Mathematics and Statistics 10(2): 378-382, 2022 http://www.hrpub.org
DOI: 10.13189/ms.2022.100212

Fractional Variational Orthogonal Collocation Method for the Solution of Fractional Fredholm Integro-Differential Equation Using Mamadu-Njoseh Polynomials

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Received December 29, 2021; Revised March 7, 2022; Accepted March 27, 2022

Cite This Paper in the following Citation Styles

(a): *[1] Jonathan Tsetimi, Ebimene James Mamadu , "Fractional Variational Orthogonal Collocation Method for the Solution of Fractional Fredholm Integro-Differential Equation Using Mamadu-Njoseh Polynomials," Mathematics and Statistics, Vol. 10, No. 2, pp. 378 - 382, 2022. DOI: 10.13189/ms.2022.100212.*

(b): *Jonathan Tsetimi, Ebimene James Mamadu (2022). Fractional Variational Orthogonal Collocation Method for the Solution of Fractional Fredholm Integro-Differential Equation Using Mamadu-Njoseh Polynomials. Mathematics and Statistics, 10(2), 378 - 382. DOI: 10.13189/ms.2022.100212.*

Abstract The use of orthogonal polynomials as basis functions via a suitable approximation scheme for the solution of many problems in science and technology has been on the increase and quite fascinating. In many numerical schemes, the convergence depends solely on the nature of the basis function adopted. The Mamadu-Njoseh polynomials are orthogonal polynomials developed in 2016 with reference to the weight function, $w(x) = x^2 + 1, x \in [-1,1]$, which bears the same convergence rate as that of Chebyshev polynomials. Thus, in this paper, the fractional variational orthogonal collocation method (FVOCM) is proposed for the solution of fractional Fredholm integro-differential equation using Mamadu-Njoseh polynomials (MNP) as basis functions. Here, the proposed method is an elegant mixture of the variational iteration method (VIM) and the orthogonal collocation method (OCM). The VIM is one of the popular methods available to researchers in seeking the solution to both linear and nonlinear differential problems requiring neither linearization nor perturbation to arrive at the required solution. Collocating at the roots of orthogonal polynomials gives birth to the OCM. For the proposed method, the VIM is initiated to generate the required approximations whereby producing the series $\sum_{i=0}^{N} u_i, N \geq 0$, which is collocated orthogonally to derive the unknown parameters. The numerical results show that the method derives a high accurate and reliable approximation with a high convergence rate. We have also presented the existence and uniqueness of solution of the method. All computational frameworks in this research are performed via MAPLE 18 software.

Keywords Orthogonal Collocation, Mamadu-Njoseh Polynomials, Fredholm Integro-Differential Equations, Fractional Derivatives, Variational Iteration Method

1. Introduction

Fractional differential equations are very vital in real life applications especially in the field of scientific modeling. Analytic methods for solving these problems actually exist such as the Laplace transform method, however, the process of execution seems complex and elaborate. For this reason, numerical approximations have become a necessity in solving most fractional problems. Popular existing numerical procedures for fractional problems include the homotopy perturbation method, decomposition method, least square method, collocation method, variational iteration method [1-5].

The variational iteration method (VIM) is one of the popular methods available to researchers in seeking the solution to both linear and nonlinear differential problems. The method is accredited to Ji-Huan He in 1999 [7]. The method is simple to execute requiring neither linearization nor perturbation to arrive at the required solution. Readers are referred to [8-10] for more details about the VIM.

A functional equation such as (1) defined on the interval I is said to possess a collocation solution u_g in some collocation space if it satisfies the points such that the cardinality resembles the dimension of the collocation space. If there exist relevant prescribed conditions then the solution u_g must satisfy these conditions necessarily.

The adoption of orthogonal polynomials in seeking the approximate solution of differential equations can be traced back to the 1930s [11]. Let equation (1) be given and assume that Lipschitz continuous function $g: I \times \Omega \in \mathbb{R} \to \mathbb{R}$ possesses a unique solution $u \in C(I)$ for all $u_0 \in \Omega$. Let

$$I_g := \{x_n: a = x_0 < x_1 < x_2 < \cdots < x_N = b\}$$

be given mesh points on I, and set $\tau_n := \{x_n, x_{n+1}\}$ with $g := x_{n+1} - x_n$, $n = 0(1)N-1$. The function u of the problem (1) will be computed by the element of orthogonal polynomial space

$$s_k^{(0)}(I_g) := \{w \in C(I): w|_{\tau_n \in \rho_k}\ (0 \le n \le N-1)\},$$

where ρ_k denotes polynomials of degree not exceeding n. The approximation of u_g is via collocation. According to Lancoz [12], when the set of collocation points is at the zeros of orthogonal collocation, it is called the orthogonal collocation.

The aim of this paper is to seek the approximate solution of fractional Fredholm integro-differential equation (1) using Mamadu-Njoseh polynomials (MNP) as trial functions via the proposed method "fractional variational orthogonal collocation method (FVOCM). The proposed method is an elegant mixture of the variational iteration method and the orthogonal collocation method (OCM). The VIM is initiated to generate the required approximations whereby producing the series $\sum_{i=0}^{N} u_i, N \ge 0$, which in turn is collocated orthogonally to locate the unknown parameters.

2. Materials and Methods

2.1. FVOCM for Fractional Fredholm Integro-Differential Equation

Let the Fredholm integro-differential equation be given [3] as

$$D^\alpha u(x) = g(x) + \int_0^1 k(x,s)u(s)ds, \quad x \ge 0, s \le 1, \quad (1)$$

$$u^i(0) = \beta_i, \quad (n-1, n] \in \alpha, \ n \in \mathbb{N}, \quad (2)$$

where $u(x)$ is the unknown, $D^\alpha u(x)$ is the Caputo fractional derivative of $u(x)$ of order α, $g(x)$ is the non-homogeneous term, $k(x,s)$ is the nucleus of the integral, x and s are variables defined in $[0,1]$.

Let

$$u_n(x) = \sum_{i=0}^{n} a_i \varphi_i^*(x'), \ x' = \frac{1}{2}(x+1), \ x \in [0,1], \quad (3)$$

be the approximate solution of (1.1). Here, $a_i{}'s$ are the unknowns to be computed, and $\varphi_i^*(x'), i = 0(1)N$, are the Mamadu-Njoseh polynomials [13].

By the fractional variational iteration method (FVIM), we construct a correctional functional for (1) as follows:

$$u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \lambda(\tau) \Big(D^\alpha u_n(\tau) - g(\tau) - \int_0^1 k(x,s) u_n(x) ds, \Big) (d\tau)^\alpha, n \ge 0, \quad (4)$$

where $\lambda(\tau)$ is the general Lagrange multiplier, and $D^\alpha u_n(\tau)$ is the popular Caputo fractional derivative [14, 17, 18] given as

$$_aD^\alpha u_n(\tau) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{n-\alpha-1} u_n{}^n(\tau) d\tau, & (n-1) < \alpha \le n, \\ \left(\frac{d}{dx}\right)^{n-1} u_n(\tau) \text{ if } \alpha = n-1 \end{cases}, \quad (5)$$

satisfying the following properties (which are useful to this work):

i). $D^\alpha C = 0$, C is a constant,

ii). $D^\alpha x^\beta = \begin{cases} 0, \ \beta \in \mu, \ \beta < \alpha_b, & \text{and} \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \ \beta \in \mu, \ \beta \ge \alpha_b, \end{cases}$

where $\alpha_b \ge \alpha$ and $\mu = 0(1)\infty$.

Now, taking the variation δ on both sides of (4), we have

$$\delta u_{n+1}(x) = \delta u_n(x) + \frac{1}{\Gamma(1+\alpha)} \delta \int_0^x \lambda(\tau) \Big(D^\alpha u_n(\tau) - g(\tau) - \int_0^1 k(x,s) u_n(x) ds, \Big) (d\tau)^\alpha. \quad (6)$$

Solving (6) via the variational theory, we have

$$\delta u_{n+1}(x) = \delta u_n(x) + \lambda(\tau) \delta u_n^\alpha(\tau)|_{\tau=x} - \lambda^\alpha(\tau) u_n(\tau)|_{\tau=x} + \frac{1}{\Gamma(1+\alpha)} \delta \int_0^x \lambda(\tau) \Big(\int_0^1 D^\alpha u_n(\tau) \delta u_n(\tau) \Big) (d\tau)^\alpha.$$

Then, $1 - \lambda^\alpha(\tau)|_{\tau=x} = 0, D^\alpha u_n(\tau) = 0, \ \lambda(\tau)|_{\tau=x} = 0.$ Thus, the generalized Lagrange multiplier is given as

$$\lambda(\tau) = (-1)^n \frac{(\tau-x)^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}. \quad (7)$$

To kick off the iteration (4) for $n \ge 0$, the initial solution is derived using (2) and (3)

$$u_0(x) = \sum_{i=0}^{n} a_i \varphi_i^*(0). \quad (8)$$

Hence, the fractional variational scheme for (1) becomes

$$u_{n+1}(x) = \begin{cases} u_0(x) = \sum_{i=0}^{n} a_i \varphi_i^*(0), \ n+1 = 0, \\ u_n(x) + \frac{1}{\Gamma(1+\alpha)} \int_0^x \lambda(\tau) \Big(D^\alpha u_n(\tau) - g(\tau) - \int_0^1 k(x,s) u_n(x) ds, \Big) (d\tau)^\alpha, n \ge 0. \end{cases} \quad (9)$$

The resulting approximations from (9) are thus defined by the series

$$U(x) = \sum_{i=0}^{N} u_{i}, \quad N \geq 0. \tag{10}$$

The equation (10) is therefore collocated orthogonally at the zeros of $\varphi_i^*(x), i = 0(1)6$, to obtain $(n-1)$ systems of linear equations, which on solving via the Gaussian elimination method yields the $a_i{}'$ s. The estimated $a_i{}'$s are substituted into (3) to obtain the required approximate solution.

2.2. Existence and Uniqueness of Solution

Theorem 2.1. Let $v \in C(D)$ and T be a resolvent kernel linked with v. For any $g \in C(I)$, the fractional integro-differential equation (1.1) has a unique solution

$$u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \left(g(t) + \int_0^1 k(x,s) u_n(s) ds \right) dt. \tag{11}$$

Proof. Replacing x in (1.1) by w, and multiplying the equation by $T(x,w)$ and integrating over $[0,w]$ with respect to w. Using (3) and the Dirichlet formula, we have that

$$\int_0^w T(w,s) u(w) dw = \int_0^w T(w,s) \left(\frac{1}{\Gamma(\alpha)} \int_0^w (w-t)^{\alpha-1} \left(g(t) + \int_0^1 k(x,s) u_n(s) ds \right) \right) dw.$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^w T(x,s) \left(\int_0^w (x-t)^{\alpha-1} g(t) dt \right) dw + \frac{1}{\Gamma(\alpha)} \int_0^w T(x,s) \, (x-t)^{\alpha-1} \left(\int_0^1 k(x,s) u_n(s) ds \right) dw,$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^w T(w,s) \left(\int_0^w (w-t)^{\alpha-1} g(t) dt \right) dw + + \frac{1}{\Gamma(\alpha)} \int_0^w (w-t)^{\alpha-1} \left((T(x,s) - k(x,s)) u_n(s) ds, \right. \tag{12}$$

implying that

$$(Tu)(s) = \int_0^w k(x,s) u_n(s) ds = \int_0^w T(x,s) u_n(s) ds. \tag{13}$$

To show that (13) is true, let $y \in C(I)$ also be a solution. Since

$$y(w) = u_n(w) + (wu)x, w \in I,$$

multiplying both sides by $T(w,s)$ and integrating over $[0,w]$ with respect to w yields

$$\int_0^w T(w,s) y(w) dw = \int_0^w T(w,s) \, u(w) dw + + \frac{1}{\Gamma(\alpha)} \int_0^w \left(\int_s^w (T(w,s) \, k(w,s) y(s) ds \right) dw.$$

$$\int_0^w T(w,s) y(w) dw = \int_0^w T(w,s) \, u(w) dw + + \frac{1}{\Gamma(\alpha)} \int_0^w (T(x,s) - k(x,s)) \, y(s) ds. \tag{14}$$

Using the Dirichlet formula on (1.14), we obtain

$$0 = [u(x) - y(x)] - \int_0^w k(x,s) y(s) ds = [u(x) - y(x)] - [u(x) - y(x)].$$

3. Numerical Illustrations

This section considers some numerical illustrations of the proposed method to some selected linear fractional Fredholm integro-differential equations as follows:

Example 1. Given the fractional Fredholm integro-differential equation

$$D^{\alpha} u(x) = \frac{(3/8) x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xt u(t) dt, \quad x \geq 0, t \leq 1, \tag{15}$$

with the initial condition $u(0) = 0$. The exact solution is $u(x) = x^2 - x$.

Applying the proposed method at $n = 5$ to (15) with $\alpha = 0.5$, we obtain four systems of linear equations with four unknowns $a_i, i = 0(1)4$. Transforming this system into a matrix equation and solving yields the matrix inverse as shown in Figure 1a. The values of the unknowns $a_i, i = 0(1)4$, are given as

$$a_0 = 3.600000, a_1 = -6.000000,$$

$$a_2 = 2.400000 \text{, and } a_3 = -0.000000.$$

Substituting these values into (3) yields the approximate solution as

$$u(x) = -1.000000000 - 3.000000000x + + 4.000000000 (\frac{1}{2} x + \frac{1}{2})^2$$

with absolute convergence. Results are presented in Table 1 showing the comparison of results with those available in the literature Nanware *et. al.* [15], which adopted the Bernstein Polynomials as trial functions. Figure 1b compared the approximate and exact solutions for **Example 1.**

Table 1. Comparison of Errors between the FVOCM and Nanware *et. al.* [15]

x	FVOCM Error	Nanware et. al. [15] Error
0.1	0.000000	3×10^{-7}
0.2	0.000000	5×10^{-7}
0.3	0.000000	9×10^{-7}
0.4	0.000000	1.4×10^{-7}
0.5	0.000000	2.3×10^{-6}
0.6	0.000000	3.8×10^{-6}
0.7	0.000000	6.0×10^{-6}
0.8	0.000000	9.3×10^{-6}
0.9	0.000000	1.37×10^{-5}

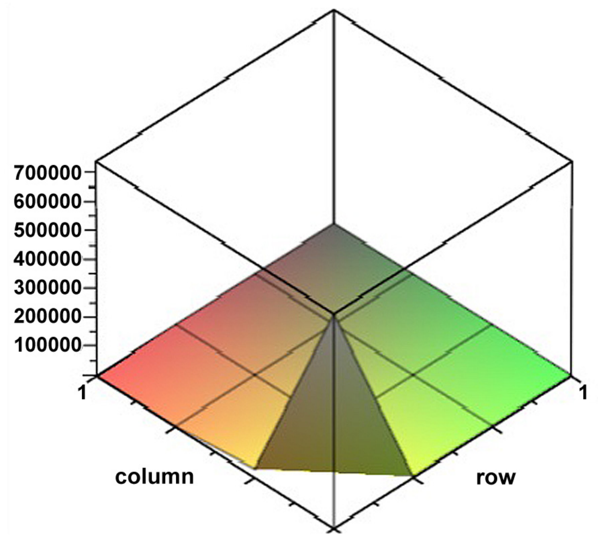

Figure 1a. Matrix inverse of Example 1

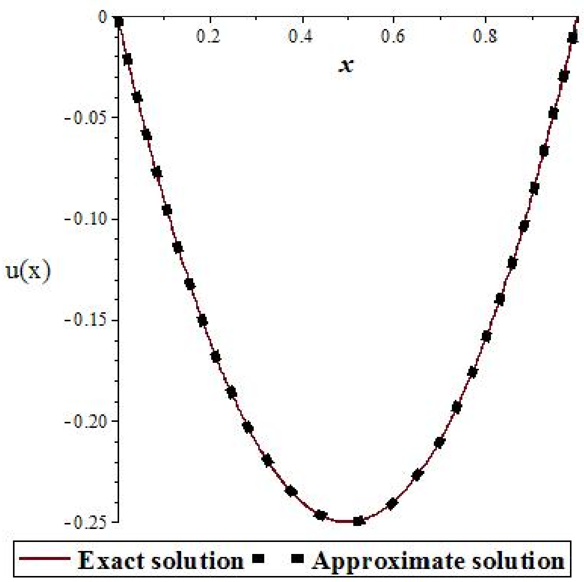

Figure 1b. Approximate solution of Example 1 as compared with Exact solution

Example 2. Given the fractional Fredholm integro-differential equation

$$D^{\alpha}u(x) = \frac{3\sqrt{3}\Gamma\left(^{2}/_{3}\right)}{\pi} - \frac{1}{5}x^2 - \frac{1}{4}x + $$
$$+\int_0^1 (xt + x^2t^2)u(t)dt, x \geq 0, t \leq, \qquad (16)$$

with the initial condition to $u(0) = u'(0) = 0$. The analytic solution is $u(x) = x^2$.

Applying the proposed method as Example 1 at $n = 5$ with $\alpha = {}^{5}/_{3}$, the numerical results are shown in Figure 2a and Figure 2b respectively with the unknowns estimated as

$$a_0 = 2.600000, \quad a_1 = -4.000000,$$
$$a_2 = 2.400000,$$
$$a_3 = 0.000000.$$

Substituting the above values into (3) we obtain the approximate solution as

$$u(x) = -1.000000000 - 2.000000000x +$$
$$+4.000000000\left(\frac{1}{2}x + \frac{1}{2}\right)^2,$$

with absolute convergence. Results are presented in Table 2 showing the comparison of results with those available in the literature Tsetimi and Mamadu [16], which adopted the Mamadu-Njoseh polynomials as trial functions via the Least square method.

Table 2. Comparison of Errors between the FVOCM and Tsetimi and Mamadu [16]

x	FVOCM Error	Tsetimi and Mamadu [16] Error
0.1	0.000000	0.000000
0.2	0.000000	0.000000
0.3	0.000000	0.000000
0.4	0.000000	0.000000
0.5	0.000000	0.000000
0.6	0.000000	0.000000
0.7	0.000000	0.000000
0.8	0.000000	0.000000
0.9	0.000000	0.000000

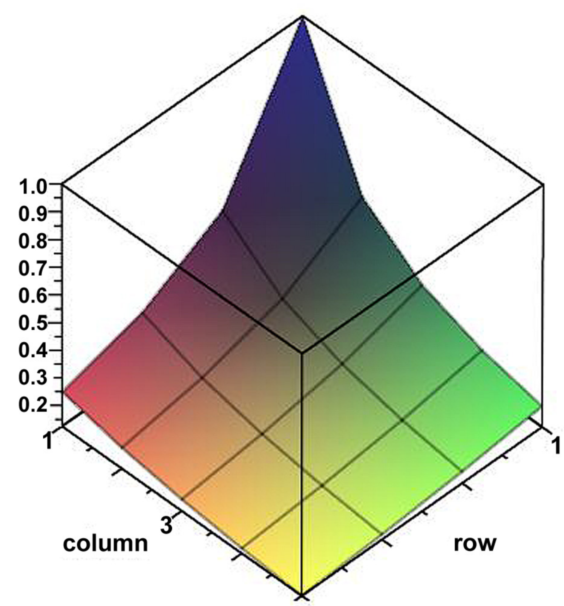

Figure 2a. Matrix inverse of Example 2

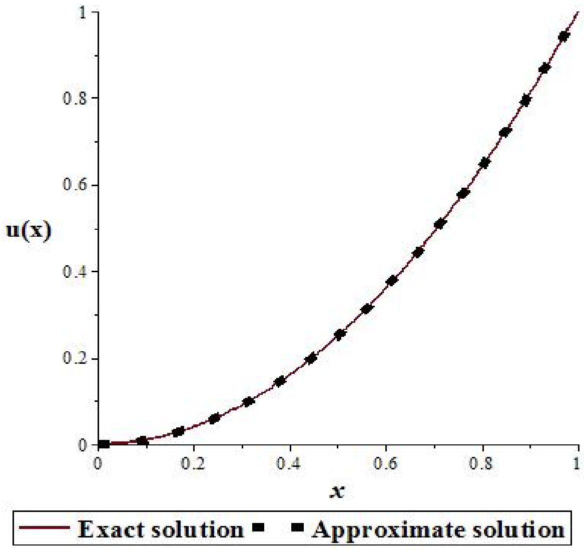

Figure 2b. Approximate solution of Example 2 as compared with Exact solution

4. Conclusion

We have successively studied the proposed method "fractional variational orthogonal collocation method using Mamadu-Njoseh polynomials" for the solution of fractional Fredholm integro-differential equation. It was observed that the method derives accurate and reliable approximations with absolute convergence.

REFERENCES

[1] Saeed R. K., Sdeq H. M., "Solving a System of Linear Fredholm Fractional Integro-Differential Equations using Homotopy Perturbation Method", Australian Journal of Basic and Applied Sciences, Vol. 4, no. 4, pp. 633–638, 2010.

[2] Mittal R. C., Nigam R., "Solution of Fractional Integrodifferential Equations by Adomain Decomposition Method," International Journal of Applied Mathematics and Mechanics, Vol. 4, no. 2, pp. 87–94, 2008.

[3] Mohammed D Sh., "Numerical solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Chebyshev Polynomial", Mathematical Problems in Engineering Volume 2014, Article ID 431965, 5pages http://dx.doi.org/10.1155/2014/431965.

[4] Bhrawy A.H., Alghamdi M.A., "A Shifted Jacobi-Gausslobatto Collocation Method For Solving Nonlinear Fractional Langevin Equation involving Two Fractional Orders in Different Intervals", Boundary Value Problems, vol. 2012, article 62, 13 pages, 2012.

[5] Tsetimi J., Mamadu, E.J., "Comparative Solutions to Fifth Order Boundary Value Problem Using Variational Iteration Decomposition Method and Variational Iteration Method with Chebychev Polynomials", Journal of the Nigerian Association of Mathematical Physics, Vol. 40, pp 39-42, 2017

[6] Mamadu E.J., Njoseh, I. N, "On the Convergences of Variation Iteration Method for Nonlinear Integro-Differential Equations", Transactions of the Nigeria Association of Mathematics Physics, Vol. 2, pp. 65-70, 2016.

[7] Abbasbandy S., and Shivanian E., "Application of the Variational Iteration Method for Systemof Nonlinear Volterra's Integro-Differential Equations", Mathematics and Computational Applications, Vol. 14, no. 2, pp. 147-158, 2009.

[8] Njoseh I.N, and Mamadu E.J., "An Application of Variation Iteration Method for the Analytical Solution of the Gas Dynamic Equation", Vol. 16, no. 1, pp. 013 – 017, 2017.

[9] Njoseh I.N, and Mamadu E.J., "Optimal Variation Iteration Method for Solving Thirteenth-Order Boundary Value Problems", Nigerian Annals Of Natural Sciences, Vol. 16, no. 1, pp. 076 – 081, 2017.

[10] Mamadu E.J., Njoseh I. N, "Power Series Variational Iteration Method for the Numerical Treatment of Fifth Order Boundary Value Problems", PK ISSN 0022- 2941; CODEN JNSMAC, Vol. 57, pp. 10-18, 2017.

[11] Brunner, H., "Collocation Methods for Volterra Integral and Related Functional Equations", Cambridge University Press, Cambridge CB2 2RU, UK, 2004.

[12] Olagunju A.S., Joseph F.L., "Third-kind Chebychev Polynomials in Collocation Methods of Solving Boundary Value Problems", IOSR Journal of Mathematics, Vol. 7, no. 2, pp. 42-47, 2013.

[13] Mamadu E.J., Njoseh, I. N, "Elzaki Transform Method for Delay Differential Method", Boson Journal of Modern Physics, Vol. 3, no. 1, pp. 2454 - 8413, 2016.

[14] Oyedepo T., Taiwo O.A., Abubakar J.U., Ogunwobi Z.O., "Numerical Studies for Solving Fractional Integro-Differential Equations by using Least Squares Method and Bernstein Polynomials", Fluid Mech Open Acc 2016, 3:3 DOI: 10.4172/2476-2296.1000142.

[15] Nanware, J.A., Goud, P.M. and Holambe, T.L., "Solution of Fractional Integro-differential Equations by Bernstein Polynomials", Malaya Journal of Matematik, Vol. S, No. 1, pp. 581-586, 2020. https://doi.org/10.26637/MJM0S20/0111

[16] Tsetimi J and Mamadu E. J. "Least Squares Method For the Solution of Fractional Fredholm Integro-Differential Equation Using Mamadu-Njoseh Polynomials", Ciencia e Technica Vitivinicola, Vol.36 no. 2, Pages: 1-11. http://ciencia-e-tecnica.org/

[17] Prapart Pue-on, "The Fractional Residual Power Series Method for Solving a System of Linear Fractional Fredholm Integro-differential Equations," Mathematics and Statistics, Vol.9, No.5, pp. 792-798, 2021. DOI: 10.13189/ms.2021.090519

[18] Yakhshiboev M. U., "Convergence Almost Everywhere of Non-convolutional Integral Operators in Lebesgue Spaces," Mathematics and Statistics, Vol. 8, No. 6, pp. 705 - 710, 2020. DOI: 10.13189/ms.2020.080611.