\frac{1}{2}$ for infinitely many terms of the Taylor expansion of n^{-x} at $1-\alpha$.

Proof. The proof simply follows the method of Theorem 2.1 in two steps. First on expanding n^{-x} at $(1-\alpha)$ a Taylor series is obtained, consequently substituting α in the Taylor series completes the proof of this corollary. □

3 The Modified Integral Representation of Zeta function

In order to prove the modified integral representation of Zeta function, we need to state the following integrals of Gamma and Zeta functions.

The integral form of Gamma function is given by

$$\Gamma(s) = \int_0^\infty e^{-\tau}\tau^{s-1}d\tau, \tag{23}$$

and the integral form of Zeta function is given by

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}dt. \tag{24}$$

We now present the modified integral representation of Zeta function in the following theorem.

Theorem 3.1. *If $s^* = \alpha + i\beta$ is a non-trivial zero of the Zeta function then for every $s = x + iy : x > \alpha$, the following results hold*

1. $\Gamma(s)\zeta(s) = (x-\alpha)^s \int_0^\infty \frac{t^{s-1}}{e^{(x-\alpha)t} - 1}dt.$

2. *If* $\int_0^\infty \frac{t^{-s^*}}{e^{(1-2\alpha)t} - 1}dt \neq 0$, *then* $\alpha = \frac{1}{2}$.

Proof. By hypothesis $s^* = \alpha + i\beta$ is a non-trivial zero of the Zeta function then by considering $\tau \to (x-\alpha)nt$, in (23) we receive

$$\Gamma(s) = n^s(x-\alpha)^s \int_0^\infty e^{-(x-\alpha)nt}t^{s-1}dt. \tag{25}$$

Now to prove the first part of the theorem for every $s = x + iy : x > \alpha$, multiplying both sides by $\Gamma(s)\zeta(s)$ and upon simple calculations we get the result asserted in the first part of our Theorem 3.1.

The second part of the this theorem is derived from first part just by replacing, $s = 1 - s^*$ in (23)

$$\Gamma(1-s^*)\zeta(1-s^*) = (1-2\alpha)^{1-s^*} \int_0^\infty \frac{t^{-s^*}}{e^{(1-2\alpha)t} - 1}dt$$

Since $\Gamma(1-s^*)\zeta(1-s^*) = 0$ therefore,

$$(1-2\alpha)^{1-s^*} \int_0^\infty \frac{t^{-s^*}}{e^{(1-2\alpha)t} - 1}dt = 0$$

but $\int_0^\infty \frac{t^{-s^*}}{e^{(1-2\alpha)t}-1}dt \neq 0$ therefore, we can write

$$(1-2\alpha)^{1-s^*} = 0.$$

The above result holds for $\alpha = \frac{1}{2}$. This statement completes the second part of our Theorem 3.1. □

In view of Theorem 3.1, we deduce the following corollary.

Corollary 3.1. *Let $s^* = \alpha + \beta i$ be a non-trivial Zeta zero, for every $s = x + iy$ and $x < \alpha$, then*

1. $\Gamma(s)\zeta(s) = (\alpha - x)^s \int_0^\infty \frac{t^{s-1}}{e^{(\alpha-x)t} - 1}dt.$

2. *If* $\int_0^\infty \frac{t^{-s^*}}{e^{(2\alpha-1)t} - 1}dt \neq 0$, *then* $\alpha = \frac{1}{2}$.

Proof. The proof simply follows the method of Theorem 3.1 only by putting $\tau \to (\alpha - x)nt$ in equation (23) that leads to produce the results stated in Corollary 3.1. □

4 Geometrical Approach

Geometric function theory (GFT) gets more important when we are able to relate the analytic properties of conformal maps to it geometric properties of their graphs. Geometrical approach is in fact, one step closer towards the application of some analytic function, see for example the graphical examples studied by [15] and [19]. In order to have a closer look on the Riemann Zeta function we study the graphical behavior of the function n^{-x} and $n^{-x} + n^{x-1}$ for all $x \in (0, 1)$ and $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$.

Analysing the function for every complex number $s = x + iy$ we observed the following numerical information from Figure 1.

$$tan\theta_n = \frac{n^{-x}}{x - \frac{1}{2}}, \quad x \neq \frac{1}{2} \tag{26}$$

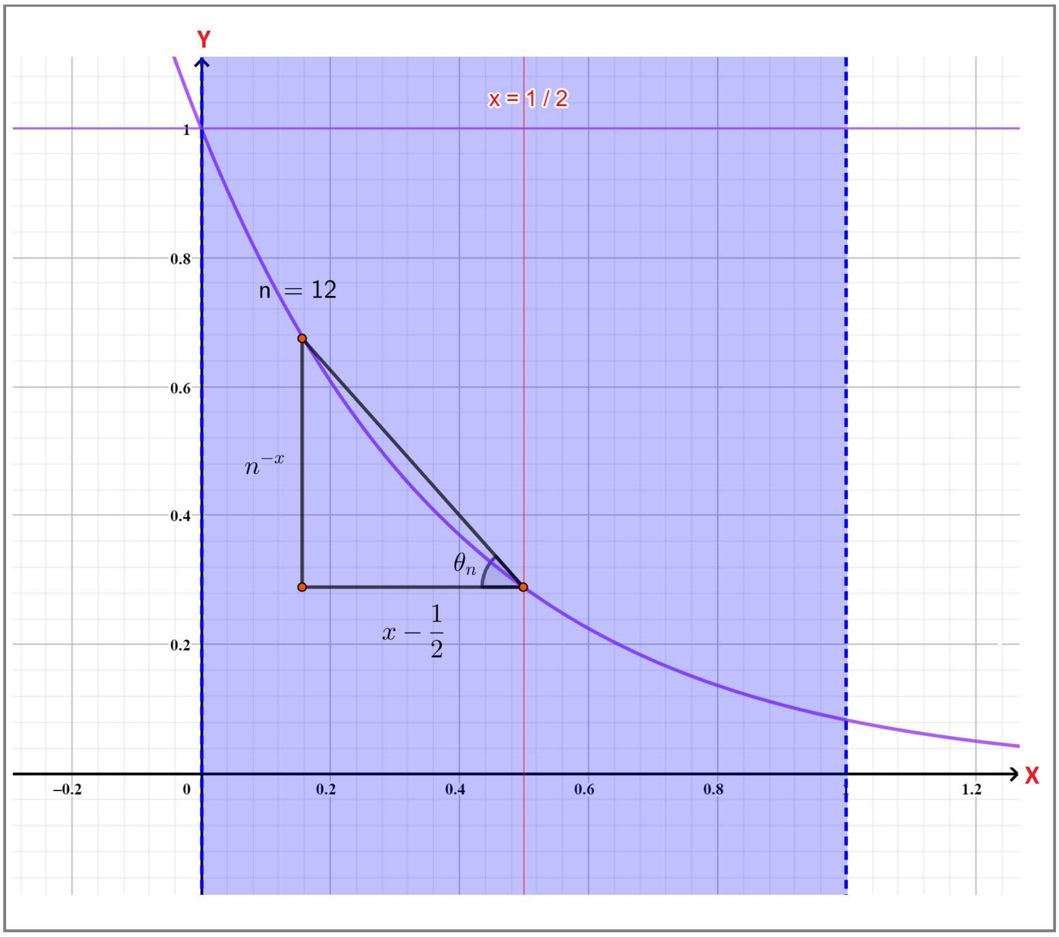

Figure 1. Graph of n^{-x}

The numerical information we obtain from Figure 2 is given by

$$tan\phi_n = \frac{n^{-x} + n^{x-1}}{x - \frac{1}{2}}, \quad x \neq \frac{1}{2} \tag{27}$$

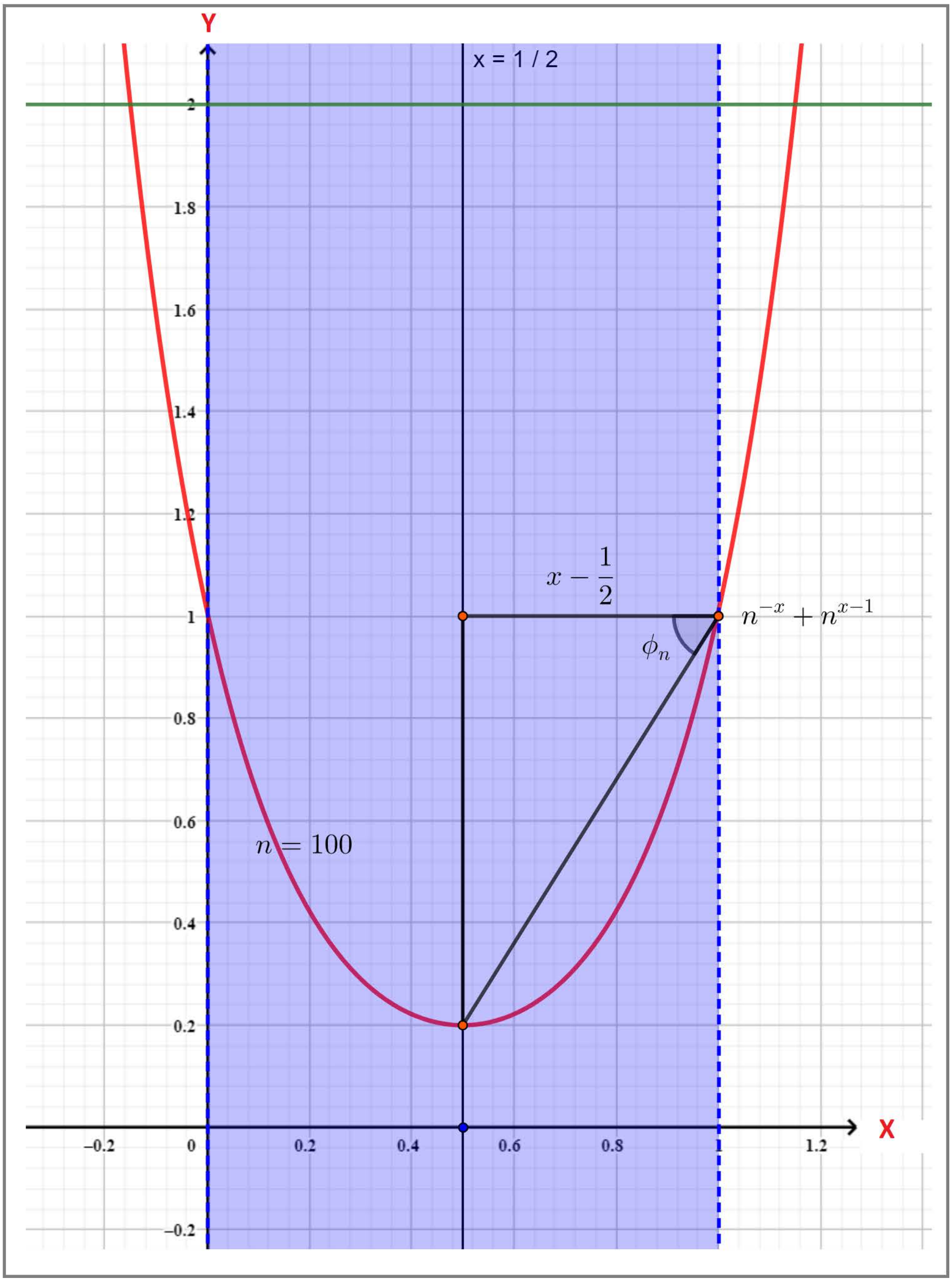

Figure 2. Graph of $n^{-x} + n^{x-1}$

In view of graphical analysis we came across a result that is presented in the form of theorem as under:

Theorem 4.1. *Let $\zeta(s)$ be defined by (2), then*

1. $$\zeta(s) = \begin{cases} \left| x - \frac{1}{2} \right| f(tan\theta_n, s), & x \neq \frac{1}{2}, \\ \zeta\left(\frac{1}{2} + it\right), & x = \frac{1}{2}. \end{cases}$$

2. *If* $\zeta(s) = 0$, *then* $\mathrm{Re}(s) = \frac{1}{2}$.

Proof. Let $s = x + iy, 0 < x < 1, n \in \mathbb{N} = \{1, 2, 3, \cdots\}$ then from (26) we have

$$n^{-x} = \begin{cases} \left|x - \frac{1}{2}\right| tan\theta_n, & x \neq \frac{1}{2} \\ n^{-x}, & x = \frac{1}{2}. \end{cases} \tag{28}$$

Recall the expression stated in Section 2 in equation 5.

$$\begin{aligned} \zeta(s) &= \left(\frac{1}{1 - 2^{1-s}}\right) \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} \\ &= \left(\frac{1}{1 - 2^{1-s}}\right) \left[\sum_{n=1}^{\infty} (-1)^{n-1} n^{-x} \cos(y \ln(n)) - i \sum_{n=1}^{\infty} (-1)^{n-1} n^{-x} \sin(y \ln(n))\right]. \end{aligned}$$

Making the use of equation (28) in above expression, the first result asserted in Theorem 4.1 is readily proved.

Now to prove the second part of the theorem using the hypothesis that if $\zeta(s) = 0$, this implies $\left|x - \frac{1}{2}\right| f(tan\theta_n, s) = 0$. Moreover, since $\zeta(s)$ is well defined that allow us to write

$$\zeta\left(\frac{1}{2} + it\right) = \lim_{x \to \frac{1}{2}} \left|x - \frac{1}{2}\right| f(tan\theta_n, s) = 0 \Rightarrow x = \frac{1}{2}. \tag{29}$$

This completes the second part of our Theorem 4.1. □

In view of Theorem 4.1, the following corollary is derived.

Corollary 4.1. *Let $\zeta(s)$ and $\zeta(1 - s)$ be respectively defined by (2) and (3), then*

1. $$\zeta(s) + \zeta(1 - s) = \begin{cases} \left|x - \frac{1}{2}\right| f(tan\theta_n, s), & x \neq \frac{1}{2} \\ \zeta\left(\frac{1}{2} + it\right) + \zeta\left(\frac{1}{2} - it\right), & x = \frac{1}{2}. \end{cases}$$

2. *If* $\zeta(s) = \zeta(1 - s) = 0$, *then* $\mathrm{Re}(s) = \frac{1}{2}$.

Proof. Manipulating equation (29), the rest proof of this corollary readily follows the method used in Theorem 4.1. □

5 Conclusion

In this article, we provided three different representations of zeta function in the critical strip by the means of Taylor series, the Incomplete Gamma function and a Geometric approach. Given that $\zeta(s) = \zeta(1 - s) = 0$, implies most probably but not certainly $Re(s) = 1/2$ in other words most probably Riemann Hypothesis holds.

Conflict of Interest

The authors declare that there is no conflict of interests.

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