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On The Unconditional Run Length Distribution and Percentiles for The s^2-chart When The In-control Process Parameter Is Estimated

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Abstract It is well known that the median is a better measure of location in skewed distributions. Run-length (RL) distribution is a skewed distribution, hence, median run-length measures chart performance better than the average run length. Some authors have advocated examination of the entire percentiles of the RL distribution in assessing chart performance. Such works already exist for Shewhart $\bar{x}-$chart, CUSUM chart, CUSUM and EWMA charts, Hotelling's chi-square, and the two simple Shewhart multivariate non-parametric charts. Similar work on S^2-chart for one- and two-sided lacks in the literature. This work stands in the gap. Therefore, a detailed and comparative study of the one-sided upper and the two-sided S^2-control charts for some m reference samples at fixed sample size and false alarm rate will be considered here using the information from the unconditional RL cdf curve and its percentiles (mainly median). The order of the RL cdf curves of the one-sided upper S^2-chart is independent of the state of the process unlike in the two-sided one. The one-sided upper chart outperformed the two-sided one both in the in-control and in detecting positive shifts. The two-sided S^2-chart is more sensitive in detecting incremental shifts than to decremental shifts.

Keywords Average Run Length (ARL), False Alarm Rate (FAR), Median Run Length (MRL), Stochastic Ordering, Pooled Variance

1 INTRODUCTION

Control charts are used to determine if a process is in a state of statistical control or not and due to their importance and usefulness in solving quality problems, they have continued to receive much attention in the literature, [20]. The most known of them are the Shewhart-type control charts for monitoring process mean and dispersion. Reference [5] stated that the R, S, and S^2 control charts are mostly used in practice to monitor the process variability. Reference [12] noted that the control limits of these charts like every other control chart depend on the in-control value(s) of the parameter(s) of the process. Reference [12] added that in a more typical situation, these parameter values are not known, instead, their estimates are obtained from the in-control Phase I samples and used in their place. Not until an adjustment is made for parameter estimation and practitioner-to-practitioner

variability, the resulting control limits obtained with the estimates are known as the unadjusted control limits and the use of them deteriorates the charts performance.
The use of the three-sigma control limits for these charts based on the normal approximation is a common practice, however, [18] stated that the results from such limits are unsatisfactory. Reference [26] noted the same thing and added that statisticians advocate the use of probability limits, instead of three-sigma control limits in the design of the charts. Reference [26], as well, recommended that S^2- chart be used in place of S-chart as the exact evaluation of the average run length (ARL) and the derivation of other results for the S-chart are less tractable compared to the S^2-chart. Reference [7], as well, discouraged the use of the three-sigma estimated limits and the use of the control limits tabulated in most textbooks and some commercial software too as most of them are based on such limits and recommended the use of the probability limits again. In line with these arguments, the probability limits and the S^2-chart will be used to achieve the aims of this paper.
It is well known that the use of estimated values affects the charts' performance as the false-alarm becomes different from the nominal value (the value it would have if the process parameters are known), see [15,21]. Note that when the process parameter, that is, the variance (σ_0^2) is known, the number of the samples until an alarm of an S^2-chart, that is the RL, follows the well known geometric distribution with the parameter being the false alarm rate (FAR) or the probability of signal (PS) depending on whether the process is in-control or not. In line with this, the average of this distribution (the ARL) is the reciprocal of the FAR, or equivalently, the reciprocal of the PS, [19]. However, when the σ_0^2 is estimated and used to calculate the control limits, the ARL is conditioned on the estimated variance. Therefore, to obtain the unconditional ARL, the conditioning-unconditioning approach in [1] is used in this work to obtain the exact distributions of all quantities that will be considered. Reference [12] noted that the variation and degradation of the charts' performance as a result of parameter estimation has been highlighted by several authors. Not until lately, the performance of a control chart is generally measured in terms of the unconditional RL and its associated characteristics, such as the unconditional ARL or unconditional FAR. For the review of works carried out in this area up to 2014, see, [5,15,21]. One of the reports, specifically by [5] noted that when the process is in control, the variability of the conditional ARL is known to be very large, which means that the value may be much different, compared to the in-control ARL when the parameter is not estimated. This is a minus for the conditional perspective and aside from the unconditional perspective being the one in common use, this is also one of the reasons for considering the conventional unconditional RL distribution and its percentiles in this work. The works by [7,9,12,14] are based on the conditional perspective.
But it is well known that the run length distribution is right-skewed and in a right-skewed distribution, the mean is far larger than the measure of the center, see [13], therefore, the median is a better measure of the central tendency. In line with this remark, [1] stated that because of the skewed nature of the run-length distribution, one might prefer the median (or some other quantiles) than the ARL as a measure of typical chart performance. Reference [10] stated that the median run length (MRL) is the 50th percentage point of the run-length distribution which denotes the median number of samples drawn on a control chart until it issues an alarm which as well means that fifty percents of the run lengths lie below the observed point. Reference [3], as well, stated that since the run-length random variable takes on only positive integer values, the distributional shape is significantly right-skewed. Reference [3] went further to say that some studies have advocated the use of other more representative measures other than the ARL for the assessment of charts performance, noting that one of such measures is the percentile of the run-length distribution. The percentiles provide wider information about the distribution and hence on the performance of a chart not provided by the mean or the expected value and its choice for assessing charts' performance is becoming a common one now. To this regard, see [2,3,16,22,24] on the run length distribution and percentiles of the Shewhart $\bar{x}$-chart, [25] for similar work on CUSUM control chart under changes in variances, [4] on the Hotelling's chi-square and the two simple Shewhart multivariate nonparametric control charts based on sign and singed-rank tests for known and estimated parameters and [8] on the adjusted Shewhart, CUSUM and EWMA control charts for sustained shifts in the process mean.
Surprisingly, works on the examination of the exact unconditional RL distribution and its percentiles of the S^2- control chart are currently unavailable in the literature as the work of [5] despite considering only the two-sided S^2-chart also based the performance criterion on the ARL. Motivated by this, this paper provides the examination of the exact unconditional RL distribution and its percentiles for one-sided upper and two-sided S^2- control charts for different Phase I samples (m) and fixed values of sample size (n) and nominal false alarm rate (α). More light was shed on the derivation of the estimators and equations used in the work unlike in [5]. Note that [12] stated that it is generally of interest to detect and signal increase when monitoring a process dispersion as that indicates process degradation. Therefore, the one-sided upper limit S^2- control chart suffices. However, detecting decreases in variabilities may be important in some situations, especially when S^2-chart is used together with the $\bar{x}$-control chart (to monitor the mean). This is because the limits of $\bar{x}$-control chart depend on the process variance estimation and a decrease in the process variance will affect the performance of the $\bar{x}$- control chart which may cause it to take longer in detecting a shift in the mean if the variance is not re-estimated and the $\bar{x}$-chart control limits recalculated, [14]. This is one of the reasons behind considering the two-sided S^2-chart in this work.
This work will be a welcome addition in the literature for research as the literature reveals that works on a comparative study of the performance of these two (the one-sided upper and the two-sided S^2-control charts) are scarce except in [14] where the comparison is based on the conditional ARL and standard deviation of the ARL when the process being monitored with the charts is only in a state of statistical control.

The remainder of this paper is structured as follows. We present the review of the conditional probability limits and signals of the one-sided upper and the two-sided S^2-control charts in Section 2. In Section 3, we present the derivations of the unconditional cdfs and the percentile distributions of the one-sided upper and two-sided S^2-control charts respectively. The results of the evaluations of the given and derived expressions will be presented in Section 4 and finally, the conclusion of the work is presented in Section 5.

2 The conditional probability limits and of the signal of one-sided upper and the two-sided S^2-control charts

2.1 The one-sided Upper S^2-chart

Let $m > 1$ reference independent random samples each of size n from Phase I analysis with which to estimate the unknown Phase I variance $\left(\sigma_0^2\right)$ be assumed available. The observations are, as well, assumed to follow a normal distribution with mean u_0 and variance σ_0^2, both unknown. Let the Phase II subgroup samples each of the same size as those in Phase I be equally assumed to follow a normal distribution with unknown mean (μ_1) and variance $\left(\sigma^2\right)$. Define the standard deviation ratio and the error factor of estimate as $\gamma = \sigma/\sigma_0$ and $W = \hat{\sigma}_0/\sigma_0$ respectively. The process is in control when $\sigma = \sigma_0$ and $\gamma = 1$ and out-of-control when $\sigma > \sigma_0$ and $\gamma > 1$. Let delta (δ) be the size of the shift in γ that makes it greater than one. With these assumptions and definitions, the α estimated upper probability limit of the one-sided upper S^2-control chart is given by $\widehat{UCL}_{one} = \frac{\chi^2_{n-1,\alpha}}{n-1}\hat{\sigma}_0^2$ and the probability of signal of the chart is given by

$$P(S^2 > \widehat{UCL}_{one}) = P\left\{S^2 > \frac{\hat{\sigma}_0^2\chi^2_{n-1}, \alpha}{n-1}\right\}$$

$$= P\left\{\frac{(n-1)S^2}{\sigma^2} > \frac{W^2}{\gamma^2}\chi^2_{n-1}, \alpha\right\}$$

$$= 1 - F_{\chi^2_{n-1}}(\frac{W^2}{\gamma^2}\chi^2_{n-1}, \alpha), \tag{1}$$

where, α is the nominal FAR, $\chi^2_{v,p}$ is $(1-p)$ quantile of the distribution of a chi-square random variable with v degree of freedom (df) and $(n-1)S^2/\sigma^2$ follows a chi-square distribution with $n-1$ df; see, [9,19]. Therefore, at an in-control state, $\gamma = 1$ and the in-control probability of signal becomes

$$1 - F_{\chi^2_{n-1}}(W^2\chi^2_{n-1}, \alpha). \tag{2}$$

It is important to note that when the parameter is known (not estimated), W reduces to one, that is, $W = 1$, therefore, under the in-control state also where $\gamma = 1$, the probability of signal simplifies out to FAR, that is,

$$FAR = 1 - F_{\chi^2_{n-1}}(\chi^2_{n-1}, \alpha).$$

We consider $\hat{\sigma}_0^2 = S_p^2$, the pooled variance of Phase I samples in this work as recommended by [17] to be the best estimator of the Phase I variance in terms of minimum mean square error (MSE). It has been noted also that pooling the sample standard deviation which its square gives the pooled variance provides lower values of the MSE than averaging them, [6]. It is well known that when the process is in control, $Y = m(n-1)S_p^2/\sigma_0^2$ follows a chi-square distribution with $m(n-1)$ df. Therefore, substituting W^2 with S_p^2/σ_0^2 in equations (1) and (2), the conditional out-of-control and the in-control probabilities of signal become

$$1 - F_{\chi^2_{n-1}}\left\{\frac{Y}{\gamma^2 m(n-1)}\chi^2_{n-1}, \alpha\right\}$$

and

$$1 - F_{\chi^2_{n-1}}\left\{\frac{Y}{m(n-1)}\chi^2_{n-1}, \alpha\right\}$$

respectively, where $\gamma > 1$.

2.2 The two-sided S^2- control chart

By using the same assumptions, definitions and procedures considered above for the one-sided S^2-chart, the α estimated probability limits of the two-sided S^2-control chart are given by $\widehat{UCL}_{two} = \hat{\sigma}_0^2\chi^2_{n-1}, \frac{\alpha}{2}/(n-1)$ and $\widehat{LCL}_{two} = \hat{\sigma}_0^2\chi^2_{n-1}, (1-\frac{\alpha}{2})/(n-1)$, while the conditional probability of signal is given by

$$P(S_i^2 > \widehat{UCL}) or P(S_i^2 < \widehat{LCL}) = 1 - \left\{F_{\chi^2_{n-1}}\left(\frac{W^2}{\gamma^2}\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{W^2}{\gamma^2}\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}, \tag{3}$$

where W^2, γ^2, α, $\chi^2_{n-1}$and $F_{\chi^2_{n-1}}$ are as already defined. Therefore, at an in-control state, $\gamma = 1$ and the in-control probability of signal becomes

$$1 - \left\{F_{\chi^2_{n-1}}\left(W^2\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(W^2\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\} \tag{4}$$

When the parameter, the Phase I variance is assumed known (not estimated) and the process is in control, $W = 1$, therefore, under $\gamma = 1$, the probability of signal simplifies to the nominal FAR, that is,

$$FAR = 1 - \left\{F_{\chi^2_{n-1}}\left(\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}$$

By subsituting S_p^2/σ_0^2 for W^2 as before in equations (3) and (4), the conditional out-of-control and the in-control probabilities of signal become

$$1 - \left\{F_{\chi^2_{n-1}}\left(\frac{Y}{\gamma^2 m(n-1)}\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{\gamma^2 m(n-1)}\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}$$

and

$$1 - \left\{F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{Y}{m(n-1)}\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}$$

respectively, where $\gamma \neq 1$ and let δ be the shift size that makes γ to either be greater or smaller than one.

3 The exact (unconditional) probability of alarm, average run length and cumulative distribution function when σ_0^2 is estimated by S_p^2.

3.1 The exact (unconditional) Probabilities of alarm and the average run length

It has been said in this paper that the unconditional probabilities of alarm, average run lengths, and the cumulative distribution functions will be derived and used in our computations. To obtain these quantities unconditionally, the conditionng-unconditioning approach over the distribution of $\hat{\sigma}_0^2$ was applied which results in integrating them over the range of the chi-square random variable Y. Therefore, the unconditional out-of-control and in-control probabilities of alarm for the one-sided upper S^2-control chart are given by

$$\int_0^\infty \left[1 - F_{\chi^2_{n-1}}\left\{\frac{y}{\gamma^2 m(n-1)}\chi^2_{n-1}, \alpha\right\}\right] f_{\chi^2_{m(n-1)}}(y)dy,$$

and

$$\int_0^\infty \left[1 - F_{\chi^2_{n-1}}\left\{\frac{y}{m(n-1)}\chi^2_{n-1}, \alpha\right\}\right] f_{\chi^2_{m(n-1)}}(y)dy,$$

where $\gamma > 1$. Therefore, the out-of-control ARL is defined as

$$ARL_\delta = \int_0^\infty \left[1 - F_{\chi^2_{n-1}}\left\{\frac{\sigma_0^2}{\sigma^2}\frac{y}{m(n-1)}\chi^2_{n-1}, \alpha\right\}\right]^{-1} f_{\chi^2_{m(n-1)}}(y)dy \tag{5}$$

and the in-control ARL is defined as

$$ARL_0 = \int_0^\infty \left[1 - F_{\chi^2_{n-1}}\left\{\frac{y}{m(n-1)}\chi^2_{n-1}, \alpha\right\}\right]^{-1} f_{\chi^2_{m(n-1)}}(y)dy \tag{6}$$

where $\gamma > 1$ as already stated and $\gamma^{-1} = \frac{\sigma_0}{\sigma}$.
For the two-sided S^2-control chart, we have the in-control and out-of-control probabilities of alarm and the in-control and out-of-control ARL defined by

$$\int_0^\infty \left[1 - \left\{F_{\chi^2_{n-1}}\left(\frac{y}{m(n-1)}\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{y}{m(n-1)}\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}\right] f_{\chi^2_{m(n-1)}}(y)dy$$

$$\int_0^\infty \left[1 - \left\{F_{\chi^2_{n-1}}\left(\frac{y}{\gamma^2 m(n-1)}\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{y}{\gamma^2 m(n-1)}\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}\right] f_{\chi^2_{m(n-1)}}(y)dy$$

$$ARL_0 = \int_0^\infty \left[1 - \left\{F_{\chi^2_{n-1}}\left(\frac{y}{m(n-1)}\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{y}{m(n-1)}\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}\right]^{-1} f_{\chi^2_{m(n-1)}}(y)dy \tag{7}$$

and

$$ARL_\delta = \int_0^\infty \left[1 - \left\{F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\frac{y}{m(n-1)}\chi^2_{n-1}, 1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\frac{y}{m(n-1)}\chi^2_{n-1}, \frac{\alpha}{2}\right)\right\}\right]^{-1} f_{\chi^2_{m(n-1)}}(y)dy \tag{8}$$

accordingly, see [23]. By considering different sizes of Phase I samples (m), the in-control and out-of-control ARLs for the one-sided upper S^2-control chart will be evaluated in simulations from equations 3 and 4 respectively. The same thing will be done in equations 7 and 8 for the in-control and out-of-control states for the two-sided S^2 chart. Note that we used Monte Carlo integration involving 10^6 simulations to evaluate the ARLs in (3), (4) (7), and (8), and the statistical software, R core (2019) was used to execute it. The results are presented in Table 1 with that of the in-control and out-of-control ARLs for the standard known S^2-chart, that is when $W = 1$ for comparison.

3.2 The exact (unconditional) cumulative distribution functions of the run lengths of the one and two-sided S^2 based control charts

Recall that given $W = \hat{\sigma_0}/\sigma_0$, the random variable N, the run length, has a geometric distribution with the probability of success (signal) given by

$$P\left(S^2 > \widehat{UCL}\right) = P\left[\frac{(n-1)S_i^2}{\sigma^2} > \frac{W^2}{\gamma^2}\chi^2_{n-1}, \alpha\right].$$

Now, by using the conditional-unconditional approach in [1] for $\bar{x}$-control chart with estimated process mean where the run length distribution is defined as

$$P(N \geq j) = E_{\bar{X}} P(N \mid \bar{X}) = E_{\bar{X}}\left[\beta\left(\delta, n; \bar{X}\right)\right]^{j-1},$$

and $E_{\bar{X}}$is the expected value of $P\left(N|\bar{X}\right)$over the distribution of $\bar{X}$, δ is the size of the shift from in-control state, N is the number of the Phase II samples sampled until an alarm and $\beta\left(\delta, n, \bar{X}\right)$ is the probability of no alarm by the chart given $\bar{X}$ and the shift size δ. Then, the exact unconditional RL distribution for the one-sided upper S^2-chart is given as

$$P(N \geq j) = \int_0^\infty \left[F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\frac{y}{m(n-1)}\chi^2_{n-1}, \alpha\right)\right]^{j-1} f_{\chi^2_{m(n-1)}}(y)dy,$$

and the RL cumulative distribution function for the one-sided upper S^2-chart is given as

$$P(N \leq a) = 1 - \int_0^\infty \left[F_{\chi^2_{n-1}}\left(\frac{\sigma_0^2}{\sigma^2}\frac{y}{m(n-1)}\chi^2_{n-1}, \alpha\right)\right]^{a} f_{\chi^2_{m(n-1)}}(y)dy \tag{9}$$

For the two-sided control chart, the exact RL distribution is

$$P(N \geq j) = \int_0^\infty \left[F_{\chi^2_{n-1}} \left\{ \frac{\sigma_0^2}{\sigma^2} \frac{y}{m(n-1)} \chi^2_{n-1}, 1-\alpha/2 \right\} - F_{\chi^2_{n-1}} \left\{ \frac{\sigma_0^2}{\sigma^2} \frac{y}{m(n-1)} \chi^2_{n-1}, \alpha/2 \right\} \right]^{j-1} f_{\chi^2_{m(n-1)}}(y) dy$$

while the RL cumulative distribution function is

$$P(N \leq a) = 1 - \int_0^\infty \left[F_{\chi^2_{n-1}} \left\{ \frac{\sigma_0^2}{\sigma^2} \frac{y}{m(n-1)} \chi^2_{n-1}, 1-\alpha/2 \right\} - F_{\chi^2_{n-1}} \left\{ \frac{\sigma_0^2}{\sigma^2} \frac{y}{m(n-1)} \chi^2_{n-1}, \alpha/2 \right\} \right]^{a} f_{\chi^2_{m(n-1)}}(y) dy \quad (10)$$

where, j, a =1,2,... is the number of Phase II samples at which the probability that the first signal is observed at or before j is obtained, $\frac{\sigma_0}{\sigma} = 1$ implies in-control state for both charts, $\frac{\sigma_0^2}{\sigma^2} < 1$, which defines only an increase in Phase II standard deviation implies an out-of-control for the one-sided upper chart and $\frac{\sigma_0}{\sigma} \neq 1$ which defines both increase or decrease in Phase II standard deviation implies an out-of-control for the two-sided chart. As well, we used Monte Carlo integration involving 10^6simulations to evaluate the unconditional cdfs in (9) and (10) for m = 30, 100, and 500. The evaluation will be done in two folds, first, when the process is in the state of control and then, when the process is in an out-of-control state. The results are plotted in Figures 1, 2, 3, and 4 respectively and the cdf curve for the standard known case is included in each figure for comparison.

3.3 The RL percentiles of the charts

One can study the statistical properties of the (S^2)-control charts including the various performance characteristics by studying the behavior of the RL cdfs in equations 9 and 10 through the shape of the curves for different values of m under an in-control and out-of-control states. Another instance is to calculate the $100pth$ RL percentiles using the idea that it is the smallest positive integer a so that the cdf at a is at least equal to p, [3]. The in-control and out-of-control percentiles can be evaluated by setting, $\sigma_0^2/\sigma^2 = 1$ and $\sigma_0^2/\sigma^2 < 1$ respectively in equations 9 and $\sigma_0^2/\sigma^2 = 1$ and $\sigma_0^2/\sigma^2 \neq 1$ in 10 are as already explained. The performance of the charts is studied here through these means. Based on this unconditional RL percentile, equations 9 and 10 can be redefined as

$$1 - \int_0^\infty \left[F_{\chi^2_{n-1}} \left(\frac{\sigma_0^2}{\sigma^2} \frac{y}{m(n-1)} \chi^2_{n-1}, \alpha \right) \right]^{a} f_{\chi^2_{m(n-1)}}(y) dy \geq p \quad (11)$$

and

$$1 - \int_0^\infty \left[F_{\chi^2_{n-1}} \left(\frac{\sigma_0^2}{\sigma^2} \frac{y}{m(n-1)} \chi^2_{n-1}, 1-\alpha/2 \right) - F_{\chi^2_{n-1}} \left(\frac{\sigma_0^2}{\sigma^2} \frac{y}{m(n-1)} \chi^2_{n-1}, \alpha/2 \right) \right]^{a} f_{\chi^2_{m(n-1)}}(y) dy \geq p \quad (12)$$

Equations 11 and 12 will be evaluated at different values of m = 20, 30, 50, and 100 and at different values of $p = 5^{th}$, 10^{th}, 20^{th}, 25^{th}, 30^{th}, 40^{th}, 50^{th}, 60^{th}, 70^{th}, 75^{th}, 80^{th}, 90^{th} and 95^{th} percentiles at an in-control state $(\sigma^2 = \sigma_0^2)$ and an out-of-control states of $(\sigma^2 > \sigma_0^2)$ by δ =0.02 and 0.05 for the one-sided chart and $(\sigma^2 > \sigma_0^2)$ and $(\sigma^2 < \sigma_0^2)$ by δ =0.02 and 0.05 for the two-sided chart respectively. Note that we fix $n = 5$ and $\alpha = 0.0027$ throughout the study. The results are presented in Table 1 alongside the values of the standard known case (when the parameter, σ_0^2 is not estimated but known) under the same in-control and out-of-control shift sizes. Note that when parameter is not estimated, the RL percentiles of the two charts can be defined to be the smallest positive integer a such that the cdf of the geometric distribution is at least equal to p. That is

$$1 - (1-\beta)^a \geq p$$

and

$$a = \frac{In(1-p)}{In(1-\beta)}$$

where β is the nominal false alarm rate. As the nominal false alarm rate is 0.0027, setting p= 0.5, the result of the evaluation is 257 which is the median RL for the in-control state.

4 Results and discussions

This paper compared the performance behaviors of the one-sided upper and two-sided S^2-control charts in terms of the unconditional RL cdf and its percentiles when the pooled variance, S_p^2 is used to estimate the Phase I in-control variance. In the light of this, the unconditional RL cdf curves in Figures 1 and 2 for one-sided upper and Figures 3 and 4 for the two-sided S^2-control charts were plotted when the process is in control and when a shift of 0.05 from the in-control state is introduced for m=30, 100 and 500.

The unconditional RL cdf curves for the one- and two-sided S^2-control charts as shown in the figures show a lot of information. First, the curves of the twoS^2-based charts show that unless m is large, say 500 and above, the unconditional RL cdfs of the charts are much different from that of the geometric distribution. Considering the in-control state, the unconditional RL cdf curves for the one-sided upper S^2 for m = 30, 100, and 500 as displayed in Figure 1 lie entirely above that of the geometric distribution up to 0.61th percentiles in approximation. But, beyond that point, they lie below it. This means that the unconditional RL cdfs of the chart are not stochastically ordered among themselves. This behavior is equally seen when compared with that of the geometric distribution. A random variable is said to be stochastically ordered, either smaller (larger) than the second when the cdf curve of that of the random variable lies entirely above (below) the cdf curve of the second random variable, [11]. Hinging on this, the cdf curves in Figure 1 for this chart is not stochastically ordered among themselves and with that of the geometric distribution and as a result, the chart for m=30, 100, and 500 perform differently among themselves and equally with that of the geometric distribution before and after the intersection point. At the upper part of the curves, the chart performance for the various m when compared to that of the standard known case (geometric distribution) seem to deteriorate more than at the lower part of it. This is because, there, the unconditional RL distributions widen away from that of the geometric distribution more than they do at the lower part. However, this remark is completely different from what is witnessed in Figure 3 for the two-sided chart under the same in-control state. Here, the unconditional RL cdf curves for m = 30, 100, and 500 lie entirely above that of the geometric distribution at all places making them stochastically ordered relative to that of the geometric distribution.
Considering an out-of-control state with a shifting size of 0.05, the unconditional cdf curves for the various Phase I samples of the one-sided upper S^2-chart in Figure 2 showed a similar characteristic with that at an in-control state in Figure 1 by intersecting at a point, however, with the two-sided chart under this state as shown in Figure 4, the chart lost its stochastic ordering and the RL cdf curves for the various m intersect among themselves and with that of the geometric distribution at 0.80th percentile approximately. With these, one can easily come to terms that the unconditional RL cdfs of the two-sided chart under different states are not consistent in terms of ordering relative to that of the geometric distribution. From the foregoing, it can be concluded that once the unconditional cdf curves of Phase II control charts for various Phase I samples used to obtain estimates for the process parameters are not stochastically ordered relative to the standard (known parameters case), difference performance is witnessed towards the tails of the distributions when compared to the standard known case.
The summary of the information contained in Table 1 is as presented here. First, attention is drawn to the huge differences between the values of the ARL and MRL for the one-sided upper S^2-chart. For instance, the in-control ARLs and MRLs for m= 20, 30, 50 and 100 are 788.90, 599.67, 487.35, 423.09 and 228, 233, 243, 249 respectively and the difference between the respective ARL and MRL reveals the magnitudes of the differences. The difference is seen in both the one and the two-sided chart both in their in-control and out-of-control states, however, it is more pronounced with the one-sided upper S^2-chart. This unbearable difference was witnessed because the RL distribution is right-skewed and the distribution of the Phase 1 estimator is chi-square which is highly a right-skewed distribution too. This is in line with works already in literature where it has been emphasized that the mean is larger than the median in right-skewed distributions, [13] and as such, the latter is preferred as a better measure of average.
Note that the standard in-control values of the ARL and MRL for the known parameter case are approximately 370 and 257 respectively. From Table 1, it is seen that as the value of m increases, the values of ARL and MRL approach their respective standard values, this is because the larger the Phase I samples used for the estimation, the more the estimates of the process parameters approach the true parameter values and the better the performance of the chart towards the case of the known parameter, see [12]. However, the approach is more in MRL than in ARL. For instance, when m = 100, the respective values of the ARL and MRL are 423.09 and 249 and 249 is nearer to 257 than 423.09 is to 370. This also shows the reason for coming to terms on why MRL is and should be preferred to the ARL. Comparing the one-sided upper S^2-chart with the two-sided one both in the in-control and in detecting the 0.02 and 0.05 increases in the variance, the former outperforms the latter. For instance, for 0.02 shift, the MRL values of the one-sided upper S^2-chart for m= 20, 30, 50, and 100 are 228, 233, 243, and 249. These values are a lot nearer to the 257 standard value when compared to 202, 216, 288, and 242 respective MRL values for the two-sided S^2-chart. A similar result is obtained with the 0.05 shift, see Table 1. By implication, the one-sided upper chart samples more observations when the process is in-control before a false alarm is raised than the two-sided limits one. Also, the one-sided upper S^2-chart raised more out-of-control alarms to the 0.02 and 0.05 increases in the process variance when compared to the two-sided limits one. This is seen in the differences between the in-control and the 0.02 and 0.05 out-of-control MRL values of the charts as those for the one-sided upper S^2 -chart are larger than those in the two-sided one.
By a way of looking into the performances of the two-sided S^2-chart more, a 0.02, 0.05 increases and a 0.02, 0.05 decreases from the in-control state are introduced in the process and the performances of the chart evaluated. From the results as presented

in Table 1, the chart is more sensitive in detecting positive (incremental) shifts than negative (decremental) shifts. This is evident when 177, 188, 201, and 211 which are the values of the MRL of the chart when a 0.02 positive shift is introduced are compared to 221, 239, 254, and 269, which are the respective values of the MRL of the chart with a 0.02 negative shift for m = 20, 30, 50 and 100. A similar conclusion is reached for the 0.05 incremental or decremental shifts, see Table 1. Again, it is normal with any good control chart that when the shift size is increased, more out-of-control alarms are expected which is shown by the reduced ARLs and MRLs of the chart. Against this norm, it is discovered that in the two-sided S^2-control chart, when a decremental shift in increased in size, rather than the ARLs and MRLs decreasing, they have kept to increase thereby not appropriately communicating the increased out-of-control situation in the process to the operator. However, this is not true with the two-sided S^2-control chart when incremental shifts are introduced as the values of the ARLs and MRLs in Table 1 are all smaller given the 0.05 shift size compared to the 0.02 shift size. This is actually because the two-sided S^2-control chart is ARL-biased as the value becomes larger when the variance is decreased than when the process is in-control and as such, the chart detects increasing variances than the decreasing ones, [27].

It is important to note that when the parameter is estimated, these two control charts based on S^{2} fall short in their performance both in the in-control and out-of-control states when compared to the performance of the same charts with known parameter case. This statement is proved when for instance, the in-control and out-of-control ARLs and MRLs of the charts for m = 20, 30, 50, and 100 are compared to those from the standard known case presented in the last seven rows of Table 1. There, one sees that the ARL and MRL for the charts with known parameter case (limit) are 370.34 and 257 as expected. However, these values are never attained by the same charts with estimated limit(s) for the values of m considered. Secondly, both the one-sided and two-sided charts based on S^{2} with known parameter case raised more out-of-control alarms to the 0.02 and 0.05 shifts in the process variance (smaller ARL and MRL) compared to when they are operating with estimated limit(s) at the various values of m considered.

5 Conclusion

The nature of the unconditional cdf curves of the one-sided upper S^2-chart for the various Phase I samples considered is not affected by whether the process is in-control or out-of-control, that is, whether the process is in-control or out-of-control, the unconditional RL cdf curves do intersect at a point. In addition to not being stochastically ordered among themselves, they are not also when compared with that of the geometric distribution for the two states of the process. The chart performs differently before and after the intersection point when compared with that of the geometric distribution. Before the point, the MRLs are closer to that of the geometric distribution than they are after the point. However, the nature of the cdf curves of the two-sided S^2-chart for the same Phase I samples considered is dependent on the state of the process. At an in-control state, the cdf curves are stochastically ordered among themselves and also with that of the geometric distribution but not when the process shifts from the in-control state. At large values of Phase I samples say, 500 and above, the cdf curves from the two charts approach that of the geometric distribution almost completely. The one-sided upper S^2-chart showed better performance both in the in-control performance and in detecting positive shifts of 0.02 and 0.05 considered in this work than the two-sided limits chart. The two-sided limits S^2- control chart is more sensitive in detecting incremental shifts than it does to decremental shifts.

REFERENCES

[1] Chakraborti, S. "Run length, average run length, and false alarm rate of Shewhart $\bar{x}$-chart: Exact derivations conditioning". Communications in Statistics, Simulation and, Computation, vol.29 no.1, pp. 61-81,2000. DOI: 10.1080/03610910008813602

[2] Chakraborti, S. "Parameter Estimation and Design Considerations in Prospective Applications of the $\bar{x}$-chart." Journal of Applied Statistics, vol.33 no.4., pp.439-459, 2006. DOI: 10.1080/02664760500163516

[3] Chakraborti, S. "Run length distribution and percentiles: The Shewhart Chart with Unknown Parameters". Quality Engineering, vol.19 no.2, pp. 119-127, 2007. DOI: 10/1080/08982110701276653

[4] Boone J. M. and Chakraborti, S. "Two simple shewhart-tpye multivariate control charts". Journal of Applied stochastic models in business and industry, vol. 28, pp.130-140, 2011. DOI: 10.1002/asmb.900

[5] Chen, G. "The run length distributions of R, S and S^2control charts when σ is estimated." The Canadian Journal of Statistics, vol. 26 no. 2, pp. 311-322,1997. DOI: 10.2307/3315513

[6] Derman, C. and Ross, S. "An Improved Estimator of σ in Quality Control". Probability in the Engineering and Informational Science, Vol. 9, No.3. pp. 411-415. 1995. DOI: 10.1017/so269964800004654

[7] Diko, M.D., Goedhart, R., Chakraborti, S., Does, R.J.M.M. and Epprecht, E.K. "Phase II control charts for monitoring dispersion when parameters are estimated". Quality Engineering, vol. 29 no. 4, pp. 605-622, 2017. DOI: 10.1080/08982112.2017.1288915

[8] Diko, D.M Goedhart, R. and Does, J.M.M. "A head-to-head comparsion of the out-of-control performance of the control charts adjusted for parameter estimatiom". Journal of Quality Engineering, vol.32 no 4, pp.1-10, 2017. DOI:10.1080/08982112.2019.1666140.

[9] Epprecht, E.K., Loureiro, L.D., Chakraborti, S. "Effect of the amount of phase I data on the phase II performance of S2 and S control charts". Journal of Quality Technology, Vol. 47 no 2, pp. 139–155,2015. DOI: 10.1080/00224065.2015.11918121

[10] Gao, H. and Khoo. M.B.C: "A study on the median run length performance of the run sum S control chart". International Journal of Mechanical Engineering and Robotics Research. vol. 8 no. 6, pp. 885-890, 2019. DOI: 10.18178/ijmerr.8.6.885-890

[11] Gibbons and Chakraborti (2003): Nonparametric Statistical Inference, 4th ed. New York, Marcel Dekker.

[12] Goedhart, R., Schoonhoven, M. and Does, R.J.M.M. "Gurranteed in-control performance for the Shewhart X and $\bar{X}$control charts". Journal of Quality Technology, vol. 49 no. 2, pp. 155-171, 2017.DOI: 10.1080/00224065.2017.11917986

[13] Gupta, S. C. (2013): Fundamentals of Statistics, Himalaya Publishing House, Mumbai

[14] Jardim, F. S. , Sarmiento, M.G.C., Epprecht, E.K. and Chakraborti, S. (2018). "Design comparison between one and two-sided S^2control charts with estimated parameter". In A.Leires, C.A. Gonzalez, I. de Brito Junior, S. Villa and H.T.Y. Yoshiaki (Eds). Operations Management for Social Good. POMS International Conference in Rio. (P.753-760). Doi: 10/1007/978-3-030-23816-2_74.

[15] Jensen, W.A, Jones-Farmer, L.A., and Woodall, W.H. "Effects of parameter estimation on control chart properties: a literature review". Journal of Quality Technology, vol. 38 no. 4, pp.349-364, 2006.DOI: 10.1080/00224065.2006.1198623

[16] Khoo, M. B. C: "Performance measures for the Shewhart $\bar{x}$-control chart". Quality Engineering, Vol. 16, No. 4, pp.585–590, 2005. DOI: 10.1081/QEN-120038020

[17] Mahmoud, M.A., Henderson, G.R., Epprecht, E.K., Woodall, W.H.: "Estimating the standard deviation in quality control applications". Journal of Quality Technology, Vol. 42, No.4, pp.348–357, 2010. DOI: 10.1080/00224065.2010.11917832

[18] Maravelakis, P.E., Panaretos, J., Psarakis, S (2002): "Effect of estimation of the process parameters on the control limits of the univariate control charts for process dispersion". Communcation in Statistics, Simulation and Computation. Vol. 31, No. 3, pp. 443–461, 2002. DOI: 10.1081/SAC-120003851

[19] Montgomery, D.C (2009) : Introduction to Statistical Quality Control, 7th edn. Wiley, Hoboken, NJ.

[20] Park, C. and Wang, M. : "A study of the $\bar{x}$ and S control charts with unequal sample sizes". www.mdpi.com/journal/Mathematics. Vol. 8, pp. 1-28, 2020. Doi: 10.3390/math8050698.

[21] Psarakis, S., Vyniou, A.K., Castagliola, P.: "Some recent developments on the effects of parameter estimation on control charts". Quality and Reliabity Engineering International. Vol.30, No.8, pp.1113-1129, 2014. DOI: 10.1002/qre.1556

[22] Radson, D., Boyd, A. H. "Graphical representation of run length distributions". Quality Engineering, Vol. 17, No. 2, pp.301–308, 2005. DOI: 10.1081/QEN-200056484

[23] Sarmiento, M. G. C., Jardim, F. S., Chakrabort, S. and Epprecht, E. K." Design of variance control charts with estimated parameters: A head to head comparison between two perspectives". Journal of Quality Technology, 2020.DOI: 10.1080/00224065.2020.1834892.

[24] Shmueli, G., Cohen, A. "Run-length distribution for control charts with runs and scans rules". Communications in Statistics, Theory, and Methods, Vol.32, No.2, pp.475–495,2003. DOI: 10.1081/STA-120018196

[25] Shu, L., Huang, W., Su, T. and Tsui, K. L. "Computation of the run-length percentiles of CUSUM control charts under changes in variances". Journal of Statistical Computation and Simulation, Vol.83,No.7,pp.1-14, 2012. DOI: 10.1080/00949655.2012.656643

[26] Zhang, L., Bebbington, M. S., Lai, C. D. and Govindaraju, K. "On Statistical Design of the S2 Control Chart, Communications in Statistics–Theory and Methods", Vol.34, No.1, pp.229–244, 2005.DOI: 10.1081/STA-200045817

[27] Zhang,L. and Govindaraju,K."On Probability limits for Phase II S control chart". International Journal of Statistics. Vol.65, No.3, pp.305-318, 2007. URL: ftp://metron.sta.uniroma1.it/RePEc/articoli/2007-3-3.pdf

Table 1. Average Run Lengths and the Percentiles of the one-sided upper and two-sided S^2-control chart, When the process variance is unknown for a Number of Phase I samples (m) and Shift Size δ at n=5 and α=0.0027

			Percentiles of the charts run length distributions												
m	shifts(δ)	ARL	5	10	20	25	30	40	50(MRL)	60	70	75	80	90	95
One-sided upper S^2-control chart with estimated limit															
20	0.0	788.90	11	24	55	74	95	150	228	334	524	625	849	1695	2998
	0.02	557.38	9	19	42	57	73	115	176	255	388	487	620	1186	2064
	0.05	348.57	4	8	18	24	31	48	70	102	151	180	223	416	673
30	0.0	599.67	13	28	61	82	104	162	233	338	476	608	768	1398	2232
	0.02	434.64	10	22	47	63	81	124	176	252	370	464	576	557	1671
	0.05	281.52	5	10	20	26	33	50	72	99	142	167	208	355	537
50	0.0	487.35	15	32	69	91	114	174	243	340	474	580	694	1154	1765
	0.02	359.69	12	25	53	70	86	130	182	253	355	424	523	857	1261
	0.05	237.50	5	10	22	28	35	51	73	99	137	162	196	310	444
100	0.0	423.09	17	35	75	96	121	180	249	340	464	546	647	1012	1406
	0.02	316.09	13	27	58	74	93	135	188	255	347	406	481	748	1052
	0.05	212.76	5	10	23	30	37	54	74	101	137	158	185	279	386
Two-sided S^2-control chart with estimated limits (Here, 0.02 and 0.05 shifts are increase in variance)															
20	0.0	325.27	13	27	59	77	97	142	202	276	376	437	520	783	1051
	0.02	297.22	11	23	51	66	85	124	177	244	336	394	472	724	986
	0.05	250.89	9	19	42	55	69	99	148	195	278	332	411	619	868
30	0.0	337.63	15	31	66	85	106	155	216	289	393	458	546	799	1066
	0.02	305.71	13	26	57	74	93	136	188	256	349	412	488	734	993
	0.05	255.98	11	21	46	59	74	109	154	205	272	333	395	609	849
50	0.0	348.65	16	34	71	93	115	167	228	305	410	476	557	818	1088
	0.02	312.45	14	29	62	81	101	146	201	269	364	426	494	737	989
	0.05	254.90	11	23	49	63	79	115	160	214	294	342	403	607	828
100	0.0	358.47	18	36	77	99	123	177	242	322	426	492	573	831	1095
	0.02	317.92	15	31	67	86	107	154	211	282	373	435	506	741	979
	0.05	256.45	12	25	53	68	84	122	167	224	298	345	409	593	803
Two-sided S^2-control chart with estimated limits (Here, 0.02 and 0.05 shifts are decreases in variance)															
20	0.0	325.27	13	27	59	77	97	142	202	276	376	437	520	783	1051
	0.02	349.74	15	30	66	86	110	160	221	301	408	476	560	834	1110
	0.05	377.60	17	36	76	100	124	180	248	333	446	514	607	886	1172
30	0.0	337.63	15	31	66	85	106	155	216	289	393	458	546	799	1066
	0.02	365.39	17	34	73	95	121	172	239	318	426	498	584	860	1150
	0.05	393.30	19	39	83	108	135	194	265	352	468	541	635	925	1213
50	0.0	348.65	16	34	71	93	115	167	228	305	410	476	557	818	1088
	0.02	379.99	18	38	80	104	129	186	254	339	448	521	607	885	1170
	0.05	411.52	21	43	90	116	144	206	282	372	489	567	664	957	1250
100	0.0	358.47	18	36	77	99	123	177	242	322	426	492	573	831	1095
	0.02	393.46	20	40	85	111	137	197	269	355	469	542	631	910	1193
	0.05	429.01	22	45	95	122	152	217	296	390	514	592	689	987	1312
Standard Known (S^2-control chart with known limit with + and - meaning increase and decrease respectively)															
	0.0	370.34	19	39	83	106	132	189	257	339	446	513	596	852	1109
One sided	0.02	279.83	14	30	62	80	100	143	194	256	337	387	450	644	837
	0.05	189.71	6	12	24	32	39	56	76	100	131	151	175	251	326
Two-sided	+0.02	323.66	17	34	72	93	115	165	225	296	390	448	521	745	968
	+0.05	253.53	13	27	57	73	90	129	176	232	305	351	408	583	758
	-0.02	410.56	21	44	92	118	146	210	285	376	494	569	660	945	1229
	-0.05	448.99	23	48	101	129	160	229	311	411	540	622	722	1033	1344

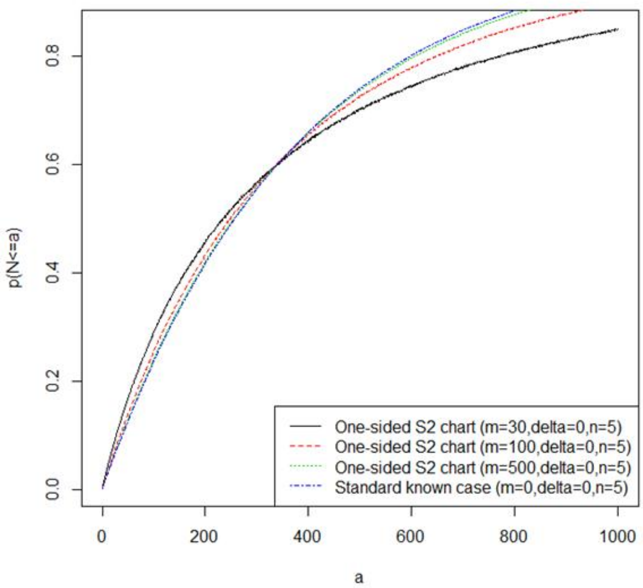

Figure 1. In-control (σ=σ_0) run length c.d.f curves of the one-sided upper S^2-control chart for m =30, 100 and 500, n =5 in each and α =0.0027

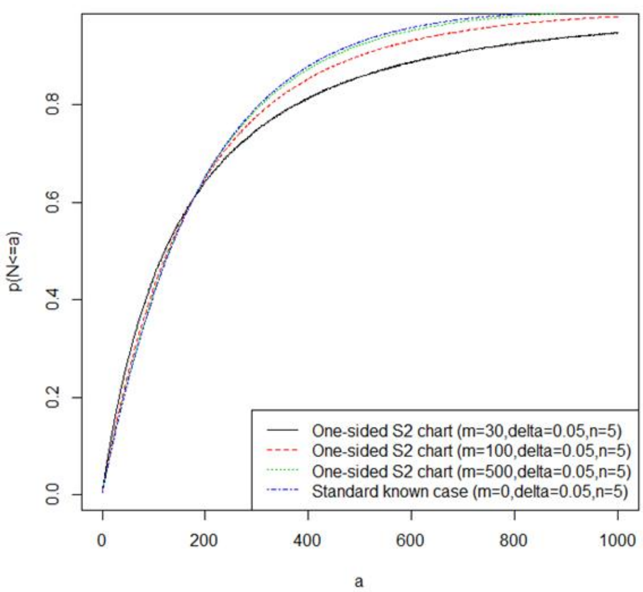

Figure 2. An out-of-control (σ >σ_0) run length c.d.f curves of the one-sided upper S^2-control chart for m = 30, 100 and 500, n = 5 in each and α =0.0027

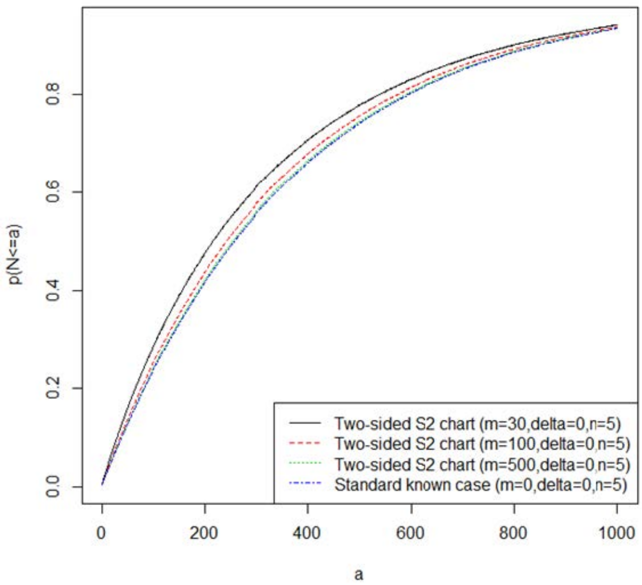

Figure 3. In-control (σ=σ_0) run length c.d.f curves of the two-sided S^2-control chart for m =30, 100 and 500, n = 5 in each and α =0.0027

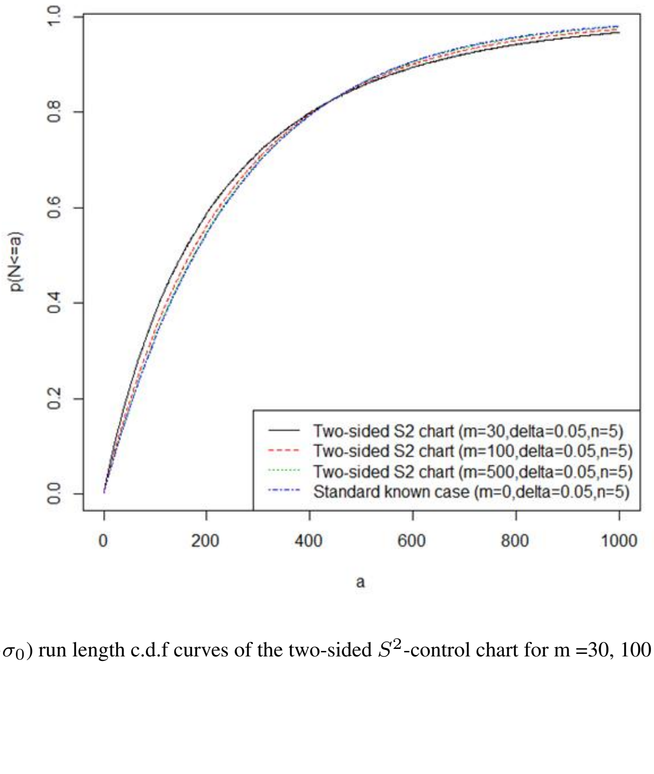

Figure 4. An out-of-control (σ >σ_0) run length c.d.f curves of the two-sided S^2-control chart for m =30, 100 and 500, n = 5 in each and α =0.0027