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Modelling of Cointegration with Student's T-errors

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Abstract Two or more non-stationary time series are said to be co-integrated if a certain linear combination of them becomes stationary. Identification of co-integrating relationships among the relevant time series helps the researchers to develop efficient forecasting methods. The classical approach of analyzing such series is to express the co-integrating time series in the form of error correction models with Gaussian errors. However, the modeling and analysis of cointegration in the presence of non-normal errors needs to be developed as most of the real time series in the field of finance and economics deviates from the assumption of normality. This paper focuses on modeling of a bivariate cointegration with a student's-t distributed error. The co-integrating vector obtained from the error correction equation is estimated using the method of maximum likelihood. A unit root test of first order non stationary process with student's t-errors is also defined. The resulting estimators are used to construct test procedures for testing the unit root and cointegration associated with two time series. The likelihood equations are all solved using numerical approaches because the estimating equations do not have an explicit solution. A simulation study is carried out to illustrate the finite sample properties of the model. The simulation experiments show that the estimates perform reasonably well. The applicability of the model is illustrated by analyzing the data on time series of Bombay stock exchange indices and crude oil prices and found that the proposed model is a good fit for the data sets.

Keywords Cointegration, Conditional MLE, Error Correction Model, Student's-t Distribution, Unit Root Test

1 Introduction

According to Wold's theorem, a single stationary time series $\{X_t\}$ with no deterministic components can be expressed as a linear combination of shocks at previous time points, $t-1, t-2, ...$, which could be approximated by a linear Autoregressive Moving Average (ARMA) model of appropriate order[1]. Box-Jenkins method of analysis assumes that a time series becomes stationary after a finite number (d) of differences and introduced a class of models known as autoregressive integrated moving average (ARIMA) models. If the underlying time series becomes stationary after d differences then the original series is said to be integrated of order d, denoted as $I(d)$. In practice, all naturally occurring time series are non-stationary and some of them may be of $I(d)$. If the time series is stationary then it is said to be of $I(0)$. The integrated series are special class of time series whose non-stationarity is due to the presence of one or more unit roots in the underlying characteristic polynomial. The unit root non-stationarity also indicates the presence of stochastic trend in the time series. Several time series from related domains may wander extensively throughout a research period. However the underlying theory will propose forces which tend to keep them together in a state of equilibrium. That is, there may exist two or more non-stationary time series, and yet a certain linear combination of those variables could be stationary. If that is the case, the original time series are said to be co-integrated.

The commonly used test procedures for cointegration in the literature are Engle and Granger two step estimator [2] and Johansen likelihood ratio test [3]. Johansen [3] developed the maximum likelihood estimator of the cointegration parameters and the likelihood ratio test for testing the presence of cointegration. Both the test are based on the assumption that the possibly cointegrated vector autoregressive(VAR) or error correction model (ECM) has normally distributed errors.The assumption of normality need not hold good if the data under study are generated by some heavy-tailed distributions. Such situations often arise in the study of financial time series. So in the present work, we focus on a bivariate cointegration model with student's-t errors.

The following is a definition of bivariate cointegration.

Let $\{X_{1t}\}$ and $\{X_{2t}\}$ be two $I(1)$ series. These series are said to be cointegrated, if there exist a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)'$ with both elements non zero such that,

$$\boldsymbol{\alpha}'[X_{1t}, X_{2t}]' = \alpha_1 X_{1t} - \alpha_2 X_{2t} \sim I(0).$$

The studies on cointegration and unit root when the error densities are non-normal is not extensively seen in the literature. [4] discussed the effects of fiscal policy on the exportation of agricultural products in nigeria using a Vector Error Correction Model (VECM) approach, when the error density follows a normally distribution. [5] discussed the performance of unit root tests in single time series process. Their study compared the performance of different unit root test such as Augmented Dickey Fuller (ADF) test, Kwiatkowski Phillips Schmidt and Shin (KPSS) test and Phillips-Perron (PP) test. All these test are developed under the assumption of normal distribution. But, since most of the series in practical situations deviate from normality , it is of interest to study the cointegration models with non-normal errors. Motivated by this, [6] discussed the estimation procedure for a bivariate cointegration model when the errors are generated by a constant conditional correlation model. Also, [7] discussed a unit process and bivariate cointegration model of first order for $I(1)$ processes which allows for logistic error.

In the literature, there are various standard non-normal distributions, each of which may require separate attention. The main objective of the present study is to explore the possibility of employing non-normal error distribution, specifically a bivariate student's t-distribution for modelling unit root and cointegrating time series. The Student's t- distribution can be a useful theoretical tool in the area of applied statistics. Lee et al. [8] examined the performance of Johansen's tests compared with Dickey Fuller tests when cointegration errors are fitted by GARCH(1,1) model with normal and Student's t error distributions. Tiku et al. [9] discusses an autoregressive models in time series with non normal errors represented by a member of a wide family of symmetric Student's t-distributions. Creal et al. [10] introduced the multivariate Student's t generalised autoregressive score model for volatilities and correlations, where the multivariate normal distribution is a special case. Here we propose the modelling of two cointegrating time series with the errors generated from a bivariate Student's t-distribution.

The rest of the paper is organised as follows. The next section briefly discusses the cointegration model with bivariate Student's t-errors. We discuss the estimation procedure in Section 3. Section 4 deals with the unit root test and cointegration test associated with the autoregressive model of order 1. To evaluate the accuracy of the estimators and test statistic, we carried out a simulation study in section 5. Finally, an application of the proposed model is illustrated in section 6.

2 Model Description

Let $\{X_{1t}\}$ and $\{X_{2t}\}$ be two cointegrating time series and both are of $I(1)$. For further analysis, it is customary to express the series in the form of ECM Following [2], let us write

$$X_{1t} + \beta X_{2t} = u_{1t}, u_{1t} = u_{1t-1} + a_{1t} \tag{1}$$

$$X_{1t} + \alpha X_{2t} = u_{2t}, u_{2t} = \phi u_{2t-1} + a_{2t}, |\phi| < 1, \tag{2}$$

It can be shown that the reduced form for the process in (1) and (2) will make the variables X_{1t} and X_{2t} as linear combinations of u_{1t} and u_{2t} and therefore both the series will be nonstationary (Integrated of order 1). That is,

$$X_{1t} = \left(\frac{\alpha}{\alpha - \beta}\right) u_{1t} - \left(\frac{\beta}{\alpha - \beta}\right) u_{2t} \tag{3}$$

and

$$X_{2t} = \left(\frac{1}{\alpha - \beta}\right) u_{2t} - \left(\frac{1}{\alpha - \beta}\right) u_{1t}. \tag{4}$$

From (3) and (4), it is clear that $\{X_{1t}\}$ and $\{X_{2t}\}$ are non stationary as they are linear combinations of a stationary and a non stationary series. Since $\{X_{1t}\}$ and $\{X_{2t}\}$ are integrated series, equation (2) describes a stationary linear combination of the nonstationary variables. Thus the variables X_{1t} and X_{2t} are cointegrated and hence we can say that they have a long run relationship in equilibrium. But if $\phi \to 1$, then the series are uncorrelated random walks and hence they are no longer cointegrated. The model given above has been studied by [2] in detail with possibly correlated white noise.

To analyse further,it is convenient to reparameterise the model in (1) and (2) by subtracting the lagged values from both sides. Let Δ be a difference operator defined by $\Delta X_t = X_t - X_{t-1}$. Applying Δ on X_{1t} and X_{2t} of both sides of equations (1) and (2) and after some algebra we will get,

$$\Delta X_{1t} = \delta\beta Z_{t-1} + \eta_{1t} \tag{5}$$

$$\Delta X_{2t} = -\delta Z_{t-1} + \eta_{2t}, \tag{6}$$

where $Z_{t-1} = X_{1,t-1} + \alpha X_{2,t-1}$ is the stationary cointegrating relationship and $\delta = \frac{1-\phi}{\alpha-\beta}$. This model was studied by [2] for obtaining the stationary cointegrating relationship between the time series.

Assume that the error variable $\eta_t = (\eta_{1t}, \eta_{2t})^{'}$ in (5) and (6) follows a bivariate Student's t-distribution with the probability density function (pdf) given by,

$$f(\eta_{1t}, \eta_{2t}) = \frac{1}{2\pi\sqrt{1-\rho^2}} \left[1 + \frac{1}{v(1-\rho^2)}\left(\eta_{1t}^2 - 2\rho\eta_{1t}\eta_{2t} + \eta_{2t}^2\right)\right]^{-\frac{(\nu+2)}{2}},$$

where $\nu > 0, -1 < \rho < 1, x, y > 0$.

It should be noted that both the marginal distributions are student's t-distribution with the same degrees of freedom ν (see [11]) with the pdf

$$f(\eta_{it}) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{(\eta_{it})^2}{\nu}\right)^{-\left(\frac{\nu+1}{2}\right)}, i = 1, 2.$$

The above bivariate distribution can be obtained by the transformation,

$$Y = \left(\frac{\sqrt{s}}{\nu}\right)^{-1} Z + \mu,$$

where Z is $N(0, \Sigma)$, Σ is variance covariance matrix and $s \sim \chi_r^2$. Here s and Z are independent. The case with ν=1 reduces to a bivariate Cauchy distribution.

3 Conditional MLE for ECM with bivariate student's t-errors

In this section, we focus on the problem of maximum likelihood estimation of cointegrated model with bivariate Student's t-distributed errors. When there exist an explicit density function for the error random variables, one can use the conditional maximum likelihood estimation for estimating the model parameters. Let the parameter vector to be estimated are the elements of $\theta = (\rho, \alpha, \beta, \delta, \nu)^{'}$. The explicit form of the conditional log likelihood function for $\theta = (\rho, \alpha, \beta, \delta, \nu)^{'}$, given $(x_{10}, x_{20}) = (0, 0)$ based on sample $(x_{1t}, x_{2t}), t = 1, 2, \cdots n$ of size n is given by ,

$$\begin{aligned} L_n(\theta) = & -n \log\left(1-\rho^2\right)^{1/2} \\ & -\frac{(\nu+2)}{2}\sum_{t=1}^{n} \log\left(1 + \frac{1}{\nu(1-\rho^2)}\left[\left(\eta_{1t}^2 - 2\rho\eta_{1t}\eta_{2t} + \eta_{2t}^2\right)\right]\right), \end{aligned}$$

where (x_{1t}, x_{2t}) are obtained from the equations (5) and (6). The form of the log likelihood function suggests that we have to maximise it by using some numerical methods. Here we obtain the parameter estimates by using a two step estimation procedure. First we obtain the estimates of $\rho, \alpha, \beta, \delta$ by solving the respective score equations. Then we use the profile likelihood technique for estimating the parameter ν. That is, the estimate of ν is obtained by maximizing the log likelihood function using $\hat{\rho}, \hat{\alpha}, \hat{\beta}, \hat{\delta}$. The discussion on profile likelihood and its asymptotic properties can be seen in [12].

On differentiating the log-likelihood function with respect to the parameters $\rho, \alpha, \beta, \delta$, we get the four equations given by,

$$\frac{\partial \log l}{\partial \rho} = 0 \Rightarrow n\frac{\rho}{(1-\rho^2)} - \sum_{t=1}^{n} \frac{(\nu+2)\left[\rho\frac{\left(\eta_{1t}^2 - 2\rho\eta_{1t}\eta_{2t} + \eta_{2t}^2\right)}{\nu(1-\rho^2)^2} - \frac{\eta_{1t}\eta_{2t}}{\nu(1-\rho^2)}\right]}{\left(1 + \frac{1}{\nu(1-\rho^2)}\left[\left(\eta_{1t}^2 - 2\rho\eta_{1t}\eta_{2t} + \eta_{2t}^2\right)\right]\right)} = 0. \quad (7)$$

$$\frac{\partial \log l}{\partial \delta}=0 \Rightarrow \sum_{t=1}^{n} \frac{(\nu+2)\left[2z_{t-1}A(\theta)(\beta+\rho)-2z_{t-1}B(\theta)(1+\beta\rho)\right]}{2\nu(1-\rho^2)\left[1+\frac{(A(\theta))^2+(B(\theta))-2\rho A(\theta)B(\theta)}{\nu(1-\hat{\rho}^2)}\right]}=0. \tag{8}$$

$$\frac{\partial \log l}{\partial \beta}=0 \Rightarrow \sum_{t=1}^{n} \frac{(\nu+2)\left[2z_{t-1}\delta A(\theta)-2z_{t-1}\delta\rho B(\theta)\right]}{2\nu(1-\rho^2)\left[1+\frac{(A(\theta))^2+(B(\theta))-2\rho A(\theta)B(\theta)}{\nu(1-\rho^2)}\right]}=0. \tag{9}$$

$$\frac{\partial \log l}{\partial \alpha}=0 \Rightarrow \sum_{t=1}^{n} \frac{(\nu+2)\left[2x_{2t-1}\delta(\beta+\rho)A(\theta)+2x_{2t-1}\delta(1+\beta\rho)B(\theta)\right]}{2\nu(1-\rho^2)\left[1+\frac{(A(\theta))^2+(B(\theta))-2\rho A(\theta)B(\theta)}{\nu(1-\rho^2)}\right]}=0, \tag{10}$$

where $A(\theta)=\Delta x_{1t}-\delta\beta z_{t-1}, B(\theta)=\Delta x_{2t}+\delta z_{t-1}$. Since there do not exist analytically closed form expressions for the estimators, we have to maximize the score functions by using some numerical methods. We have solved these equations by using Newton Raphson method for obtaining the parameter estimates. Let the estimates of ρ, α ,β ,δ be $\hat{\rho}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\delta}$ respectively. Once the model parameters are estimated, the profile likelihood function $L\left(\nu,\hat{\alpha},\hat{\beta},\hat{\delta},\hat{\rho}\right)$ is maximized over ν to get $\hat{\nu}$. The profile log likelihood function is given by,

$$\begin{aligned}L_n(\theta)=&-n\log\left(1-\hat{\rho}^2\right)^{\frac{1}{2}}\\&-\frac{(\nu+2)}{2}\sum_{t=1}^{n}\log\left(1+\frac{1}{\nu(1-\hat{\rho}^2)}\left(A(\hat{\theta})^2-2\rho A(\hat{\theta})B(\hat{\theta})+B(\hat{\theta})^2\right)\right),\end{aligned} \tag{11}$$

where $A(\hat{\theta})=\Delta x_{1t}-\hat{\delta}\hat{\beta}\hat{z}_{t-1}, B(\hat{\theta})=\Delta x_{2t}+\hat{\delta}\hat{z}_{t-1}$.

The properties of first order autoregressive models with student's t- distributed errors are needed to develop a test procedure for cointegration model in (5) and (6), which we discuss in section 4.

4 Unit root test for Auto Regressive model of order one with Student's t-errors

A sequence that contains one or more roots of its characteristic polynomial that are equal to one is called a unit root process. The simplest model that may contain a unit root is the Auto Regressive model of order one (AR(1)),

$$X_t=\phi X_{t-1}+a_t, \tag{12}$$

where a_t denotes a serially uncorrelated white noise with 0 mean and constant variance.

If ϕ= 1, equation (12) becomes a random walk without drift model, that is, a non-stationary process. If, $|\phi|<1$, then the series $\{X_t\}$ is stationary. Dickey et al [13] developed a test procedure to determine whether a variable has a unit root, or equivalently, the variable follows a random walk model. Here, our interest is to analyse the first order autoregressive equation in the presence of Student's t-errors. We consider the first order autoregressive process $\{X_t\}$ with the errors $\{a_t\}$ is a sequence of Student's t-random variables.

Our interest is to find the maximum likelihood estimator of ϕ, say $\hat{\phi}$ and the test procedure for unit root under the hypothesis $H_0:\phi=1$ against $H_1:|\phi|<1$. Suppose n observations are available for the analysis and we obtain the likelihood function based on n observations generated by the model (12). The corresponding joint density function of $(a_1,a_2\cdots\cdots,a_n)$ is given by,

$$\prod_{t=1}^{n}\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\left(1+\frac{a_t^2}{\nu}\right)^{-\left(\frac{\nu+1}{2}\right)}.$$

And the conditional likelihood of ϕ given x_0 for the model(12) is,

$$L(\phi \mid x_0) = \left(\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \right)^n \prod_{t=1}^{n} \left(1 + \frac{(x_t - \phi x_{t-1})^2}{\nu} \right)^{-\left(\frac{\nu+1}{2}\right)}.$$

The corresponding log-likelihood function is,

$$\begin{aligned} L_n(\phi \mid x_0) = & -\left(\frac{\nu+1}{2}\right) \sum_{t=1}^{n} \log \left(1 + \frac{(x_t - \phi x_{t-1})^2}{\nu} \right) \\ & + n \log \Gamma\left(\frac{\nu+1}{2}\right) - n \log \sqrt{\nu\pi} - n \log \Gamma\left(\frac{\nu}{2}\right) \end{aligned} \tag{13}$$

The critical points of the above log likelihood function can be obtained by solving the score functions given by,

$$\frac{\partial L(\phi \mid x_0)}{\partial \phi} = 0 \Rightarrow \left(\frac{\nu+1}{2}\right) \sum_{t=1}^{T} 2x_{t-1} \frac{(x_t - \phi x_{t-1})}{\nu\left(1 + \frac{(x_t - \phi x_{t-1})^2}{\nu}\right)} = 0 \tag{14}$$

We obtain the estimator of ϕ numerically by using the Newton Raphson method. Once the estimate of ϕ is obtained, then the profile log likelihood function

$$\begin{aligned} L_n\left(\hat{\phi} \mid x_0\right) = & -\left(\frac{\nu+1}{2}\right) \sum_{t=1}^{n} \log \left(1 + \frac{\left(x_t - \hat{\phi} x_{t-1}\right)^2}{\nu} \right) \\ & + n \log \Gamma\left(\frac{\nu+1}{2}\right) - n \log \sqrt{\nu\pi} - n \log \Gamma\left(\frac{\nu}{2}\right) \end{aligned} \tag{15}$$

is maximized over ν to get $\hat{\nu}$. We maximize the above profile log likelihood function by using numerical methods for obtaining the estimate of ν.

Now let us consider the hypothesis $H_0 : \phi = 1$ against the alternative $H_1 : |\phi| < 1$, that is the time series $\{X_t\}$ was generated by a stationary model. Under H_0, the maximum value of the likelihood function is

$$L_0 = \left(\frac{\Gamma\left(\frac{\hat{\nu}+1}{2}\right)}{\sqrt{\hat{\nu}\pi}\Gamma\left(\frac{\hat{\nu}}{2}\right)} \right)^n \prod_{t=1}^{n} \left(1 + \frac{(x_t - x_{t-1})^2}{\hat{\nu}} \right)^{-\left(\frac{\hat{\nu}+1}{2}\right)}$$

and under the alternative, the maximum value of likelihood function is,

$$L_1 = \left(\frac{\Gamma\left(\frac{\hat{\nu}+1}{2}\right)}{\sqrt{\hat{\nu}\pi}\Gamma\left(\frac{\hat{\nu}}{2}\right)} \right)^n \prod_{t=1}^{n} \left(1 + \frac{\left(x_t - \hat{\phi} x_{t-1}\right)^2}{\hat{\nu}} \right)^{-\left(\frac{\hat{\nu}+1}{2}\right)}.$$

Hence, for $\hat{\phi} \in H_1$, the likelihood ratio test statistic is

$$-2\log\lambda = -2\log \prod_{t=1}^{n} \left(\frac{1 + \frac{\left(x_t - \hat{\phi} x_{t-1}\right)^2}{\hat{\nu}}}{1 + \frac{(x_t - x_{t-1})^2}{\hat{\nu}}} \right)^{\left(\frac{\hat{\nu}+1}{2}\right)}$$

$$= -2\left(\frac{\hat{\nu}+1}{2}\right)\left(\sum_{t=1}^{n} Log\left(1+\frac{\left(x_t-\hat{\phi}x_{t-1}\right)^2}{\hat{\nu}}\right)-\sum_{t=1}^{n}\log\left(1+\frac{(x_t-x_{t-1})^2}{\hat{\nu}}\right)\right). \qquad (16)$$

We reject H_0 if $-2log\lambda$ is either too small or too large.

4.1 Test for cointegration

We know that, if $\phi \to 1$ in (12) , then the time series contains a unit root. Here one can extend the idea of testing the presence of unit root for testing the presence of cointegration using the approach discussed in [2]. First, we have to confirm all the variables are integrated of same order and are non stationary in nature. Once the series are confirmed to be non stationary with same order of integration, we can test for the presence of cointegration by using the residuals from the fitted error correction model. Here the hypothesis of interest for testing the cointegration is, $H_0 : \delta = 0$ against $\delta \neq 0$ in the model (5 and 6). Once we obtain the parameter estimates from the data, we tests the residuals from the error correction model using the test procedure described below. We denote the residuals from the error correction model (5) and (6) by, $\hat{\eta}_{1t} = \Delta x_{1t} - \hat{\beta}\hat{\delta}\hat{z}_{t-1}$ and $\hat{\eta}_{2t} = \Delta x_{2t} + \hat{\delta}\hat{z}_{t-1}$, where $\hat{z}_{t-1} = x_{1,t-1} + \hat{\alpha}x_{2,t-1}$, $\hat{\alpha}$ and $\hat{\beta}$, $\hat{\delta}$ are the parameter estimates of α, β and δ . And, if the residuals obtained from the ECM are stationary, (ie; the null hypothesis of no cointegration is rejected) then we can conclude that the variables will be cointegrated.

Now we will obtain the likelihood ratio test statistic for testing the presence of cointegration in an error correction model. Under the null hypothesis, the maximum value of the likelihood function is,

$$L_0 = \prod_{t=1}^{n}\frac{1}{\sqrt{1-\hat{\rho}_0^2}}\left(1+\frac{1}{\hat{\nu}(1-\hat{\rho}_0^2)}\left(\Delta x_{1t}^2 - 2\hat{\rho}_0\Delta x_{1t}\Delta x_{2t} + \Delta x_{2t}^2\right)\right)^{-\frac{(\hat{\nu}+2)}{2}},$$

and under the alternative hypothesis, the maximum value of the likelihood function is,

$$L_1 = \prod_{t=1}^{n}\frac{1}{\sqrt{1-\hat{\rho}_1^2}}\left(\frac{1}{\hat{\nu}(1-\hat{\rho}_1^2)}\left(\left(A_1(\hat{\theta})\right)^2 - 2\hat{\rho}_1 A_1(\hat{\theta})B_1(\hat{\theta}) + \left(B_1(\hat{\theta})\right)^2\right)\right)^{-\frac{(\hat{\nu}+2)}{2}}.$$

The likelihood ratio test statistic obtained is given by,

$$\begin{aligned} log\left(L_1/L_0\right) &= n\log\left[\frac{1-\hat{\rho}_0^2}{1-\hat{\rho}_1^2}\right] \\ &- (\hat{\nu}+2)\sum_{t=1}^{n}\log\left(\frac{1+\frac{1}{\hat{\nu}(1-\hat{\rho}_1^2)}\left(\left(A_1(\hat{\theta})\right)^2 - 2\hat{\rho}_1 A_1(\hat{\theta})B_1(\hat{\theta}) + \left(B_1(\hat{\theta})\right)^2\right)}{1+\frac{1}{\hat{\nu}(1-\hat{\rho}_0^2)}\left(\Delta x_{1t}^2 - 2\hat{\rho}_0\Delta x_{1t}\Delta x_{2t} + \Delta x_{2t}^2\right)}\right). \end{aligned} \qquad (17)$$

Here we reject the null hypothesis of no cointegration if $log\left(L_1/L_0\right)$ is large in absolute.

To evaluate the performance of the estimators and test statistic, a simulation study is carried out, which is illustrated in section 5.

5 Simulation study for t distributed errors

Let us begin with a simulation study for the first order autoregressive model with Student's t-distributed errors. For the study, we generate the error random variable from a Student's t-distribution. Then for specified values of the model parameter, we simulated the sequence $\{x_t\}$, t=1,2,...,n using the relation described in (12). Based on this sample, we obtain the maximum likelihood estimates by solving the equation (14) and (15). For the given values of the model parameter, we repeated the experiment 100 times for computing the estimates and then averaged them over the repetitions. Using the estimate of ϕ, we obtained the maximum likelihood estimate of ν by maximizing the profile log likelihood function. Next we compute the likelihood ratio test statistic in equation (16) for various sample sizes and for different parameter values. Finally we compute the number of rejections in 500 trials for testing the null hypothesis of interest. The numerical computations are carried out for various value of the model parameter and are summarised in Tables 1 to 4.

Table 1. The average estimates and the corresponding root mean squares errors of the MLE for $\nu = 3$

n	ϕ	$\hat{\phi}$	$\hat{\nu}$
300	-0.5	-0.5058(0.0017)	3.0882(0.3451)
	-0.3	-0.2857(0.0038)	3.1144(0.3380)
	0.3	0.8991(0.0026)	3.1082(0.2145)
	0.5	0.4994(0.0022)	3.0703(0.2514)
	0.7	0.6929(0.0020)	3.0560(0.3270)
	0.9	0.8941(0.0007)	3.1036(0.2802)
500	-0.5	-0.430(0.0009)	3.0564(0.1084)
	-0.3	-0.3002(0.0012)	3.0130(0.1031)
	0.3	0.2992(0.0014)	3.0976(0.1511)
	0.5	0.5011(0.0012)	3.0391(0.1563)
	0.7	0.6987(0.0008)	3.0473(0.1489)
	0.9	0.9020(0.0003)	3.0724(0.1832)

Table 2. The average estimates and the corresponding root mean squares errors of the MLE for $\nu = 4$

n	ϕ	$\hat{\phi}$	$\hat{\nu}$
300	-0.5	-0.5055(0.0011)	4.0858(0.2152)
	-0.3	-0.3048(0.0037)	4.0941(0.3121)
	0.3	0.3005(0.0013)	4.0702(0.2112)
	0.5	0.4859(0.0021)	4.1221(0.1999)
	0.7	0.6999(0.0024)	4.0927(0.2100)
	0.9	0.8980(0.0006)	4.0854(0.1221)
500	-0.5	-0.4975(0.0015)	4.0521(0.1132)
	-0.3	-0.2973(0.0016)	4.0139(0.1523)
	0.3	0.2932(0.0008)	4.0891(0.1021)
	0.5	0.4926(0.0019)	4.1020 (0.1989)
	0.7	0.7004(0.0013)	4.0698(0.1654)
	0.9	0.8996(0.0003)	4.0541(0.0989)

Table 3. No of rejections in 500 trials of the hypothesis $H_0 : \phi = 1$ against $H_1 : \phi < 1$ using the test statistic given in (16) for different values of ϕ and $\nu = 3$

n	50			100			250			500		
level of significance	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
ϕ=-0.5	500	500	500	500	500	500	500	500	500	500	500	500
ϕ=-0.2	500	500	500	500	500	500	500	500	500	500	500	500
ϕ=0.5	500	500	500	500	500	500	500	500	500	500	500	500
ϕ=0.8	356	417	453	489	500	500	500	500	500	500	500	500
ϕ=0.9	184	209	324	365	427	461	452	473	492	499	500	500
ϕ=0.95	63	106	170	192	228	333	428	478	493	491	496	498

Next we carried out a simulation study to evaluate the performance of the error correction model based on bivariate Student's

Table 4. No of rejections in 500 trials of the hypothesis $H_0 : \phi = 1$ against $H_1 : \phi < 1$ using the test statistic given in (16) for different values of ϕ and $\nu = 4$

n	50			100			250			500		
level of significance	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
ϕ=-0.5	500	500	500	500	500	500	500	500	500	500	500	500
ϕ=-0.2	500	500	500	500	500	500	500	500	500	500	500	500
ϕ=0.5	500	500	500	500	500	500	500	500	500	500	500	500
ϕ=0.8	337	395	417	458	472	477	498	500	500	500	500	500
ϕ=0.9	167	236	265	328	385	396	409	435	438	498	499	500
ϕ=0.95	80	139	169	152	238	266	407	434	439	449	438	463

t-distributed errors. For the study, we generate a sample of size, say n, from the bivariate Student's t-distribution and then genrerate the ECM in (5) and (6). Finally we obtained the MLE of the parameters by solving the likelihood equations in (7) to (10). Using the estimated parameter values of $\rho, \alpha, \beta, \delta$, we maximized the profile log likelihood function in (11) and obtained the estimate of ν. We then repeated the experiment 100 times for computing the estimates and then averaged them over the repetitions. Once the parameters of ECM being estimated, we test for cointegration using the residuals from the error correction model. We use the test statistic given in (17) to compute the number of rejections of the null hypothesis under the various alternatives. Table 5 and 6 corresponds to the parameter estimates of the error correction model associated with t distribution with $\nu = 3$ and $\nu = 4$. Table 7 and 8 gives the number of rejections of the null hypothesis under the various alternatives.

Table 5. The average estimates and the corresponding root mean squared errors of MLE for $\nu = 3$

n	ρ	α	β	δ	$\hat{\rho}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\nu}$
150	-0.2	2	3	-0.6	-0.1599(0.0244)	2.0010(0.0068)	3.0048(0.0025)	-0.6013(0.0001)	2.6894(0.2216)
	-0.5	3	1	0.2	-0.4369(0.0084)	3.0036(0.0032)	1.0046(0.0044)	0.2016(0.0022)	2.6516(0.3512)
	0.2	1.8	0.5	1	0.1618(0.0088)	1.801(0.0001)	0.4992(0.0009)	1.0031(0.0005)	2.7346(0.2052)
	0.5	1.4	1.5	-7	0.4355(0.0089)	1.3999(0.0006)	1.5000(0.0009)	-6.7570(0.0121)	2.6910(0.2342)
	0.9	0.2	3.5	-0.18	0.8779(0.0007)	0.2038(0.0001)	3.4834(0.0140)	-0.1816(0.0001)	2.6817(0.2878)
300	-0.2	2	3	-0.6	-0.1772(0.0184)	1.9999(0.0031)	2.9997(0.0001)	-0.6001(0.00001)	2.8292(0.1165)
	-0.5	3	1	0.2	-0.4899(0.0053)	2.9917(0.0010)	0.9847(0.0034)	0.1985(0.0001)	2.8032(0.1621)
	0.2	1.8	0.5	1	0.1733(0.0048)	1.7999(0.0042)	0.5029(0.0003)	1.0006(0.0002)	2.8452(0.1123)
	0.5	1.4	1.5	-7	0.4426(0.0055)	1.3999(0.0005)	1.3999(0.0001)	1.4996(0.0002)	2.8012(0.2012)
	0.9	0.2	3.5	-0.18	0.8790(0.0006)	0.2017(0.0001)	3.4966(0.0073)	-0.1811(0.00009)	2.8321(0.1325)

Table 6. The average estimates and the corresponding root mean squared errors of MLE for $\nu = 4$

n	ρ	α	β	δ	$\hat{\rho}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\nu}$
150	-0.2	2	3	-0.6	-0.1907(0.0061)	2.0017(0.0002)	3.0080(0.0032)	-0.601(0.0001)	4.0538(0.2012)
	-0.5	3	1	0.2	-0.4974(0.0044)	3.0050(0.0020)	0.9813(0.0032)	0.2001(0.0002)	3.9830(0.2136)
	0.2	1.8	0.5	1	0.1933(0.0073)	1.8009(0.00002)	0.4945(0.0007)	0.9993(0.0004)	4.1559(0.2013)
	0.5	1.4	1.5	-7	0.4914(0.0047)	1.3999(0.0005)	1.4998(0.0001)	-7.0004(0.0001)	3.9764(0.1962)
	0.9	0.2	3.5	-0.18	0.8964(0.0002)	0.205(0.0001)	3.5130(0.0014)	-0.1822(0.00002)	4.0261(0.1032)
300	-0.2	2	3	-0.6	-0.2045(0.0028)	2.0013(0.0001)	3.0048(0.0016)	-0.6008(0.00006)	4.0025(0.1145)
	-0.5	3	1	0.2	-0.4940(0.0024)	2.9949(0.0017)	0.9985(0.0009)	0.1999(0.00009)	4.0706(0.1098)
	0.2	1.8	0.5	1	0.1869(0.0038)	1.8004(0.00001)	0.5005(0.0003)	0.9990(0.0001)	4.1018(0.1021)
	0.5	1.4	1.5	-7	0.4982(0.0020)	1.3999(0.00004)	1.5001(0.00005)	-7.0003(0.0060)	4.0778(0.1213)
	0.9	0.2	3.5	-0.18	0.9016(0.0003)	0.2026(0.00009)	3.4999(0.0040)	-0.1817(0.00001)	3.9870(0.1026)

Note that from Tables 1 and 3, for series of length 300, estimates are reasonably satisfactory and become more accurate with increasing sample size. From Tables 3, 4, 7 and 8, it can be seen that as ϕ becomes closer to 1, the number of rejections of the null hypothesis of unit root and no cointegration becomes smaller. For example, in a length of 50 series in Table 3, the hypothesis $H_0 : \phi = 0.95$ was rejected 63 times at the 0.01 significance level, while it was rejected 184 times when ϕ was 0.9. Hence we claim that the derived test statistic is powerful for testing the presence of unit root in an observed nonstationary time series.

Table 7. No of rejections in 500 trials of the hypothesis $H_0 : \phi = 1$ (δ=0) against the alternative of $\phi < 1$($\delta \neq 0$) for $\nu = 3$

n	50			100			150		
level of significance	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
ϕ=0.5	500	500	500	500	500	500	500	500	500
ϕ= 0.8	426	456	468	500	500	500	500	500	500
ϕ= 0.9	242	298	342	424	448	465	500	500	500
ϕ= 0.95	101	137	177	223	258	314	361	391	422
ϕ= 0.99	38	42	45	80	85	92	125	130	135

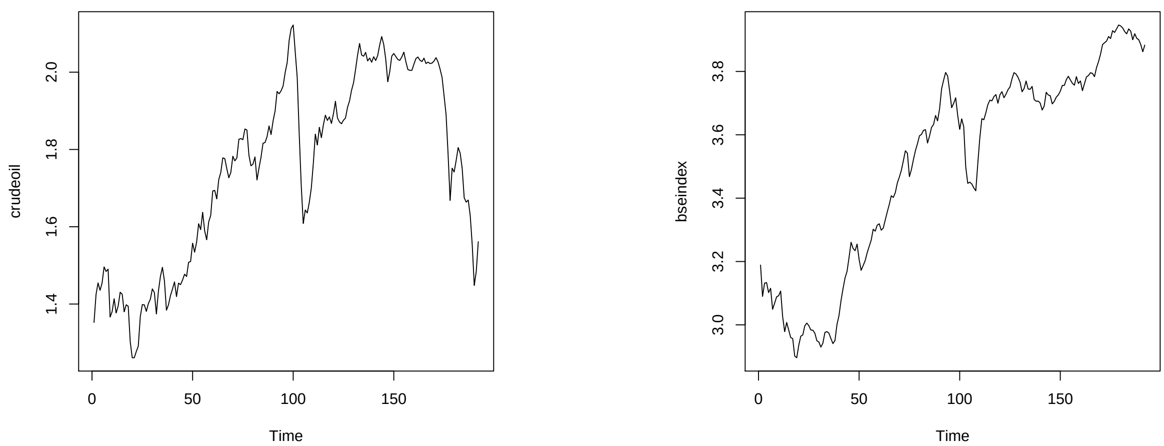
Table 8. No of rejections in 500 trials of the hypothesis $H_0 : \phi = 1$ (δ=0) against the alternative of $\phi < 1$($\delta \neq 0$) for $\nu = 4$

n	50			100			150		
level of significance	0.01	0.05	0.1	0.01	0.05	0.1	0.01	0.05	0.1
ϕ=0.5	500	500	500	500	500	500	500	500	500
ϕ= 0.8	440	452	500	500	500	500	500	500	500
ϕ= 0.9	285	312	350	451	468	481	500	500	500
ϕ= 0.95	111	132	170	218	241	311	350	381	410
ϕ= 0.99	42	46	50	85	98	101	128	140	144

6 Data Analysis

In this section, we illustrate the analysis of cointegration model with bivariate Student's t-errors with identical marginals using the real data set. The data set consists of 192 monthly observation of crude oil price and Bombay stock exchange index for the period 2000 to 2016. All the variables are transformed in to their natural logarithm. The data set is downloaded from the website of Ministry of Petroleum and Natural gas, Govt. of India and the website of Reserve Bank of India.

Figure 1 provides the time series plot of the log transformed data and it indicates that the time series is non stationary. We



(a) Time plot of crude oil price **(b)** Time plot of BSE index

Figure 1. Time series plot

performed a unit root test developed for Student's t error variables to the data set in order to test whether the series is stationary or not.

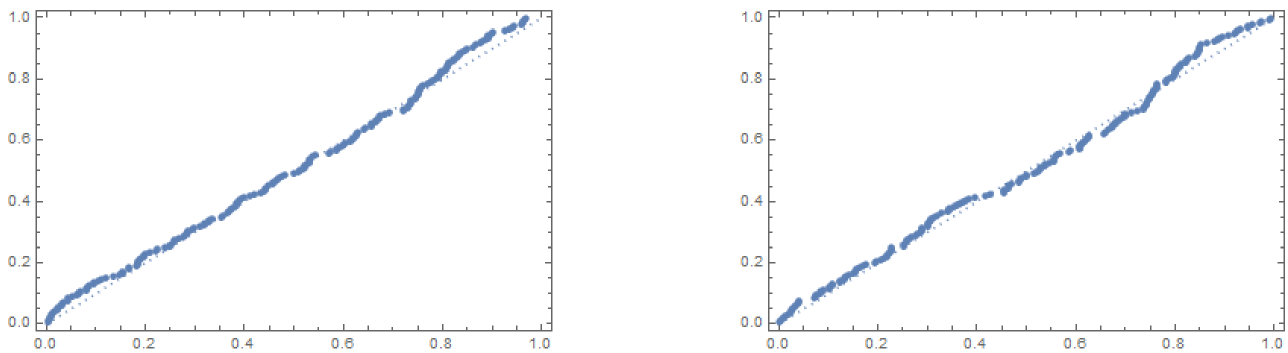
The test statistic value obtained for crude oil and BSE series are 0.00029 and 0.0015 respectively. From table 9, we can conclude that the null of hypothesis of unit root cannot be rejected at any given level of significance. So we perform a maximum likelihood estimation described in section 3 in order to find the parameter estimates of an error correction model of order 1. The parameter estimates are obtained as $\hat{\rho} = 0.2425$, $\hat{\alpha} = -2.8930$, $\hat{\beta} = -0.6960$ and $\hat{\delta} = 0.00207$ and $\hat{\nu} = 4.54$. The value of the test statistic obtained for the error correction equation model is -90 and from the student's t-table for two

Table 9. Emperical level of the test

Nominal level	0.2	0.1	0.05
Assymptotic Value	0.978	1.638	2.353

tailed test we can conclude that we reject the null hypothesis of no cointegration at 5 percent level of significance, implying that the residuals from the ECM are stationary. Thus the cointegrating vector parameter estimate provides an estimate of a long run relationship. That is, $X_{1t} - 2.893X_{2t}$ is the cointegrating relationship and the cointegrated vector is $[1, -2.893]^{'}$. Using the parameter estimates of the ECM, we tested whether the marginals of the residuals follow a Student's t-distribution using Kolmogorov-Smirnov test. The p values obtained for the series are 0.638 and 0.562 respectively, which indicates that Student's t-distribution is suitable for the residuals.

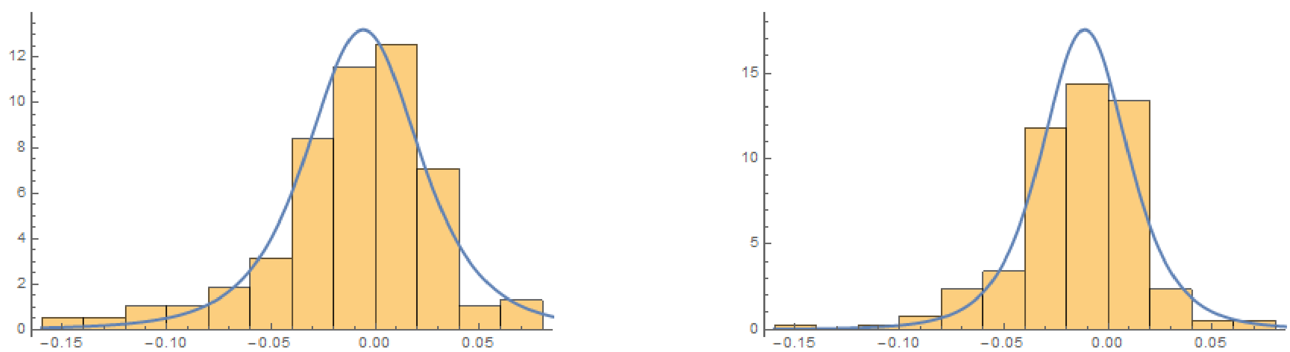
PP-plot and histogram for the marginals given in Figure 2 and 3 also confirms that the chosen distribution is suitable for the



(a) PP- plot of residuals of crude oil series

(b) PP- plot of residuals of BSE series

Figure 2. Monthly Rubber data



(a) Histogram of crude oil price with ν=3.69

(b) Histogram of BSE index with ν=4.35

Figure 3. Quarterly Oil price data

residual series.

Since we have assumed a bivariate student's t-distribution for the errors, it is also important to check whether the joint distribution of the residual series is following this assumption. We have examined the assumption of bivariate student's t-distribution for the residual series using a bivariate Kolmogorov Smirnov test [14]. Under the null hypothesis, where the residuals follows a bivariate student's t-distribution, the p value obtained for the test is 0.99, which indicates that bivariate student's t-distribution is suitable for modelling the joint residual series. We have also obtained the bivariate skewness and kurtosis measures to confirm whether the fitted distribution has heavy tails [15]. These measures are found to be 0.56 and 8 respectively, which confirms the presence of a heavy tailed distribution for the joint residual series.

7 Conclusions

We developed a bivariate cointegration model generated from a bivariate student's-t distributed errors. The estimation method and testing procedure for the presence of unit root and cointegration are developed. The model parameters are estimated using the method of maximum likelihood. Simulation experiments and data analysis are conducted to illustrate the applications of the proposed model.

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