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Explicit Formulas and Numerical Integral Equation of ARL for $SARX(P,r)_L$ Model Based on CUSUM Chart

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Abstract The Cumulative Sum (CUSUM) chart is widely used and has many applications in different fields such as finance, medical, engineering, and other fields. In real applications, there are many situations in which the observations of random processes are serially correlated, such as a hospital admission in the medical field, a share price in the economic field, or a daily rainfall in the environmental field. The common characteristic of control charts that has been used to evaluate the performance of control charts is the Average Run Length (ARL). The primary goals of this paper are to derive the explicit formula and develop the numerical integral equation of the ARL for the CUSUM chart when observations are seasonal autoregressive models with exogenous variable, $SARX(P,r)_L$ with exponential white noise. The Fredholm Integral Equation has been used for solving the explicit formula of ARL, and we used numerical methods including the Midpoint rule, the Trapezoidal rule, the Simpson's rule, and the Gaussian rule to approximate the numerical integral equation of ARL. The uniqueness of solutions is guaranteed by using Banach's Fixed Point Theorem. In addition, the proposed explicit formula was compared with their numerical methods in terms of the absolute percentage difference to verify the accuracy of the ARL results and the computational time (CPU). The results obtained indicate that the ARL from the explicit formula is close to the numerical integral equation with an absolute percentage difference of less than 1%. We found an excellent agreement between the explicit formulas and the numerical integral equation solutions. An important conclusion of this study was that the explicit formulas outperformed the numerical integral equation methods in terms of CPU time. Consequently, the proposed explicit formulas and the numerical integral equation have been the alternative methods for finding the ARL of the CUSUM control chart and would be of use in fields like biology, engineering, physics, medical, and social sciences, among others.

Keywords Explicit Formulas, Numerical Integral Equation, Seasonal Autoregressive with Exogenous Variable Model

1. Introduction

The main intention of statistical process control (SPC) is to provide a technique for improving productivity. Control charts are one of the efficient tools of SPC for detecting changes in the mean or variations in the process. The SPC charts, such as the Shewhart chart, were first introduced by Shewhart [1], the Cumulative Sum (CUSUM) chart, introduced by Page [2], and the Exponentially Weighted Moving Average (EWMA) chart, proposed by Roberts [3]. These are used to monitor product quality and to detect the occurrence of special causes that may be indicative of out-of-control situations. The literature concerning control of small shifts $(\delta < 1.5\sigma)$ is recommended, such as Hawkins and Olwell [4] and Lucas [5] to show that the CUSUM chart is more efficient than the Shewhart chart in terms of detection of small changes in the process means. However, the choice of control charts depends on the quality characteristics to be measured in the process. There are many situations in which the process is serially correlated, such as chemical processes. Consequently, the purpose of this paper is to study the Fredholm integral equations method to derive a closed-form solution of the Average Run Length for the CUSUM chart when observations are seasonal autoregressive with an exogenous variable; $SARX(P,r)_L$ model with exponential distribution white noise.

A time series is a sequence of data points that follows a sample over time with equally spaced time points, such as the hourly air temperature or the daily closing price of a stock's market capitalization. In this research, we will focus on a time series that is affected by seasonal factors such as the month of the year or the quarter of the year. Many time series have a seasonal pattern that occurs, such as in economics, business, or medicine. Several studies, for example, Etuk [6] presented a model of Nigerian GDP series based on a seasonal autoregressive integrated moving average (SARIMA) model. Similarly, Ayinde and Abdulwahab [7] proposed a predictive model for Nigerian monthly crude oil exportation data by using a SARIMA model. Later, Doguwa and Alade [8] used a SARIMA model and a seasonal autoregressive integrated moving average with exogenous variables (SARIMAX) model to identify the model of short-term headline inflation data, which found an autoregressive integrated moving average with exogenous variables model has the capacity to identify the underlying patterns in time series and to quantify the impact of environmental influences.

The Average Run Length (ARL) is the average number of subgroups taken until a control chart signals that the process is out of control. It is used to measure the performance of the control chart for detecting shifts in the mean. When the process is in an in-control state, the average number of samples taken from an in-control process until the control chart falsely signals out-of-control is called in-control ARL (ARL_0). While the process is out-of-control, the average number of samples that fall within the control limits before an alarm signal is given that the process is out-of-control. It is called out-of-control ARL (ARL_1). Also, there are several methods for estimating the ARL, such as the Monte Carlo simulations method (MC), the Markov Chain approach (MCA), and the Integral Equation approach (IE). The Monte Carlo simulation method is used for checking the accuracy of analytical results, but it is very time consuming to run. Roberts [3] used the Monte Carlo simulation method to estimate the ARL of the EWMA chart. Later, Crowder [9] used the Integral Equations approach to find the ARL when observations have a Gaussian distribution. After that, Lucas and Saccucci [10] evaluated the ARL of the EWMA chart by using a finite-state Markov chain approach. The limitations of the MCA and MC methods led this research to be interested in estimating the ARL by the Integral Equation method. This method has been discussed in the literature, e.g., Areepong and Novikov [11] proposed the analytical solution of the ARL and the average delay (AD) for the EWMA control chart when observations are of Exponential distribution. Recently, Mititelu et al. [12] proved the close form of ARL by the Fredholm Integral Equation for one-sided EWMA charts with Laplace distribution and the CUSUM chart with Hyperexponential distribution.

Generally, control charts are designed under the assumption that observations are independent and identically distributed. However, in some situations, the process is autocorrelated, such as in chemical processes, and thus it needs to be monitored by appropriate control charts. Several researchers, including Vanbrackle and Reynolds [13] proposed Integral Equation methods for evaluating the ARL of control chart when the process is serially correlated. They evaluated the ARL of EWMA and CUSUM charts by using an Integral Equation method and a Markov Chain approach when the observations are from an AR(1) model with additional random error. Later, Busaba et al. [14] proposed the analytical solution of ARL for the CUSUM chart in the case of the observations being a stationary AR(1) model. After that, Petcharat et al. [15] derived explicit formulas of ARL for EWMA and CUSUM charts when observations are the moving average (MA) model of order q with exponential white noise. The value of q is called the order of the MA model. Later, Phanyaem et al. [16] presented the explicit formulas of ARL for the CUSUM chart when observations are from an ARMA(1,1) model. In addition, Phanyaem [17] proposed the explicit formulas for the ARL of the CUSUM chart based on the $SARMA(1,1)_L$ model. Recently, Piyapatr and Lili [18] presented the explicit formulas of the ARL of the CUSUM chart for an autoregressive integrated moving average; ARIMA(p,d,q) model with an exponential white noise and compared it with the numerical integration.

The invention of an explicit formula for the seasonal autoregressive with an exogenous variable, the $SARX(P,r)_L$ model of the CUSUM control chart, which has yet to be established by any researcher, is the originality provided. Furthermore, the results were also compared to those obtained using the four numerical integral approaches. The rest of the paper is organized as follows: The materials and methods are presented in Section 2. The results from the explicit formula and the numerical integral equation are proposed in Section 3. Finally, in Section 4, we provided the conclusion.

2. Materials and Methods

In this section, we present the characteristics of the CUSUM chart, which was first introduced by Page [2] in 1954 and is widely used and powerful tool for monitoring and detecting the mean in the process.

2.1. CUSUM Chart for $SARX(P,r)_L$ Model

The sequential observations $Y_1, Y_2, Y_3, \ldots$ in this paper are those from a seasonal autoregressive with an exogenous variable, the SARX(P,r)$_L$ model with exponential white noise. The CUSUM statistics at the time t is denoted by C_t defined as follows:

$$C_t = \max(C_{t-1} + Y_t - a,\ 0); \quad t = 1,2,... \tag{1}$$

where Y_t is a sequence of the SARX(P,r)$_L$ model, $C_0 = u$ is an initial value, and a is a reference value of the CUSUM chart.

The following recursion describes the general form of the SARX(P,r)$_L$ model.

$$Y_t = \sum_{i=1}^{r} \beta_i X_{it} + \mu + \phi_1 Y_{t-L} + ... + \phi_P Y_{t-PL} + \varepsilon_t \tag{2}$$

where X_t is an exogenous variable,

β_i is a coefficient of X_t,

ϕ_i is an autoregressive coefficient, i = 1, 2,…, P

ε_t is an exponential white noise.

Let $Y_{t-L}, Y_{t-2L}, \ldots, Y_{t-PL}$ be the SARX(P,r)$_L$ model's initial values. The stopping time of the CUSUM control chart is given by

$$\tau_b = \inf\{t > 0;\ C_t > b\}, \tag{3}$$

where τ_b is the stopping time

b is the constant parameter as the upper control limit.

2.2. Explicit Formula of ARL for SARX(P,r)$_L$ Model

This section is devoted to our analytical derivation as applied to the CUSUM chart for the SARX(P,r)$_L$ model with an exponential distribution of white noise. We use Integral Equation to find the analytical explicit formula of ARL for the CUSUM chart and compare the results to the numerical integral equation (NIE) method. We assume that the lower control limit is 0 and the upper control limit is b.

Suppose $H(u)$ denotes the ARL for the SARX(P,r)$_L$ model with an initial value of $C_0 = u$. As a result, we define the function $H(u)$ as follows:

$$ARL = H(u) = \mathbb{E}_\infty(\tau_b) < \infty. \tag{4}$$

where $\mathbb{E}_\infty$ (.) is the expectation under the density function $f(x,\alpha)$.

The solution of the integral equation is as follows:

$$H(u) = 1 + \mathbb{E}_C\left[I\{0 < C_1 < b\}H(C_1)\right] + \mathbb{P}_C\{C_1 = 0\}H(0).$$

To extend the function into the Fredholm Integral Equations of the second kind.

$$H(u) = 1 + H(0)F(a-u-Y_t) + \int_0^b L(y) f(y+a-u-Y_t)dy.$$

The ARL of the CUSUM chart is:

$$H(u) = 1 + \alpha e^{\alpha(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL})} \int_0^b H(y)e^{-\alpha y}dy$$

$$+(1-e^{-\alpha(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})})H(0) \tag{5}$$

In the next step, we check the uniqueness of the solution by using Banach's Fixed Point Theorem.

Banach's Fixed Point Theorem

Let $(\mathbb{M}, d)$ be a complete metric space and the mapping $T:\mathbb{M}\to\mathbb{M}$ be a *contraction*, then T is unique on Fixed Point. In other words, the Banach's Fixed Point theorem states that for a contractive mapping T on a complete metric space, there exists a unique solution to the fixed point equation $T(u) = u$.

Proof.

To prove that T is a contraction. We have the inequality for any $u \in I$ and $H_1, H_2 \in C(I)$.

$$\|T(H_1) - T(H_2)\| \le q\|H_1 - H_2\|$$

where q is a positive constant.

According to the operator T, we can write as:

$$T(H(u)) = 1 + \alpha e^{\alpha(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL})} \int_0^b H(y)e^{-\alpha y}dy$$

$$+(1-e^{-\alpha(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})})H(0)$$

So, we have that;

$$\|T(H_1)-T(H_2)\|_1 = \sup_{u\in I}\Big|H_1(0)-H_2(0)(1-e^{-\alpha(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})})$$

$$+\alpha e^{\alpha(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL})} \int_0^b \left(H_1(y)-H_2(y)\right)e^{-\alpha y}\,dy\Big|$$

$$\le \sup_{u\in I}\|H_1(0)-H_2(0)\|_1 (1-e^{-\alpha(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})})$$

$$+\|H_1-H_2\|_1 \alpha e^{\alpha(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL})} \int_0^b e^{-\alpha y}\,dy\Big|$$

$$= \|H_1-H_2\|_1 \sup_{u\in I}(1-e^{-\alpha(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})-\alpha b})$$

$$= \|H_1-H_2\|_1 (1-e^{-\alpha(-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})-\alpha b})$$

$$= q_1\|H_1-H_2\|_1$$

where $q = (1-e^{-\alpha(-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}+b)}) < 1.$

By the triangular inequality;

$$|H_1(0)-H_2(0)| \le \sup_{x\in[0,b]}|H_1(u)-H_2(u)| = \|H_1-H_2\|.$$

So that, $\|T(H_1)-T(H_2)\| \leq q\|H_1-H_2\|$, the operator T is a contraction. According to Banach's Fixed Point Theorem if the operator T is contraction, the fixed point equation $T(H(u))=H(u)$ has a unique solution.

Consequently, we derive the explicit formula of ARL for the CUSUM chart when observations are $\text{SARX}(P,r)_L$ model with exponential white noise.

Firstly, we let d be a constant as follows:

$$d=\int_0^b H(y)e^{-\alpha y}dy.$$

Thus, the function $H(u)$ can be written as

$$H(u)=1+\alpha e^{\alpha(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL})}d$$
$$+(1-e^{-\alpha(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})})H(0). \quad (6)$$

For $u=0,$ thus we have $H(0)$ in the following form:

$$H(0)=1+\alpha e^{\alpha(-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL})}d$$
$$+(1-e^{-\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})})H(0)$$

Hence, substituting $H(0)$ into equation (6) as follows:

$$H(u)=1+\alpha d+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})}-e^{\alpha u}. \quad (7)$$

Define the constant d ;

$$d=\int_0^b H(y)e^{-\alpha y}dy.$$

$$=\frac{e^{\alpha b}}{\alpha}(1-e^{-\alpha b})(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})})-be^{\alpha b}$$

Substitute the following constant d into (7):

$$H(u)=e^{\alpha b}(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})}-\alpha b)-e^{\alpha u}.$$

Consequently, we obtain the explicit formulas of ARL by solving the Integral Equations for the in-control state, where the parameter $\alpha=\alpha_0$. Thus, the explicit formula for ARL_0 is

$$ARL_0=e^{\alpha_0 b}(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})}-\alpha_0 b)-e^{\alpha_0 u}.$$

On the other hand, the process is in an out-of-control state, the parameter $\alpha=\alpha_1$; where $\alpha_1=\alpha_0(1+\delta)$. The explicit formula for ARL_1 can be written as follows:

$$ARL_1=e^{\alpha_1 b}(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})}-\alpha_1 b)-e^{\alpha_1 u}.$$

where $0\leq\phi_i\leq 1$ is autoregressive coefficient; $i=1,2,\ldots,P$ and α is a parameter of the exponential distribution, X_{it} is an exogenous variable, β_i is the coefficient of X_{it} and b is the constant parameter as the upper control limit.

2.3. Numerical Integration of ARL for $\text{SARX}(P,r)_L$ Model

Sometimes, the integral equations are difficult to solve. For this reason, the methods of numerical integration are used to solve such problems. In this section, the numerical integral equation method called the NIE method, is introduced by Champ and Rigdon [19]. In this paper, we analyze four different approaches to finding an approximation of an integral equation, including the Midpoint rule, the Trapezoidal rule, the Simpson's rule, and the Gaussian rule.

Let $y \sim Exp(\alpha)$ and $f(y)$ be the probability density functions of the exponential distribution, and $F(y)$ be the cumulative density function of the exponential distribution.

$$F(y)=1-e^{-\alpha y}$$

$$\text{and} \quad f(y)=\frac{dF(y)}{du}=\alpha e^{-\alpha y}.$$

All of these numerical approximation methods consist of dividing the interval [0, b] on the same m subinterval and calculating the area of the function for each subinterval using specific formulas.

There are a few basic methods of numerical integration, differentiating in a way of approximation as follows:

2.3.1. Midpoint Rule

Let the function $f(y)$ be over the finite interval $[0,b]$. The value $W(y)=1$ is chosen and a set of equally spaced points is used. The interval $[0,b]$ is subdivided into m subintervals $\{[y_{k-1},y_k],k=1,2,...,m\}$ of equal width h = b/m by using equally spaced points $y_k=y_0+kh$ for $k=1,2,...,m$ where $y_0=0$ and $y_m=b.$ In the case of the midpoint rule, the set of nodes is chosen as the midpoints m_k of the subintervals. Midpoints are given by

$$m_k=a+(k-\frac{1}{2})b.$$

The weights w_k for each midpoints were chosen to be 1. On each subinterval, the midpoint rule will integrate a constant function exactly. For integrating a general function $f(y)$ over an interval $[0,b]$, the composite midpoint rule for subintervals is obtained by combining the rules for the subintervals and the result is:

$$M(f,h)=h\sum_{k=1}^{m} f\left(a+\left(k-\frac{1}{2}\right)h\right).$$

Thus, the composite midpoint rule approximation for the integral is given by:

$$\int_0^b f(y)dy \approx h\sum_{k=1}^{m} f\left(a+\left(k-\frac{1}{2}\right)h\right).$$

2.3.2. Trapezoidal Rule

For the trapezoidal rule, the function $W(y)=1,$ the interval of integration $[0,b]$ is finite and a set of points is equally spaced. Thus, the interval is subdivided into m subintervals

$\{[y_{k-1},y_k],\ k=1,2,...,m\}$ of equal width $h=\dfrac{b-0}{2m}$ by using equally space points $y_k=y_0+kh$ for $k=0,1,\ ...,\ m$ where

$y_0=0$ and $y_m=b.$ The weights are chosen so that, on each subinterval of width $h,$ a polynomial of degree 1 will be integrated exactly by the rule. By combining the rules, this option provides equal width $h/2$ at the end points of a subinterval. The composite rule is:

$$T(f,h)=\frac{h}{2}(f(0)-f(b))+h\sum_{k=1}^{m} f(y_k).$$

Thus, the composite trapezoidal rule approximation for the integral is given by:

$$\int_0^b f(y)dy \approx \frac{h}{2}(f(0)-f(b))+h\sum_{k=1}^{m} f(y_k).$$

2.3.3. Simpson's Rule

For Simpson's rule, the function $W(y)=1$, the interval of integration $[0,b]$ is finite and the set of points is equally spaced. The interval is subdivided into $2m$ subintervals $[y_{k-1},y_k],\quad k=1,2,....,2m$ of equal width $h=b/2m$ by using equally spaced points $y_k=y_0+kh$ for $k=1,2,...,2m$ where $y_0=0$ and $y_{2m}=b.$ The composite Simpson's rule is obtained by combining the rules for the subintervals of width $2h.$ The composite rule is

$$S(f,h)=\frac{h}{3}(f(0)-f(b))+\frac{2h}{3}\sum_{k=1}^{m} f(y_{2k})+\frac{4h}{3}\sum_{k=1}^{m} f(y_{2k-1}).$$

Thus, the composite Simpson's rule approximation for the integral is given by:

$$\int_0^b f(y)dy \approx \frac{h}{3}(f(0)-f(b))+\frac{2h}{3}\sum_{k=1}^{m} f(y_{2k})+\frac{4h}{3}\sum_{k=1}^{m} f(y_{2k-1}).$$

2.3.4. Gaussian Rule

In Gaussian rules, the integration interval can be infinite, the weight function $W(y)$ might not equal 1, and the set of points $\{y_k, k=1,2,...,n\}$ is equally spaced. Gaussian of the following form:

$$\int_0^b W(y)f(y)dy \approx \sum_{k=1}^{m} w_k f(a_k)$$

where a_k is a set of point, $0\le\ a_1\le a_2\le ...\le a_m\le b$ and w_k is a set of constant weight, $w_k\ =b/m\ge 0.$

The numerical approximation to the integral equation is denoted by $\tilde{H}(a_i),$ which can be written as:

$$\tilde{H}(a_i)=1+\tilde{H}(0)\,\mathrm{F}(a-a_i-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

$$+\sum_{j=1}^{m} w_j\tilde{H}(a_j)f(a_j+a-a_i-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

A system of m linear equations $\tilde{H}(a_1),\tilde{H}(a_2),..,\tilde{H}(a_m),$ can be written as

$$\tilde{H}(a_1)=1+\tilde{H}(a_1)[\mathrm{F}(a-a_1-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

$$+\ w_1 f(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})]$$

$$+\sum_{j=2}^{m} w_j\tilde{H}(a_j)f(a_j+a-a_1-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

$$\tilde{H}(a_2)=1+\tilde{H}(a_1)[\mathrm{F}(a-a_2-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

$$+w_1 f(a_1+a-a_2-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})]$$

$$+\sum_{j=2}^{m} w_j\tilde{H}(a_j)f(a_j+a-a_2-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

$$\vdots$$

$$\tilde{H}(a_m)=1+\tilde{H}(a_1)[\mathrm{F}(a-a_m-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

$$+\ w_1 f(a_1+a-a_m-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})]$$

$$+\sum_{j=2}^{m} w_j\tilde{H}(a_j)f(a_j+a-a_m-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL})$$

It can be rewritten in matrix form as follows:

$$\mathbf{H}_{m\times 1}=\mathbf{1}_{m\times 1}+\mathbf{R}_{m\times m}\mathbf{H}_{m\times 1}$$

where $\mathbf{H}_{m\times 1}=\begin{pmatrix}\tilde{H}(a_1)\\ \tilde{H}(a_2)\\ \vdots\\ \tilde{H}(a_m)\end{pmatrix}$, $\quad\mathbf{1}_{m\times 1}=\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$

and $\mathbf{I}_m$ = $diag(1,1,…,1)$ is the unit matrix of order m. If there exists an $(\mathbf{I}_m - \mathbf{R}_{mxm})^{-1}$, the solution of the matrix equation is as follows:

$$\mathbf{H}_{m\times 1}=(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}.$$

Therefore, the numerical integration of ARL for the CUSUM chart based on the $\mathrm{SARX(P,r)_L}$ model is as follows:

$$\tilde{H}(u) = 1 + \tilde{H}(a_1)[F(a - u - \mu - \sum_{i=1}^{r}\beta_i X_{it} - \phi_1 Y_{t-L} - ... - \phi_P Y_{t-PL})$$
$$+ w_1 f(a_1 + a - u - \mu - \sum_{i=1}^{r}\beta_i X_{it} - \phi_1 Y_{t-L} - ... - \phi_P Y_{t-PL})]$$
$$+ \sum_{j=2}^{m} w_j \tilde{H}(a_j) f(a_j + a - u - \mu - \sum_{i=1}^{r}\beta_i X_{it} - \phi_1 Y_{t-L} - ... - \phi_P Y_{t-PL}) \quad (8)$$

3. Results

In this section, we present the results obtained from the explicit formulas of ARL for the CUSUM control chart when observations are the seasonal autoregressive with exogenous variables, $SARX(P,r)_L$ model. The explicit formula is compared with numerical integral equation (NIE) methods using the midpoint rule with 500 nodes to verify which method is better with the absolute percentage difference and the CPU times.

The absolute percentage difference can be computed as follows:

$$Diff\,(\%) = \frac{\left|ARL_{Explicit\ Formulas} - ARL_{NIE}\right|}{ARL_{Explicit\ Formulas}} \times 100.$$

where $ARL_{Explicit\ Formulas}$ is the ARL from explicit formula

ARL_{NIE} is the ARL from NIE method.

In these simulation studies, we also computed at ARL_0 = 370 and 500, and determined the exponential white noise parameter when the process is in an in-control state, given the in-control parameter $\alpha = \alpha_0 = 1$. The process is out-of-control state, given the out-of-control parameter $\alpha = \alpha_1 = \alpha_0\,(1+\delta)$ where δ = 0.00, 1.50, 1.60, 1.70, 1.80, 1.90, 2.00, 2.50, and 3.00 respectively. Table 1-2 present the parameters of reference values a and the value of the upper control limit b of the CUSUM chart based on the $SARX(1,1)_4$, $SARX(2,1)_4$ and $SARX(3,1)_4$ models when ARL_0 = 370 and ARL_0 = 500, respectively. The absolute percentage difference, Diff(%), and the CPU time are used to compare the performance of the explicit formula and the NIE methods for the CUSUM chart. From Table 1-2, the absolute percentage difference is up to 0.3% by the NIE method for the case of the division points m = 500, and the CPU times are approximately 11-14 minutes, whereas the CPU times from the explicit formula are less than 1 second.

Table 3-4 show the ARLs on the $SARX(2,1)_4$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$ and $\beta_1 = 0.1$ by using the explicit formula and the NIE methods, including the Midpoint rule, Trapezoidal rule, Simpson's rule, and Gaussian rule on the CUSUM chart with the reference values a = 2.50 and the value of the upper control limit b = 4.151 for ARL_0 = 370 and a = 3.00, and b = 3.723 for ARL_0 = 500, respectively. The results from Table 3-4, show that the ARL from the explicit formulas and the NIE methods are slightly different, and the CPU time from the explicit formula is much faster than the NIE methods for all situations.

Table 5-6 show the ARLs on the $SARX(2,2)_4$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$ and $\beta_1 = 0.1$, $\beta_2 = 0.1$ by using the explicit formula and the NIE methods, including the Midpoint rule, Trapezoidal rule, Simpson's rule and Gaussian rule on the CUSUM chart with the reference values a = 3.00 and the value of the upper control limit b = 3.529 for ARL_0 = 370 and a = 2.50, and b = 4.731 for ARL_0 = 500, respectively. It can be seen that the ARLs from the explicit formula are close to the NIE methods and the explicit formula can reduce in the CPU time better than the NIE methods.

Finally, Table 7-8 show the ARLs on the $SARX(3,2)_4$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\phi_3 = 0.10$, and $\beta_1 = 0.1$, $\beta_2 = 0.1$ by using the explicit formula and the NIE methods, including the Midpoint rule, Trapezoidal rule, Simpson's rule, and Gaussian rule on the CUSUM chart with the parameter values for ARL_0 = 370 being a = 2.50 and b = 4.585, and the parameter values for ARL_0 = 500 are a = 3.00, and b = 4.004. The results from Table 7-8, show that the ARL from the explicit formulas and the NIE methods are slightly different and the CPU time from the explicit formula is less than the NIE methods.

In addition, we compare the ARL_1 of the CUSUM chart based on the $SARX(2,1)_4$ model by using the explicit formula and the NIE methods with ARL_0 = 370 in Fig. 1A and ARL_0 = 500 in Fig. 1B. The results show that the ARL_1 from the explicit formula is close to the NIE methods.

In Fig. 2A and Fig. 2B, we compare the ARL_1 of the CUSUM chart based on the $SARX(2,2)_4$ model by using the explicit formula and the NIE methods, given ARL_0 = 370 and ARL_0 = 500, respectively. The results show that the ARL_1 from the explicit formula and the NIE methods differ only slightly.

In Fig. 3A and Fig. 3B, we compare the ARL_1 of the CUSUM chart based on the $SARX(3,2)_4$ model by using the explicit formula and the NIE methods, given ARL_0 = 370 and ARL_0 = 500, respectively. The results show that the ARL_1 from the explicit formula is close to the NIE methods.

Table 1. Comparisons of ARL_0 of $SARX(P,r)_L$ model for the CUSUM chart using the explicit formula and the NIE methods for $ARL_0 = 370$

Model: SARX(1,1)$_4$ model with $\phi_1 = 0.10$, $\beta_1 = 0.1$				
a	*b*	Explicit Formula	NIE	Diff(%)
2.50	3.976	370.267 (0.01)	368.999 **(11.68)**[a]	0.342
3.00	3.270	370.236 (0.01)	369.096 **(12.90)**	0.308
Model: SARX(2,1)$_4$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\beta_1 = 0.1$				
a	*b*	Explicit Formula	NIE	Diff(%)
2.50	4.151	370.267 (0.01)	368.929 (12.51)	0.361
3.00	3.397	370.195 (0.01)	369.02 (11.69)	0.317
Model: SARX(3,1)$_4$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\phi_3 = 0.10$, $\beta_1 = 0.1$				
a	*b*	Explicit Formula	NIE	Diff(%)
2.50	4.349	370.136 (0.01)	368.778 (11.71)	0.367
3.00	3.529	370.045 (0.01)	368.836 (13.02)	0.327

[a.] The values in parentheses are CPU times (Minutes)

Table 2. Comparisons of ARL_0 of $SARX(P,r)_L$ model for the CUSUM chart using the explicit formula and the NIE methods for $ARL_0 = 500$

Model: SARX(1,1)$_4$ model with $\phi_1 = 0.10$, $\beta_1 = 0.1$				
a	*b*	Explicit Formula	NIE	Diff(%)
2.50	4.326	500.158 (0.01)	498.209 (11.39)	0.390
3.00	3.592	500.224 (0.01)	498.523 (12.26)	0.340
Model: SARX(2,1)$_4$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\beta_1 = 0.1$				
a	*b*	Explicit Formula	NIE	Diff(%)
2.50	4.515	500.773 (0.01)	498.775 (12.82)	0.399
3.00	3.723	500.429 (0.01)	498.676 (11.33)	0.350
Model: SARX(3,1)$_4$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\phi_3 = 0.10$, $\beta_1 = 0.1$				
a	*b*	Explicit Formula	NIE	Diff(%)
2.50	4.731	500.270 (0.01)	498.234 (14.39)	0.407
3.00	3.859	500.136 (0.01)	498.335 (12.81)	0.360

[a.] The values in parentheses are CPU times (Minutes)

Table 3. Comparison of ARL computed using explicit formulas against the numerical integral equation for the SARX(2,1)$_4$ model given ARL$_0$ = 370, ϕ_1 = 0.10, ϕ_2 = 0.10, β_1 = 0.1, a = 2.5 and b = 4.151

δ	Explicit Formula	Numerical Integration Equation			
		Midpoint Rule	**Simpson Rule**	**Trapezoidal Rule**	**Gaussian Rule**
0.00	370.267 (0.01)	370**.000** (11.68)	370**.000** (106.03)	370**.000** (14.11)	370**.000** (12.07)
1.50	7.718 (0.01)	7.917 (12.66)	7.891 (89.56)	7.963 (14.18)	8.025 (12.25)
1.60	7.090 (0.01)	7.262 (12.05)	7.240 (106.48)	7.300 (14.17)	7.351 (11.12)
1.70	6.560 (0.01)	6.709 (11.85)	6.691 (101.86)	6.741 (14.36)	6.784 (11.18)
1.80	6.108 (0.01)	6.237 (11.87)	6.222 (102.87)	6.264 (13.75)	6.300 (11.01)
1.90	5.718 (0.01)	5.831 (11.96)	5.818 (96.43)	5.855 (14.45)	5.885 (11.02)
2.00	5.380 (0.01)	5.479 (11.66)	5.468 (89.07)	5.499 (14.33)	5.526 (11.74)
2.50	4.199 (0.01)	4.253 (12.57)	4.247 (106.71)	4.264 (13.44)	4.278 (12.69)
3.00	3.502 (0.01)	3.533 (12.38)	3.530 (102.98)	3.540 (12.96)	3.548 (13.07)

[a] The values in parentheses are CPU times (Minutes)

Table 4. Comparison of ARL computed using explicit formulas against the numerical integral equation for the SARX(2,1)$_4$ model given ARL$_0$ = 500, ϕ_1 = 0.10, ϕ_2 = 0.10, β_1 = 0.1, a = 3.0 and b = 3.723

δ	Explicit Formula	Numerical Integration Equation			
		Midpoint Rule	**Simpson Rule**	**Trapezoidal Rule**	**Gaussian Rule**
0.00	500.429 (0.01)	500.000 (11.33)	500.000 (98.27)	500.000 (13.70)	500.000 (14.42)
1.50	9.395 (0.01)	9.387 (13.84)	9.357 (117.77)	9.428 (13.75)	9.476 (14.25)
1.60	8.550 (0.01)	8.543 (13.15)	8.519 (116.05)	8.577 (13.95)	8.617 (14.62)
1.70	7.840 (0.01)	7.835 (12.94)	7.815 (107.90)	7.863 (14.10)	7.897 (14.11)
1.80	7.239 (0.01)	7.234 (12.86)	7.217 (105.91)	7.258 (1**4**.21)	7.286 (14.54)
1.90	6.724 (0.01)	6.720 (13.55)	6.705 (106.73)	6.750 (13.66)	6.765 (**14.42**)
2.00	6.279 (0.01)	6.275 (13.44)	6.260 (110.16)	6.293 (14.56)	6.314 (1**4**.96)
2.50	4.751 (0.01)	4.749 (12.97)	4.742 (108.69)	4.758 (14.65)	4.796 (14.28)
3.00	3.873 (0.01)	3.872 (12.96)	3.868 (94.67)	3.878 (14.99)	3.884 (14.30)

[a] The values in parentheses are CPU times (Minutes)

Table 5. Comparison of ARL computed using explicit formulas against the numerical integral equation for the $SARX(2,2)_4$ model given $ARL_0 = 370$, $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\beta_1 = 0.1$, $\beta_2 = 0.1$, $a = 3.0$ and $b = 3.529$

δ	Explicit Formula	Numerical Integration Equation			
		Midpoint Rule	**Simpson Rule**	**Trapezoidal Rule**	**Gaussian Rule**
0.00	370.045 (0.01)	370.000 **(14.82)**	370.000 **(103.11)**	370.000 **(14.64)**	370.000 **(15.40)**
1.50	8.426 (0.01)	8.420 **(14.53)**	8.397 **(116.98)**	8.452 **(14.79)**	8.490 **(15.42)**
1.60	7.705 (0.01)	7.700 **(13.84)**	7.681 **(106.11)**	7.727 **(14.78)**	7.758 (15.00)
1.70	7.097 (0.01)	7.092 **(13.85)**	7.076 **(106.97)**	7.115 **(15.07)**	7.141 **(15.39)**
1.80	6.132 (0.01)	6.574 **(13.71)**	6.561 **(107.80)**	6.594 **(14.85)**	6.616 **(15.67)**
1.90	6.133 (0.01)	6.129 **(13.94)**	6.118 **(101.79)**	6.146 **(15.01)**	6.166 (15.42)
2.00	5.746 (0.01)	5.743 **(14.33)**	5.733 **(126.75)**	5.758 **(14.75)**	5.775 **(15.06)**
2.50	4.408 (0.01)	4.406 **(14.54)**	4.401 **(105.30)**	4.414 (14.43)	4.423 **(16.44)**
3.00	3.629 (0.01)	3.628 **(14.66)**	3.625 **(121.62)**	3.633 (14.75)	3.639 **(14.98)**

[a.] The values in parentheses are CPU times (Minutes)

Table 6. Comparison of ARL computed using explicit formulas against the numerical integral equation for the $SARX(2,2)_4$ model given $ARL_0 = 500$, $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\beta_1 = 0.1$, $\beta_2 = 0.1$, $a = 2.5$ and $b = 4.731$

δ	Explicit Formula	Numerical Integration Equation			
		Midpoint Rule	**Simpson Rule**	**Trapezoidal Rule**	**Gaussian Rule**
0.00	500.270 (0.01)	500.000 **(14.65)**	500.000 **(112.20)**	500.000 **(14.84)**	500.000 **(14.95)**
1.50	7.957 (0.01)	7.954 **(14.12)**	7.921 **(135.90)**	8.044 **(14.99)**	8.174 **(14.92)**
1.60	7.311 (0.01)	7.309 **(12.94)**	7.281 **(129.12)**	7.382 **(14.72)**	7.488 **(14.84)**
1.70	6.767 (0.01)	6.765 **(13.22)**	6.742 **(123.29)**	6.826 **(15.05)**	6.914 **(15.89)**
1.80	6.304 (0.01)	6.302 **(13.49)**	6.283 **(117.06)**	6.353 **(15.48)**	6.426 **(16.52)**
1.90	5.905 (0.01)	5.904 **(14.06)**	5.887 **(121.22)**	5.947 **(15.32)**	6.009 **(16.45)**
2.00	5.559 (0.01)	5.558 **(14.01)**	5.543 **(112.61)**	5.595 **(14.98)**	5.467 **(16.34)**
2.50	4.351 (0.01)	4.350 **(14.36)**	4.342 **(104.05)**	4.369 **(15.05)**	4.396 **(14.81)**
3.00	3.636 (0.01)	3.636 **(14.76)**	3.631 **(105.29)**	3.647 **(15.26)**	3.662 (15.13)

[a.] The values in parentheses are CPU times (Minutes)

Table 7. Comparison of ARL computed using explicit formulas against the numerical integral equation for the SARX(3,2)$_4$ model given ARL$_0$ = 370, $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\phi_3 = 0.10$, $\beta_1 = 0.1$, $\beta_2 = 0.1$, $a = 2.5$ and $b = 4.585$

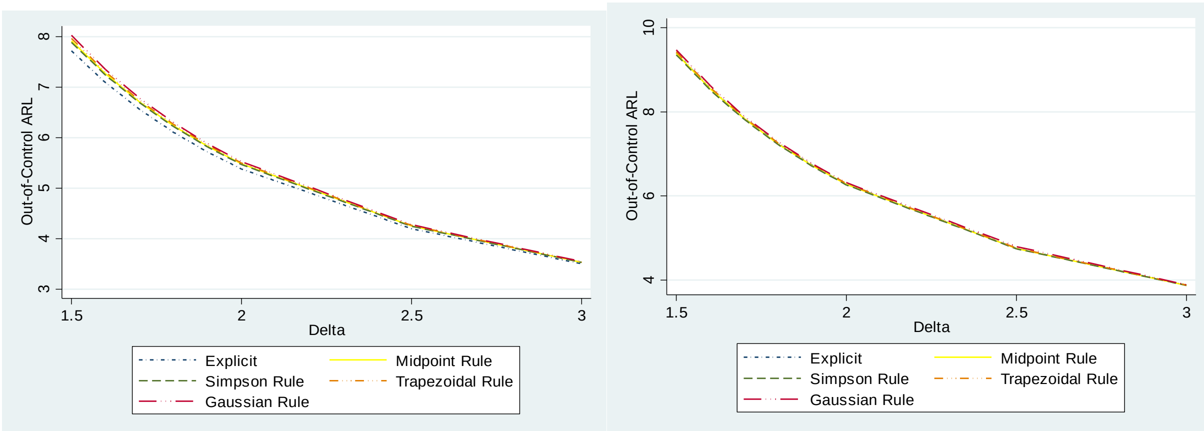
δ	Explicit Formula	Numerical Integration Equation			
		Midpoint Rule	**Simpson Rule**	**Trapezoidal Rule**	**Gaussian Rule**
0.00	370.091 (0.01)	**370.000** **(14.91)**	**370.000** **(125.73)**	**370.000** **(15.41)**	370.000 **(17.89)**
1.50	7.218 (0.01)	7.219 (15.10)	7.189 (123.08)	7.291 (16.52)	7.401 **(17.11)**
1.60	6.665 (0.01)	6.664 (15.41)	6.641 (126.23)	6.726 (16.78)	6.816 **(17.21)**
1.70	6.197 (0.01)	6.196 (15.41)	6.177 (124.66)	6.248 (16.45)	6.322 **(17.57)**
1.80	5.797 (0.01)	5.796 (16.19)	5.776 (105.36)	5.839 (16.67)	5.902 **(17.56)**
1.90	5.451 (0.01)	5.450 (13.19)	5.436 (108.50)	5.486 (16.28)	5.539 **(17.65)**
2.00	5.149 (0.01)	5.148 (13.74)	5.135 (149.65)	5.179 (15.66)	5.225 **(17.95)**
2.50	4.083 (0.01)	4.083 (14.74)	4.076 (133.18)	4.099 (13.97)	4.122 **(17.02)**
3.00	3.443 (0.01)	3.443 (14.78)	3.439 (125.31)	3.457 (14.81)	3.466 **(16.44)**

[a.] The values in parentheses are CPU times (Minutes)

Table 8. Comparison of ARL computed using explicit formulas against the numerical integral equation for the SARX(3,2)$_4$ model given ARL$_0$ = 500, $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\phi_3 = 0.10$, $\beta_1 = 0.1$, $\beta_2 = 0.1$, $a = 3.0$ and $b = 4.004$

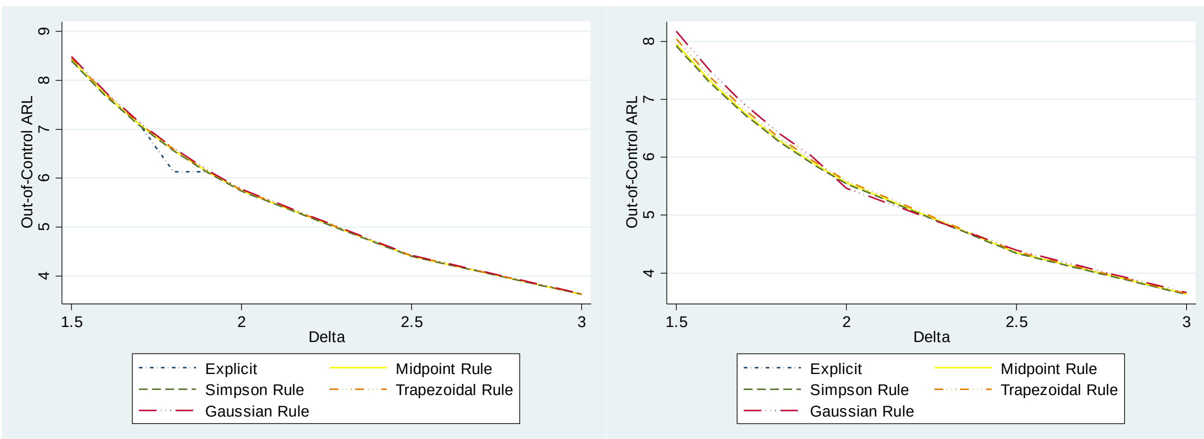
δ	Explicit Formula	Numerical Integration Equation			
		Midpoint Rule	**Simpson Rule**	**Trapezoidal Rule**	**Gaussian Rule**
0.00	500.419 (0.01)	500.000 **(15.17)**	500.000 **(116.19)**	500.000 **(16.79)**	500.000 **(17.64)**
1.50	9.009 (0.01)	9.002 **(16.00)**	8.971 **(109.72)**	9.054 **(17.10)**	9.119 **(17.73)**
1.60	8.213 (0.01)	8.208 **(15.93)**	8.182 **(111.02)**	8.250 **(18.15)**	8.302 **(16.89)**
1.70	7.545 (0.01)	7.541 **(15.99)**	7.519 **(110.70)**	7.576 **(15.89)**	7.621 **(16.77)**
1.80	6.979 (0.01)	6.795 **(16.33)**	6.957 **(108.47)**	7.005 **(16.04)**	7.043 **(16.44)**
1.90	6.493 (0.01)	6.490 **(15.90)**	6.475 **(115.33)**	6.515 **(16.19)**	6.548 **(16.85)**
2.00	6.073 (0.01)	6.071 **(15.73)**	6.058 **(129.54)**	6.093 **(16.13)**	6.121 **(16.74)**
2.50	4.629 (0.01)	4.628 **(14.95)**	4.621 **(127.75)**	4.640 **(15.99)**	4.655 **(16.44)**
3.00	3.797 (0.01)	3.796 **(15.35)**	3.792 **(128.23)**	3.803 **(16.27)**	3.811 **(17.72)**

[a.] The values in parentheses are CPU times (Minutes)



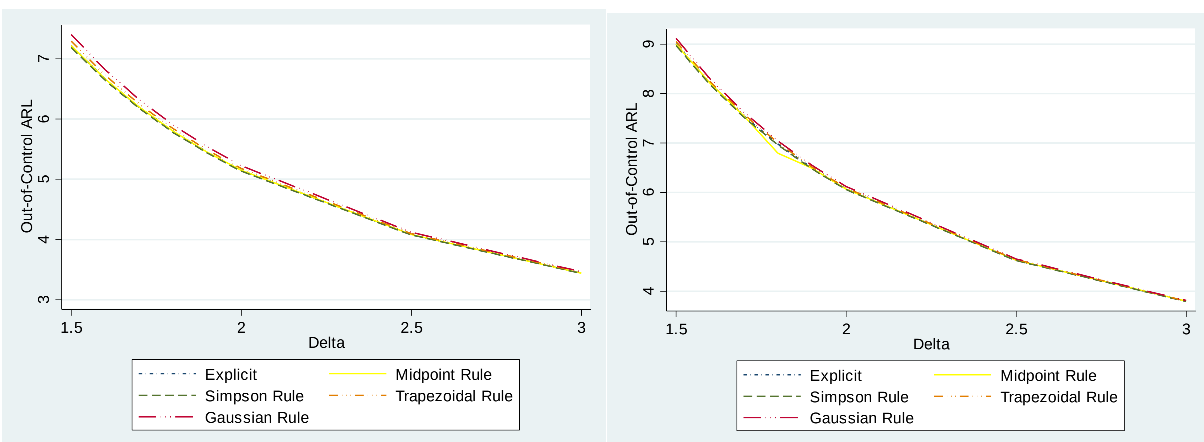
(A) (B)

Figure 1. Comparison of ARL_1 of various methods for the $SARX(2,1)_4$ model with $ARL_0 = 370$ in Part A and $ARL_0 = 500$ in Part B



(A) (B)

Figure 2. Comparison of ARL_1 of various methods for the $SARX(2,2)_4$ model with $ARL_0 = 370$ in Part A and $ARL_0 = 500$ in Part B



(A) (B)

Figure 3. Comparison of ARL_1 of various methods for the $SARX(3,2)_4$ model with $ARL_0 = 370$ in Part A and $ARL_0 = 500$ in Part B

4. Conclusion

In this paper, we proposed the explicit formula for the ARL of the CUSUM chart for a $SARX(P,r)_L$ model with exponential white noise. We derived the explicit formula by using the integral equation technique and using the Banach's Fixed Point theorem to guarantee the existence and uniqueness of the solution. The explicit formula is close to the numerical integral equation (NIE) method with an absolute percentage difference of less than 1%. Consequently, the explicit formula is sufficiently accurate and easy to calculate in comparison with the NIE method. In addition, the CPU time for calculating the explicit formula takes less than 1 minute to find the exact value of the ARL and decrease process time. Thus, the explicit formula can reduce the CPU time much better than the NIE methods.

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