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Some Inequalities for n-times Differentiable Strongly Convex Functions

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Abstract The theory of inequality is in a process of continuous development and has become a quite effective and powerful tool in various branches of mathematics to solve many problems. Convex functions are closely related to the theory of inequality, and many important inequalities are the results of the applications of convex functions. Recently, the results obtained for convex functions have been tried to be extended for strongly convex functions. In our previous studies, the perturbed trapezoid inequality obtained for convex functions has been extended to the functions that can be differentiated n-times. This study deals with some general identities introduced for n-times differentiable strongly convex functions. Besides, new inequalities related to general perturbed trapezoid inequality are constructed. These inequalities are obtained for the classes of functions which n^{th} derivatives of absolute values of the mentioned functions are strongly convex. It is seen that new classes of strongly convex functions turn into those obtained for convex functions under certain conditions. Considering the upper bounds obtained for strongly convex functions, it is concluded that it is better than those obtained for convex functions.

Keywords Convex Functions, Strongly Convex Functions, Perturbed Trapezoid Inequalities

1. Introduction

Convex functions are essential in many areas of mathematics, such as optimization theory, mathematical analysis, function theory, functional analysis, mathematical economics, etc. Convex functions were first suggested by Jensen centuries ago. Since then, many extensions and generalizations have been made in the theory of inequalities for convex functions. Examples of these are strongly convex [1], logarithmically convex [2], tgs-convex [3], s-convex [4], h-convex [5], etc.

Strongly convex functions form the basis of this study. Firstly, Polyak [1] has suggested strongly convex functions. These functions used to minimize a function are of great importance in optimization and approximation theory, and mathematical economics [6]. Some features of strongly convex functions are like well-known results for convex functions. Some generalizations and extensions are suggested for strongly convex functions as well as convex functions from past to present. Merentes and Nikodem [7] suggested a few further properties of strongly convex functions. Adamek [8] introduced F-strongly convex functions for non-negative function F defined on a real vector space. Moradi et al. [9] obtained new inequalities involving Jensen type inequalities for strongly convex function. Bakula and Nikodem [10] presented similar of the converse Jensen inequality for strongly convex and strongly midconvex functions. Song et al. [11] introduced some new integral inequalities of Jensen's type for the classes of F-strongly convex function. Recently, many studies have focused on strongly generalized convex functions. Awan et al. [12] presented the notion of strongly generalized convex functions called strongly η-convex function. Mishra and Sharma [13] suggested the notion of strongly generalized convex functions of higher order.

In this study, the perturbed trapezoid inequalities are considered for n times differentiable strongly convex functions. The new classes of strongly convex functions turn into those obtained for convex functions under certain conditions.

Definition 1.1 [14]: A function $\psi : I \to \mathbb{R}$ is convex on I if the inequality

$$\psi\left(tk+(1-t)l\right) \le t\psi(k)+(1-t)\psi(l) \tag{1}$$

is held for all $k,l \in I$ and $t \in [0,1]$. It is said that ψ is concave if $(-\psi)$ is convex. For numerical integration, trapezoid inequality is introduced as

$$\left|\int_k^l \psi(x)\,dx - \frac{1}{2}(l-k)\left(\psi(k)+\psi(l)\right)\right| \le \frac{1}{12}M_2(l-k)^3 \tag{2}$$

where $\psi : [k,l] \to \mathbb{R}$ is assumed to be twice differentiable on (k,l) with the second derivative bounded on (k,l) by $M_2 = \sup_{x\in(k,l)}\left|\psi''(x)\right| < +\infty$.

Definition 1.2 [1]: A function $\psi : [k,l] \to \mathbb{R}_0$ is said to be strongly convex with modulus $c \ge 0$, if

$$\begin{aligned}&\psi\left(\lambda x+(1-\lambda)y\right)\\ &\le \lambda\psi(x)+(1-\lambda)\psi(y)-c\lambda(1-\lambda)(x-y)^2\end{aligned} \tag{3}$$

is held for all $x,y \in [k,l]$ and $\lambda \in [0,1]$.

Lemma 1.1 [15]: Let $\psi : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be n times differentiable mapping on I°, $k,l \in I^\circ$ with $k<l$ where n is even number. If $\psi^{(n)} \in L[k,l]$, then equality in the following holds

$$\begin{aligned}&\frac{1}{l-k}\int_k^l \psi(x)\,dx - \frac{\psi(k)+\psi(l)}{2} + \cdots \\
&-\frac{(l-k)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}.\left[\psi^{(n-4)}(k)+\psi^{(n-4)}(l)\right]\\
&+\frac{(l-k)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}.\left[\psi^{(n-3)}(l)-\psi^{(n-3)}(k)\right]\\
&-\frac{(l-k)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}.\left[\psi^{(n-2)}(k)+\psi^{(n-2)}(l)\right]\\
&+\frac{(l-k)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}.\left[\psi^{(n-1)}(l)-\psi^{(n-1)}(k)\right]\\
&=\frac{(l-k)^n}{2.n!.a_n}\int_0^1\left(a_nt^n+\cdots+a_1t+a_0\right)\left[\psi^{(n)}\left(kt+(1-t)l\right)+\psi^{(n)}\left(lt+(1-t)k\right)\right]dt\end{aligned} \tag{4}$$

The following notations and conventions are used throughout this study. Let us consider as $I=[0,\infty)\subset\mathbb{R}=(-\infty,+\infty)$, $k,l \in I$ with $0<k<l$ and $\psi^{(n)} \in L[k,l]$. In the next section, some results related to the perturbed trapezoid inequality are introduced.

2. Main Results

Theorem 2.1: Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be n times differentiable mapping on I°, $k,l \in I^\circ$, $k<l$, with modulus c, c_1, c_2 where n is even number. If $\left|\psi^{(n)}\right|$ is strongly convex on $[k,l]$, then the inequality in the following holds:

$$\left|\frac{1}{l-k}\int_k^l \psi(x)\,dx-\frac{\psi(k)+\psi(l)}{2}+\cdots\right. \tag{5}$$
$$-\frac{(l-k)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}.\left[\psi^{(n-4)}(k)+\psi^{(n-4)}(l)\right]$$
$$+\frac{(l-k)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}.\left[\psi^{(n-3)}(l)-\psi^{(n-3)}(k)\right]$$
$$\left.-\frac{(l-k)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}.\left[\psi^{(n-2)}(k)+\psi^{(n-2)}(l)\right]+\frac{(l-k)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}.\left[\psi^{(n-1)}(l)-\psi^{(n-1)}(k)\right]\right|$$
$$\leq\frac{(l-k)^n}{2n!.|a_n|}.\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]\left[\psi^{(n)}(k)+\psi^{(n)}(l)\right]-\frac{c(l-k)^{n+2}}{2n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]$$

Proof 2.1: From Lemma 1.1 and $\left|\psi^{(n)}\right|$ is a strongly convex function, it is concluded that

$$\left|\frac{1}{l-k}\int_k^l \psi(x)\,dx-\frac{\psi(k)+\psi(l)}{2}+\cdots\right.$$
$$-\frac{(l-k)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}.\left[\psi^{(n-4)}(k)+\psi^{(n-4)}(l)\right]$$
$$+\frac{(l-k)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}.\left[\psi^{(n-3)}(l)-\psi^{(n-3)}(k)\right]$$
$$\left.-\frac{(l-k)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}.\left[\psi^{(n-2)}(k)+\psi^{(n-2)}(l)\right]+\frac{(l-k)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}.\left[\psi^{(n-1)}(l)-\psi^{(n-1)}(k)\right]\right|$$
$$=\left|\frac{(l-k)^n}{2.n!.a_n}.\left\{\int_0^1\left(a_nt^n+\cdots+a_1t+a_0\right)\left[\psi^{(n)}(tk+(1-t)l)+\psi^{(n)}(tl+(1-t)k)\right]dt\right\}\right|$$
$$\leq\frac{(l-k)^n}{2.n!.|a_n|}.\left\{\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left[\left|\psi^{(n)}(tk+(1-t)l)\right|+\left|\psi^{(n)}(tl+(1-t)k)\right|\right]dt\right\}$$
$$\leq\frac{(l-k)^n}{2.n!.|a_n|}.\left\{\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left[t\left|\psi^{(n)}(k)\right|+(1-t)\left|\psi^{(n)}(l)\right|-c_1t(1-t)(k-l)^2+t\left|\psi^{(n)}(l)\right|\right.\right.$$
$$\left.\left.+(1-t)\left|\psi^{(n)}(k)\right|-c_2t(1-t)(k-l)^2\right]dt\right\}$$
$$\leq\frac{(l-k)^n}{2.n!.|a_n|}.\left\{\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left[\left|\psi^{(n)}(k)\right|+\left|\psi^{(n)}(l)\right|-ct(1-t)(k-l)^2\right]dt\right\}$$
$$\leq\frac{(l-k)^n}{2n!.|a_n|}.\left[\left|\psi^{(n)}(k)\right|+\left|\psi^{(n)}(l)\right|\right]$$
$$\times\left\{\int_0^1\left[|a_n|t^n+\cdots+|a_1|t^2+|a_0|\right]dt-c(l-k)^2\int_0^1\left[|a_n|\left(t^{n+1}-t^{n+2}\right)+\cdots+|a_1|\left(t^2-t^3\right)+|a_0|\left(t-t^2\right)\right]dt\right\}$$
$$\leq\frac{(l-k)^n}{2n!.|a_n|}.\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]\left[\psi^{(n)}(k)+\psi^{(n)}(l)\right]-\frac{c(l-k)^{n+2}}{2n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]$$

Then, the theorem is proved.

Corollary 2.1: If the modulus c is assumed as zero, Theorem 2.1 is reduced to Theorem 3 [15] given for convex functions.

Theorem 2.2: Let $\psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be n times differentiable mapping on I°, $k, l \in I^{\circ}$ with $k < l$, and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and modulus c, c_1, c_2 where n is even number. If the mapping $\left|\psi^{(n)}\right|^q$ is strongly convex on the interval $[k, l]$, thus one obtains

$$\left| \frac{1}{l-k} \int_k^l \psi(x)\,dx - \frac{\psi(k)+\psi(l)}{2} + \cdots \right. \tag{6}$$

$$-\frac{(l-k)^{n-4}\left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n}.\left[\psi^{(n-4)}(k) + \psi^{(n-4)}(l)\right]$$

$$+\frac{(l-k)^{n-3}\left[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4a_2\right]}{2.n!.a_n}.\left[\psi^{(n-3)}(l) - \psi^{(n-3)}(k)\right]$$

$$\left. -\frac{(l-k)^{n-2}\left[n.a_n + \cdots + 2a_2\right]}{2.n!.a_n}.\left[\psi^{(n-2)}(k) + \psi^{(n-2)}(l)\right] + \frac{(l-k)^{n-1}\left[a_n + \cdots + a_1 + 2a_0\right]}{2.n!.a_n}.\left[\psi^{(n-1)}(l) - \psi^{(n-1)}(k)\right] \right|$$

$$\leq \frac{(l-k)^n}{n!.|a_n|}.\left[\sum_{i=0}^{n} \frac{|a_i|}{(i\,p+1)^{\frac{1}{p}}}\right].\left[\frac{\left|\psi^{(n)}(k)\right|^q + \left|\psi^{(n)}(l)\right|^q}{2} - \frac{c(k-l)^2}{6}\right]^{\frac{1}{q}}$$

Proof 2.2: Using Lemma 1.1, Definition 1.2, Hölder's integral inequality, and Minkowski's integral inequality [16,17], we construct

$$\left| \frac{1}{l-k} \int_k^l \psi(x)\,dx - \frac{\psi(k)+\psi(l)}{2} + \cdots \right. \tag{7}$$

$$-\frac{(l-k)^{n-4}\left[n.(n-1).(n-2).a_n + \cdots + 4.3.2.a_4\right]}{2.n!.a_n}.\left[\psi^{(n-4)}(k) + \psi^{(n-4)}(l)\right]$$

$$+\frac{(l-k)^{n-3}\left[n.(n-1).a_n + \cdots + 4.3.a_4 + 3.2.a_3 + 4a_2\right]}{2.n!.a_n}.\left[\psi^{(n-3)}(l) - \psi^{(n-3)}(k)\right]$$

$$\left. -\frac{(l-k)^{n-2}\left[n.a_n + \cdots + 2a_2\right]}{2.n!.a_n}.\left[\psi^{(n-2)}(k) + \psi^{(n-2)}(l)\right] + \frac{(l-k)^{n-1}\left[a_n + \cdots + a_1 + 2a_0\right]}{2.n!.a_n}.\left[\psi^{(n-1)}(l) - \psi^{(n-1)}(k)\right] \right|$$

$$= \left| \frac{(l-k)^n}{2.n!.a_n} \int_0^1 \left(a_n t^n + \cdots + a_1 t + a_0\right)\left[\psi^{(n)}(tk + (1-t)l) + \psi^{(n)}(tl + (1-t)k)\right] dt \right|$$

$$\leq \frac{(l-k)^n}{2.n!.|a_n|}\left[\int_0^1 \left|a_n t^n + \cdots + a_1 t + a_0\right|\left|\psi^{(n)}(tk + (1-t)l)\right| dt + \int_0^1 \left|a_n t^n + \cdots + a_1 t + a_0\right|\left|\psi^{(n)}(tl + (1-t)k)\right| dt\right]$$

$$\leq \frac{(l-k)^n}{2.n!.|a_n|}\left[\left(\int_0^1 \left|a_n t^n + \cdots + a_1 t + a_0\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\psi^{(n)}(tk + (1-t)l)\right|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 \left|a_n t^n + \cdots + a_1 t + a_0\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|\psi^{(n)}(tl + (1-t)k)\right|^q dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{(l-k)^n}{n!.|a_n|}.\left[\sum_{i=0}^{n} \frac{|a_i|}{(i\,p+1)^{\frac{1}{p}}}\right].\left[\frac{\left|\psi^{(n)}(k)\right|^q + \left|\psi^{(n)}(l)\right|^q}{2} - \frac{c(k-l)^2}{6}\right]^{\frac{1}{q}}$$

such that $\frac{1}{p}+\frac{1}{q}=1$. Considering the strongly convexity of $\left|\psi^{(n)}\right|^q$, then

$$\begin{aligned}\int_0^1\left|\psi^{(n)}\left(tk+(1-t)l\right)\right|^q dt &\le \int_0^1\left[t\left|\psi^{(n)}(k)\right|^q+(1-t)\left|\psi^{(n)}(l)\right|^q-c_1t(1-t)(k-l)^2\right]dt\\ &=\frac{\left|\psi^{(n)}(k)\right|^q+\left|\psi^{(n)}(l)\right|^q}{2}-\frac{c_1(k-l)^2}{6}\end{aligned}\tag{8}$$

$$\begin{aligned}\int_0^1\left|\psi^{(n)}\left(tl+(1-t)k\right)\right|^q dt &\le \int_0^1\left[t\left|\psi^{(n)}(l)\right|^q+(1-t)\left|\psi^{(n)}(k)\right|^q-c_2t(1-t)(a-b)^2\right]dt\\ &=\frac{\left|\psi^{(n)}(k)\right|^q+\left|\psi^{(n)}(l)\right|^q}{2}-\frac{c_2(k-l)^2}{6}\end{aligned}\tag{9}$$

is obtained. Using the Minkowski's inequality, we have

$$\begin{aligned}\left(\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|^p dt\right)^{\frac{1}{p}} &\le \left(\int_0^1|a_n|^p t^{np}dt\right)^{\frac{1}{p}}+\cdots+\left(\int_0^1|a_1|^p t^p dt\right)^{\frac{1}{p}}+\left(\int_0^1|a_0|^p dt\right)^{\frac{1}{p}}\\ &=\frac{|a_n|}{(np+1)^{1/p}}+\cdots+\frac{|a_1|}{(p+1)^{1/p}}+|a_0|=\sum_{i=0}^{n}\frac{|a_i|}{(ip+1)^{1/p}}\end{aligned}\tag{10}$$

Substituting (8), (9), and (10) to (7), then (6) is obtained.

Corollary 2.2: From Theorem 2.1, Theorem 4 [15] given for convex functions is obtained if c is assumed as zero.

Theorem 2.3: Let $\psi: I\subseteq\mathbb{R}\to\mathbb{R}$ be n times differentiable mapping on I°, $k,l\in I^\circ$ with $k<l$ and $p>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, with modulus c_1,c_2, where n is even number. If the mapping $\left|\psi^{(n)}\right|^p$ is strongly convex on $[k,l]$, thus the inequality in the following holds:

$$\begin{aligned}&\left|\frac{1}{l-k}\int_k^l\psi(x)dx-\frac{\psi(k)+\psi(l)}{2}+\cdots\right.\\ &-\frac{(l-k)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}.\left[\psi^{(n-4)}(k)+\psi^{(n-4)}(l)\right]\\ &+\frac{(l-k)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}.\left[\psi^{(n-3)}(l)-\psi^{(n-3)}(k)\right]\\ &\left.-\frac{(l-k)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}.\left[\psi^{(n-2)}(k)+\psi^{(n-2)}(l)\right]+\frac{(l-k)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}.\left[\psi^{(n-1)}(l)-\psi^{(n-1)}(k)\right]\right|\\ &\le\frac{(l-k)^n}{2.n!.|a_n|}\left[\sum_{i=0}^{n}\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}\\ &\times\left\{\left[\left(\sum_{i=0}^{n}\frac{|a_i|}{i+2}\right)\left|\psi^{(n)}(k)\right|^p+\left(\sum_{i=0}^{n}\frac{|a_i|}{(i+1)(i+2)}\right)\left|\psi^{(n)}(l)\right|^p-c_1(k-l)^2\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]^{\frac{1}{p}}\right.\\ &\left.+\left[\left(\sum_{i=0}^{n}\frac{|a_i|}{i+2}\right)\left|\psi^{(n)}(l)\right|^p+\left(\sum_{i=0}^{n}\frac{|a_i|}{(i+1)(i+2)}\right)\left|\psi^{(n)}(k)\right|^p-c_2(k-l)^2\sum_{i=0}^{n}\frac{|a_i|}{(i+2)(i+3)}\right]^{\frac{1}{p}}\right\}\end{aligned}\tag{11}$$

Proof 2.3: Applying Lemma 1.1, Definition 1.2, and power mean integral inequality yields

$$\left|\frac{1}{l-k}\int_k^l \psi(x)dx-\frac{\psi(k)+\psi(l)}{2}+\cdots\right.$$
$$-\frac{(l-k)^{n-4}\left[n.(n-1).(n-2).a_n+\cdots+4.3.2.a_4\right]}{2.n!.a_n}.\left[\psi^{(n-4)}(k)+\psi^{(n-4)}(l)\right]$$
$$+\frac{(l-k)^{n-3}\left[n.(n-1).a_n+\cdots+4.3.a_4+3.2.a_3+4a_2\right]}{2.n!.a_n}.\left[\psi^{(n-3)}(l)-\psi^{(n-3)}(k)\right]$$
$$\left.-\frac{(l-k)^{n-2}\left[n.a_n+\cdots+2a_2\right]}{2.n!.a_n}.\left[\psi^{(n-2)}(k)+\psi^{(n-2)}(l)\right]+\frac{(l-k)^{n-1}\left[a_n+\cdots+a_1+2a_0\right]}{2.n!.a_n}.\left[\psi^{(n-1)}(l)-\psi^{(n-1)}(k)\right]\right|$$
$$=\left|\frac{(l-k)^n}{2.n!.a_n}\int_0^1\left(a_nt^n+\cdots+a_1t+a_0\right)\left[\psi^{(n)}(tk+(1-t)l)+\psi^{(n)}(tl+(1-t)k)\right]dt\right|$$
$$\le\frac{(l-k)^n}{2.n!.|a_n|}\left\{\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left|\psi^{(n)}(tk+(1-t)l)\right|dt+\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left|\psi^{(n)}(tl+(1-t)k)\right|dt\right\}$$
$$\le\frac{(l-k)^n}{2.n!.|a_n|}\left(\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|dt\right)^{1-\frac{1}{p}}\left\{\left(\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left|\psi^{(n)}(tk+(1-t)l)\right|^p dt\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left|\psi^{(n)}(tl+(1-t)k)\right|^p dt\right)^{\frac{1}{p}}\right\}$$
$$\le\frac{(l-k)^n}{2.n!.|a_n|}\left(|a_n|\int_0^1t^ndt+\cdots+|a_1|\int_0^1tdt+\int_0^1|a_0|dt\right)^{1-\frac{1}{p}}$$
$$\times\left\{\left(\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left(t\left|\psi^{(n)}(k)\right|^p+(1-t)\left|\psi^{(n)}(l)\right|^p-c_1t(1-t)(k-l)^2\right)dt\right)^{\frac{1}{p}}\right.$$
$$\left.+\left(\int_0^1\left|a_nt^n+\cdots+a_1t+a_0\right|\left(t\left|\psi^{(n)}(l)\right|^p+(1-t)\left|\psi^{(n)}(k)\right|^p-c_2t(1-t)(k-l)^2\right)dt\right)^{\frac{1}{p}}\right\}$$
$$\le\frac{(l-k)^n}{2.n!.|a_n|}\left[\sum_{i=0}^n\frac{|a_i|}{i+1}\right]^{1-\frac{1}{p}}$$
$$\times\left\{\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+2}\right)\left|\psi^{(n)}(k)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|}{(i+1)(i+2)}\right)\left|\psi^{(n)}(l)\right|^p-c_1(k-l)^2\sum_{i=0}^n\frac{|a_i|}{(i+2)(i+3)}\right]^{\frac{1}{p}}\right.$$
$$\left.+\left[\left(\sum_{i=0}^n\frac{|a_i|}{i+2}\right)\left|\psi^{(n)}(l)\right|^p+\left(\sum_{i=0}^n\frac{|a_i|}{(i+1)(i+2)}\right)\left|\psi^{(n)}(k)\right|^p-c_2(k-l)^2\sum_{i=0}^n\frac{|a_i|}{(i+2)(i+3)}\right]^{\frac{1}{p}}\right\}$$

The proof is completed.

Corollary 2.3: Under conditions of Theorem 2.1, if $c=0$, it is reduced to Theorem 5 [15] given for convex functions.

3. Conclusions

The inequalities form the basis of many studies performed in mathematics. Inequalities are frequently used in almost all branches of mathematics and other branches of science. The purpose of mathematical inequalities is to set lower and upper bounds for functions whose value is not known exactly, or to directly set numerical limits to these functions. Thus, it is possible to find the approximate values of these functions at the desired points. This subject, which has increased in popularity, has become the focus of attention of mathematicians. The inequalities that contribute to this theory are convex function classes.

In this study, by obtaining a perturbed trapezoidal equality for n^{th} order differentiable functions, the strongly convexity definitions and perturbed trapezoidal integral inequalities are presented. It is concluded that the inequalities obtained with the strongly convex function classes have a much better upper bound than the inequalities obtained with the convex function classes.

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Author Contributions

All authors contributed to the study conception and design. All authors read and approved the final manuscript.

REFERENCES

[1] Polyak, B. T., “Existence theorems and convergence of minimizing sequences in extremum problems with restrictions,” Soviet Math. Dokl. 7: 72-75, 1966.

[2] J. E. Pečarić, F. Proschan, Y. L. Tong, “Convex Functions, Partial Orderings and Statistical Applications,” Boston, Academic Press, 1992.

[3] Tunç, M., Göv, E., Şanal, Ü., “On tgs-convex function and their inequalities,” Facta Universitatis Ser. Math. Inform. 30(5): 679-691, 2015.

[4] Hudzik, H. & Maligranda, L., “Some remarks on *s*-convex functions,” Aequationes Mathematicae 48: 100-111, 1994.

[5] Varošanec, S., “On h-convexity,” Journal of Mathematical Analysis and Applications 326(1): 303-311, 2007.

[6] A. W. Roberts & D. E. Varberg, “Convex functions,” New York, Academic Press, 1973.

[7] Merentes, N. & Nikodem, K., “Remarks on strongly convex functions,” Aequat. Math. 80(1-2): 193-199, 2010.

[8] Adamek, M., “On a problem connected with strongly convex functions,” Mathematical Inequalities & Applications 19(4): 1287-1293, 2016.

[9] Moradi, H. R., Omidvar, M. E., Khan, M. A, Nikodem, K, “Around Jensen’s inequality for strongly convex functions,” Aequationes Mathematicae 92(1): 25-37, 2018.

[10] Bakula, M. K. & Nikodem, K., “On the converse Jensen inequality for strongly convex functions,” Journal of Mathematical Analysis and Applications 434(1): 516-522, 2016.

[11] Song, Y-Q., Khan, M. A., Ulla, S. Z., Chu, Y-M., “Integral inequalities involving strongly convex functions,” Journal of Function Spaces 2018: 1-8, 2018.

[12] Awan, M. U., Noor, M. A., Noor, K. I., Safdar, F., “On strongly generalized convex functions,” Filomat 31(18): 5783-5790, 2017.

[13] Mishra, S. K. & Sharma, N., “On strongly generalized convex functions of higher order,” Mathematical Inequalities & Applications 22(1): 111-121, 2019.

[14] D. S. Mitrinović, J. Pečarić, A. M. Fink, “Classical and New Inequalities in Analysis,” Kluwer Academic, Dordrecht, 1993.

[15] Dönmez Demir, D. & Şanal, G., “Perturbed trapezoid inequalities for *n* th order differentiable convex functions and their applications,” AIMS Mathematics 5(6): 5495-5509, 2020.

[16] Dragomir, S. S., Cerone, P., Sofo, A., “Some remarks on the trapezoid rule in numerical integration,” Indian J. Pure Appl. Math. 31(5): 475-494, 2000.

[17] Ma, X. F. & Wang, L. C., “Two mapping related to Minkowski's inequalities,” Journal of Inequalities in Pure and Applied Mathematics 10(3): 1-16, 2009.