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A Simulation of an Elastic Filament Using Kirchhoff Model

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Abstract This paper presents numerical simulations and comparisons between different approaches concerning elastic thin rods. Elastic rods are ideal for modeling the stretching, bending, and twisting deformations of such long and thin elastic materials. The static solution of Kirchhoff's equations [2] is produced using ODE45 solver where Kirchhoff and reference system equations are combined instantaneously. Solutions using formulations are based on Euler's elastica theory [1] which determines the deformed centerline of the rod by solving a boundary-value problem, on the Discreet Elastic Rod method using Bishop frame (DER) [5,6] which is based on discrete differential geometry, it starts with a discrete energy formulation and obtains the forces and equations of motion by taking the derivative of energies. Instead of discretizing smooth equations, DER solves discrete equations and obeys geometrical exactness. Using DER we measure torsion as the difference of angles between the material and the Bishop frame of the rod so that no additional degree of freedom is needed to represent the torsional behavior. We found excellent agreement between our Kirchhoff-based solution and numerical results obtained by the other methods. In our numerical results, we include simulation of the rod under the action of the terminal moment and illustrations of the gravity effects.

Keywords Kirchhoff Equation, ODE45, Fernet–Serret Frame, Bishop Frame, Discreet Elastic Rod

1. Introduction

Elastic rods have many interesting applications, at large length scale they are used to study the mechanics of macro structural elements such as marine cables. At small length scales, these theories have been used to model biological threads, hair beams [15], DNA molecules [16] rubber bands and microfibers [17]. They are ideal for modeling the stretching, bending, and twisting deformations of such long, thin elastic materials. In a seminal work [2] Gustav Kirchhoff proposed a new theory in order to model the bending and torsion of a rod. In the 20th century, the Cosserat brothers introduced the directory theory which was an extension of Kirchhoff's rod theory in [3] and [4]. The rod theory is also considered as an example of a Cosserat rod theory. As far as we are aware that the analytic solutions of Kirchhoff equations for static rods are well studied. In [7] the most general analytical solution for a static Kirchhoff model is presented. In [8, 9] a time-dependent perturbation scheme is employed to study the stability of equilibrium solutions of the Kirchhoff rod model. This technique has been extensively used, to study DNA rings [10, 11]. Despite the success of the above examples, many contributions toward considering equilibrium solutions of the Kirchhoff model subject to some particular boundary conditions have been made, see [12–14] and references therein. This list is by no means exhaustive, and research in this direction is actively ongoing. The present contribution might be helpful in this endeavor.

The paper is organized as follows. In the next section of this paper, we present a short introduction to the general Kirchhoff rod model, we consider separately the geometric description of the rod (subsection 2.1), its deformation geometry (subsection 2.2), and lastly, we derive the equilibrium equations for a Kirchhoff rod. In Section 3 we describe our implementation procedure and present some numerical examples, including simulation of the rod under the action of the terminal moment, illustrations of the gravity force, we also consider examples of helices with different orientations, the first one is clamped vertically, while the other one is a horizontal helix. Finally, Section 4 contains our main results.

2. Kirchhoff Rod Model

In this study, the composition of the rod is assumed to be isotropic and linear elastic. According to the physics of Kirchhoff, the rod Γ is a deformable body *in which* the *length*, dominates the *other two dimensions* of the *cross-section*. We divide the rod into thin disks of length ds and cross-section $A(s)$. We apply the conservation law of the linear and angular momentum to every disk. We will study a segment of a rod with length ds. A segment of the rod can be evaluated by the arc-length parameter $s \in [0, L]$. Let us denote with A the cross section of the rod.

2.1. Geometric Description

The configuration of the rod as shown in figure 1 is a parameterized space curve $\mathbf{r}(s,t)$ along with a parameterized orthonormal basis $\mathbf{d}_1(s,t), \mathbf{d}_2(s,t), \mathbf{d}_3(s,t)$.

Vectors $\mathbf{d}_1(s,t), \mathbf{d}_2(s,t)$ describe the material section at point s at time t.

$$(s,t) \mapsto \boldsymbol{r}(s,t),\ (s,t) \in [s_1, s_2] \times R \to R^3,$$

$$\mathbf{d}_1(s,t), \mathbf{d}_2(s,t), \mathbf{d}_3(s,t) = \mathbf{d}_2(s,t) \times \mathbf{d}_1(s,t) \in R^3 \qquad (1)$$

First notice that, the frame $\mathbf{d}_1(s,t), \mathbf{d}_2(s,t), \mathbf{d}_3(s,t)$ is defined externally to the curve, different from the Fernet–Serret frame which is defined by $\mathbf{r}(s,t)$ and its derivative.

2.2. Geometry of Deformation

We define $\mathbf{v}(s,t) := \dfrac{\partial \mathbf{r}(s,t)}{\partial s}$, as a vector with three components with respect to the triad $\{\mathbf{d}_i(s,t)\}_{i=1,2,3}$. So, we write, $\mathbf{v} = \sum_{i=1}^{3} v_i \cdot \mathbf{d}_i$, if we multiply this equation with $\mathbf{d}_1$, we obtain the first component of the vector $\mathbf{v}$

$$\mathbf{v} \cdot \mathbf{d}_1 = \left(\sum_{i=1}^{3} v_i \cdot \mathbf{d}_i \right) \cdot \mathbf{d}_1 \to \mathbf{d}_i \cdot \mathbf{d}_j = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \Rightarrow v_1 = \mathbf{v} \cdot \mathbf{d}_1.$$

In general, we have $v_i = \mathbf{v} \cdot \mathbf{d}_i$.

As they $(\{\mathbf{d}_i(s,t)\}_{i=1,2,3})$ are orthonormal' basis, they satisfy kinematic relation

$$\frac{\partial \mathbf{d}_i(s,t)}{\partial s} = \mathbf{u} \times \mathbf{d}_i(s,t) \qquad (2)$$

$$\frac{\partial \mathbf{d}_i(s,t)}{\partial t} = \boldsymbol{\omega} \times \mathbf{d}_i(s,t) \qquad (3)$$

where $\mathbf{u}$ is Darboux vector.

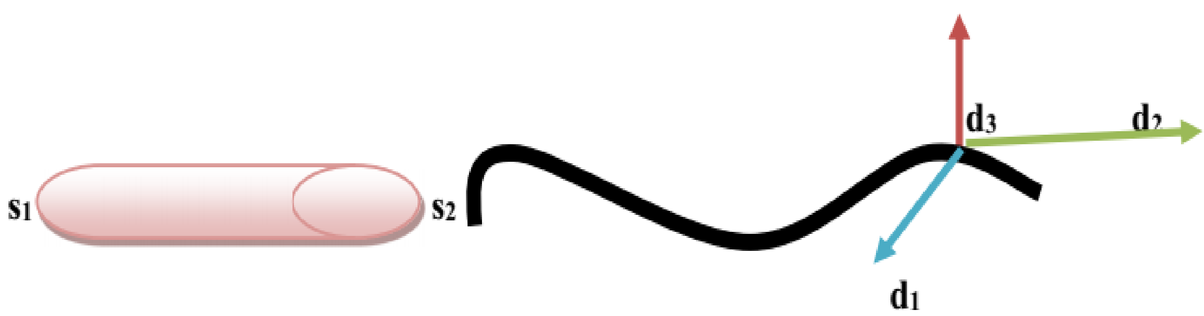

Figure 1. Geometric Configuration of a Rod

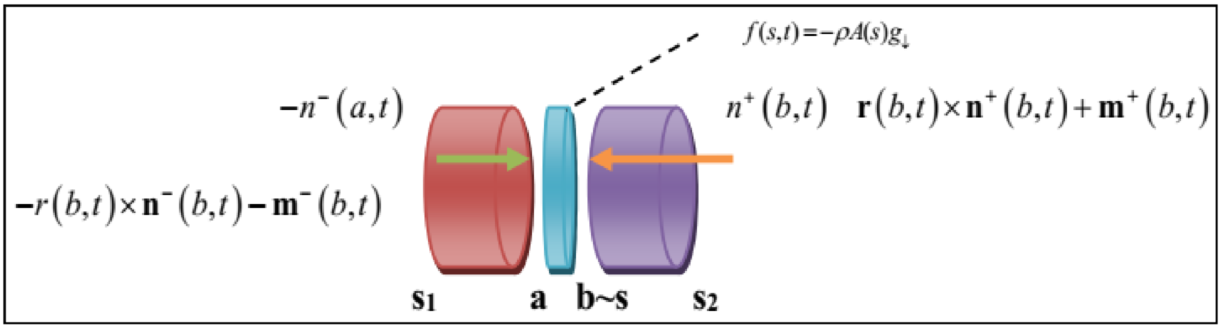

Figure 2. The forces that are acting on the material cross section [**a, b**], and a body force such as for example gravity, that is $f(s,t)=-\rho A(s)g_{\downarrow}$.

2.3. Mechanics of the Rod

We now consider the balance laws that allow us to determine which configurations are equilibrium shapes. We suppose the forces acting on the cross section $[a,\ \mathrm{b}]$ (see figure 2), consist a contact force $n^{+}(b,t)$

By requiring that the rate of change of the linear momentum equals the net of resultant forces acting on $[a,\ \mathrm{b}]$ we find the equation of the motion, that is

$$\mathbf{n}^{+}(b,t)-\mathbf{n}^{-}(a,t)+\int_a^b \mathbf{f}(s,t)ds=\int_a^b(\rho A)(s)\mathbf{r}_{tt}(s,t)ds \quad (4)$$

The equation (4) is true for every $[\mathbf{a},\mathbf{b}]\subset[s_1,s_2]$. Since n^{+} is a continuous quantity, from this condition we have $\mathbf{n}^{+}(a,t)=\lim_{b\to a}\left[\mathbf{n}^{+}(b,t)\right]$. Taking the limit $b\to a$ of the equation

$$\lim_{b\to a}\left[\mathbf{n}^{+}(b,t)-\mathbf{n}^{-}(a,t)\right]+\lim_{b\to a}\int_a^b \mathbf{f}(s,t)ds=\lim_{b\to a}\int_a^b(\rho A)(s)\mathbf{r}_{tt}(s,t)ds$$

$$\lim_{b\to a}\left[\mathbf{n}^{+}(b,t)-\mathbf{n}^{-}(a,t)\right]=0$$

$$\lim_{b\to a}\mathbf{n}^{+}(b,t)-\mathbf{n}^{-}(a,t)=0$$

$$\mathbf{n}^{+}(a,t)=\mathbf{n}^{-}(a,t)$$

We drop the superscripts $\mathbf{n}(b,t)-\mathbf{n}(a,t)+\int_a^b \mathbf{f}(s,t)ds=\int_a^b(\rho A)(s)\mathbf{r}_{tt}(s,t)ds$, and we write the equation again in the following way

$$\mathbf{n}(b,t)-\mathbf{n}(a,t)+\int_a^b \mathbf{f}(s,t)ds=\int_a^b(\rho A)(s)\mathbf{r}_{tt}(s,t)ds$$

The above equation holds for every $s_1<a<s<s_2$, that is

$$\mathbf{n}(s,t)-\mathbf{n}(a,t)+\int_a^s \mathbf{f}(s,t)ds=\int_a^s(\rho A)(s)\mathbf{r}_{tt}(s,t)ds$$

We differentiate the above equation with respect to s

$$\frac{\partial}{\partial s}\left(\mathbf{n}(s,t)-\mathbf{n}(a,t)+\int_a^s \mathbf{f}(s,t)ds\right)=\frac{\partial}{\partial s}\left(\int_a^s(\rho A)(s)\mathbf{r}_{tt}(s,t)ds\right)$$

Using some elementary calculation, we have

$$\frac{\partial\mathbf{n}(s,t)}{\partial s}+\mathbf{f}(s,t)=(\rho A)(s)\mathbf{r}_{tt}(s,t),\forall s\in[s_1 s_2] \quad (5)$$

We also assume a resultant contact torque $\mathbf{r}(b,t)\times\mathbf{n}^{+}(b,t)+\mathbf{m}^{+}(b,t)$ acting on the cross section $[a,\ \mathrm{b}]$ exerted on $[a,\ \mathrm{b}]$ by $]\mathrm{b},\ s_2]$ and $-\mathbf{r}(b,t)\times\mathbf{n}^{-}(b,t)-\mathbf{m}^{-}(b,t)$ exerted on $[a,\ \mathrm{b}]$ by $[\mathrm{s}_1,\ \mathrm{a}[$.

We assume that, the result of all other torques acting on $[a,\ \mathrm{b}]$ has the form

$$\int_a^b \mathbf{r}(s,t)\times f(s,t)ds$$

The Angular Impulse – Momentum Law on $[a,\ \mathrm{b}]$:

$$\mathbf{m}^{+}(b,t)-\mathbf{m}^{-}(a,t)+\mathbf{r}(b,t)\times\mathbf{n}^{+}(b,t)-\mathbf{r}(a,t)\times\mathbf{n}^{-}(a,t)+ \\ +\int_a^b \mathbf{r}(s,t)\times\mathbf{f}(s,t)ds=\frac{\partial}{\partial t}\left[\int_a^b \mathbf{r}(s,t)\times(\rho A)(s)\mathbf{r}_t(s,t)ds\right] \quad (6)$$

Using elementary calculations leads to:

$$\frac{\partial\mathbf{m}(s,t)}{\partial s}+\frac{\partial\left[\mathbf{r}(s,t)\times\mathbf{n}(s,t)\right]}{\partial s}+\mathbf{r}(s,t)\times\mathbf{f}(s,t) \\ =(\rho A)(s)\mathbf{r}(s,t)\times\mathbf{r}_{tt}(s,t) \quad (7)$$

Equations (5) and (7) are the equilibrium form for a special Cosserat rod. Substituting the equation (5) into the equation (7), we have

$$\frac{\partial\mathbf{m}(s,t)}{\partial s}+\frac{\partial\mathbf{r}(s,t)}{\partial s}\times\mathbf{n}(s,t)=0$$

Kinematics equation combined with two balance laws gives the equations for a Cosserat rod

$$\begin{cases} \dfrac{\partial \mathbf{r}(s,t)}{\partial t} = \mathbf{v} \\ \dfrac{\partial \mathbf{d}_i(s,t)}{\partial s} = \mathbf{u} \times \mathbf{d}_i(s,t) \\ \dfrac{\partial \mathbf{n}(s,t)}{\partial s} + \mathbf{f}(s,t) = (\rho A)(s)\mathbf{r}_{tt}(s,t), \forall s \in [s_1 s_2] \\ \dfrac{\partial \mathbf{m}(s,t)}{\partial s} + \dfrac{\partial \mathbf{r}(s,t)}{\partial s} \times \mathbf{n}(s,t) = 0, \forall s \in [s_1 s_2] \end{cases} \quad (8)$$

Let us consider now the central axis of the cross section as in figure 3 which is in analogy to center of mass of all points making up the cross section. Our cross section has a thickness $[a,\ \mathrm{b}]$.

$\mathbf{u}_j(x_1, x_2, s, t)$ represents the location of the material point j in the current configuration with respect to an reference frame and (x_1, x_2, s) is the location of the same material point in the reference configuration at time t. In general, we have

$$\mathbf{u}(x_1, x_2, s, t) = \mathbf{r}(s,t) + Q(\theta,t)\left(x^1\mathbf{e}_1 + x^2\mathbf{e}_2\right)$$
$$= \mathbf{r}(s,t) + x^1\mathbf{d}_1 + x^2\mathbf{d}_2$$

$\mathbf{d}_i = Q\mathbf{e}_i$ where $\mathbf{r}(s,t)$ represents the position of the centerline and $Q(\theta,t)$ is the cross-section orientation. Equation (5) has the form

$$\frac{\partial \mathbf{n}(s,t)}{\partial s} + \mathbf{f}(s,t) = (\rho A)(s)\left(\mathbf{r}_{tt}(s,t) + \mathbf{g}_{tt}(x_1, x_2, t)\right) \forall s \in [s_1 s_2] \quad (9)$$

Assuming the $\mathbf{f}(s,t)$ takes value in $span\{e_1, e_3\}$, since we are working will planar problems. The resultant of all torques has the form

$$\int_a^b \left[\mathbf{r}(s,t) \times \mathbf{f}(s,t) + l(s)e_3\right] ds$$

$$\mathbf{u}_j(x_1, x_2, s, t) = \mathbf{r}(s,t) + \mathbf{g}_j(x_1, x_2, t)$$

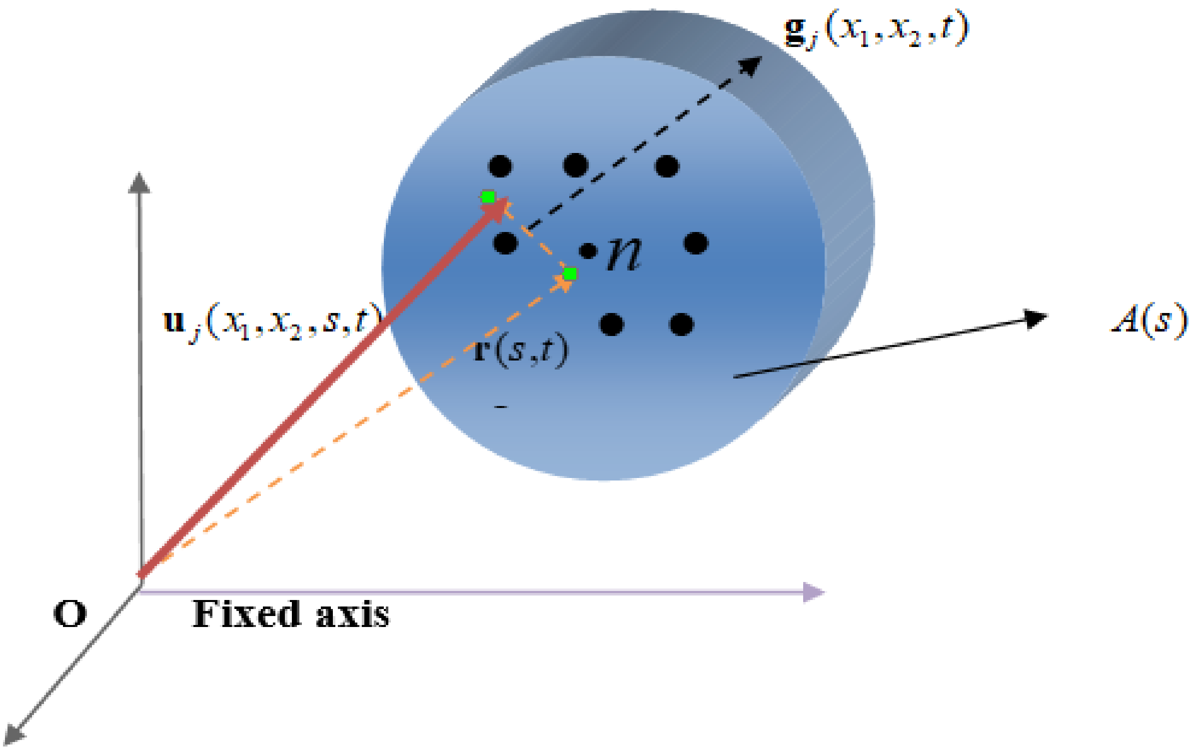

Figure 3. The location of the material point j in the current configuration related to a reference frame

In other words, we consider both the couple created by the body force $\mathbf{f}(s,t)$ and an external body couple $\mathbf{l}(s)$.

Equation (7) takes the following form

$$\frac{\partial \mathbf{m}(s,t)}{\partial s}+\frac{\partial\left[\mathbf{r}(s,t)\times\mathbf{n}(s,t)\right]}{\partial s}+\mathbf{r}(s,t)\times\mathbf{f}(s,t)+\mathbf{l}(s)$$
$$=(\rho A)(s)\mathbf{r}(s,t)\times\mathbf{r}_{tt}(s,t)+\rho\mathbf{r}(s,t)\times\mathbf{g}_{tt}(s,t) \qquad (10)$$
$$+\rho\mathbf{g}(s,t)\times\mathbf{r}_{tt}(s,t)+\rho\mathbf{h}_{\mathbf{t}}(s,t)$$

Let us consider

$$\mathbf{u}(x_1,x_2,s,t)=\mathbf{r}(s,t)+Q(\theta,t)\left(x^1e_1+x^2e_2\right)$$
$$=\mathbf{r}(s,t)+x^1\mathbf{d_1}+x^2\mathbf{d_2}$$

and assume the vectors $\mathbf{d_1},\mathbf{d_2}$ coincide with the principal directions of inertia of the cross section $[a,\ \mathrm{b}]$.

Equation (10), yields

$$\frac{\partial \mathbf{m}(s,t)}{\partial s}+\frac{\partial\left[\mathbf{r}(s,t)\times\mathbf{n}(s,t)\right]}{\partial s}+\mathbf{r}(s,t)\times\mathbf{f}(s,t)+\mathbf{l}(s)$$
$$=(\rho A)\mathbf{r}(s,t)\times\frac{\partial \mathbf{r}(s,t)}{\partial t}+\rho\left(I_1\mathbf{d_1}\times\frac{\partial^2\mathbf{d_1}}{\partial t^2}+I_2\mathbf{d}_2\times\frac{\partial^2\mathbf{d}_2}{\partial t^2}\right)$$

Substituting the equation (9) into the equation (10), we have

$$\frac{\partial \mathbf{m}(s,t)}{\partial s}+\frac{\partial \mathbf{r}(s,t)}{\partial s}\times\mathbf{n}(s,t)+\mathbf{l}(s)=$$
$$=\rho\left(I_1\mathbf{d_1}\times\frac{\partial^2\mathbf{d_1}}{\partial t^2}+I_2\mathbf{d}_2\times\frac{\partial^2\mathbf{d}_2}{\partial t^2}\right)$$

Where

$$I_1=\int_{A(s)}\left(x^2\right)^2dx_1dx_2 \quad \text{and} \quad I_2=\int_{A(s)}\left(x^1\right)^2dx_1dx_2$$

Before we compare our method with the other methods, let us briefly remind the idea behind Discreet Elastic Rod method and Euler's classic theory. The discrete elastic rod method (DER) proposed by Bergou et al. [5,6], based on discrete differential geometry, starts with a discrete energy formulation and obtains the forces and equations of motion by taking the derivative of energies. Instead of discretizing smooth equations, DER solves discrete equations and obeys geometrical exactness. While the Euler's elastica theory determines the deformed centerline of the rod by solving a boundary-value problem. In the next section, we implement and perform simulations for different scenarios.

3. Implementation and Numerical Results

Let us consider the balance laws for a special Cosserat rod

$$\frac{\partial \mathbf{n}(s,t)}{\partial s}+\mathbf{f}(s,t)=(\rho A)(s)\left(\mathbf{r}_{tt}(s,t)+\mathbf{g}_{tt}(x_1,x_2,t)\right)\forall s\in[s_1s_2]$$
$$\frac{\partial \mathbf{m}(s,t)}{\partial s}+\frac{\partial \mathbf{r}(s,t)}{\partial s}\times\mathbf{n}(s,t)+\mathbf{l}(s)= \qquad (11)$$
$$=\rho\left(I_1\mathbf{d_1}\times\frac{\partial^2\mathbf{d_1}}{\partial t^2}+I_2\mathbf{d}_2\times\frac{\partial^2\mathbf{d}_2}{\partial t^2}\right)$$

The partial differential equations (11) enhanced by constitutive relations for the moment **m** we have

$$\mathbf{m}=EI_1k_1\mathbf{d_1}+EI_2k_2\mathbf{d}_2+Dk_3\mathbf{d}_3 \Leftrightarrow \begin{pmatrix} \overbrace{EI_1k_1}^{m_1} \\ \overbrace{EI_2k_2}^{m_2} \\ \overbrace{Dk_3}^{m_3}\end{pmatrix}\begin{matrix}\leftrightarrow\mathbf{d_1}\\ \leftrightarrow\mathbf{d}_2\\ \leftrightarrow\mathbf{d_1}\end{matrix} \qquad (12)$$

Where EI_1,EI_2 are flexural rigidity, D is a torsional stiffness and E is Young's Modulus.

For a static rod, all derivative with respect to t vanish, that is

$$\frac{\partial \mathbf{n}(s,t)}{\partial s}+\mathbf{f}(s,t)=0\forall s\in[s_1s_2]$$
$$\frac{\partial \mathbf{m}(s,t)}{\partial s}+\frac{\partial \mathbf{r}(s,t)}{\partial s}\times\mathbf{n}(s,t)+\mathbf{l}(s)=0 \qquad (13)$$

The resultant of forces $\mathbf{n}(s,t)$ and the resultant of moments $\mathbf{m}(s,t)$ are vectors, with $n_i(s,t)$ and $m_i(s,t)$ we will denote components of these vectors with respect to the local basis $\{\mathbf{d_i}(s,t)\}_{i=1,2,3}$. We write

$$\mathbf{n}(s,t)=n_i(s,t)\mathbf{d_i}(s,t), \quad \mathbf{m}(s,t)=m_i(s,t)\mathbf{d_i}(s,t)$$

Equation (13) can be written

$$\frac{\partial \mathbf{n}(s,t)}{\partial s}+\mathbf{f}(s,t)=$$
$$=\left\{\begin{matrix}\overbrace{\frac{\partial n_1(s,t)}{\partial s}+n_2k_3-n_3k_2+f_1(s,t)}^{\mathbf{d_1}(s,t)}\\ \overbrace{\frac{\partial n_2(s,t)}{\partial s}-n_1k_3+n_3k_1+f_2(s,t)}^{\mathbf{d}_2(s,t)}\\ \underbrace{\frac{\partial n_3(s,t)}{\partial s}+n_1k_2-k_1n_2+f_3(s,t)}_{\mathbf{d}3(s,t)}\end{matrix}\right. \qquad (14)$$

$$\frac{\partial \mathbf{m}(s,t)}{\partial s}+\frac{\partial \mathbf{r}(s,t)}{\partial s}\times \mathbf{n}(s,t)+\mathbf{l}(s)=$$

$$=\begin{cases}\overbrace{EI_1\frac{\partial k_1(s,t)}{\partial s}+(EI_2-D)k_3k_2+n_2+l_1}^{\mathbf{d}_1(s,t)}\\ \overbrace{EI_2\frac{\partial k_2(s,t)}{\partial s}+(D-EI_1)k_3k_1-n_1+l_2}^{\mathbf{d}_2(s,t)}\\ \underbrace{D\frac{\partial k_3(s,t)}{\partial s}+(EI_1-EI_2)k_1k_2+l_3}_{\mathbf{d}3(s,t)}\end{cases}$$

Now, using initial conditions $(n_1(0),n_2(0),n_3(0))^T$ $(k_1(0),k_2(0),k_3(0))^T$ we obtain a solution of the equation (14).

But this solution is full understood if it is written with respect to the external and fixed system. It is known that there exists a transformation which takes vectors from the fixed system to the local one and vice versa (see figure 4). We write this transformation as follows

$$\mathbf{d}_i(s,t)=Q(s,t)\mathbf{e}_i \tag{15}$$

where $\mathbf{e}_i$ is a base vector of the fixed system and $Q(s,t)$ is proper rotation matrix of the form

$$Q(s,t)=\begin{pmatrix} q_{11} & q_{12} & q_{13}\\ q_{21} & q_{22} & q_{23}\\ q_{31} & q_{32} & q_{33}\end{pmatrix}$$

If we derive equation (10), we write

$$\frac{\partial \mathbf{d}_i(s,t)}{\partial s}=\frac{\partial Q(s,t)}{\partial s}\mathbf{e}_i \tag{16}$$

$$\text{where}\quad \frac{\partial Q(s,t)}{\partial s}=\mathbf{u}\times Q(s,t) \tag{17}$$

If we substitute equation (17) and (15) to (16), we obtain equation (2)

$$\frac{\partial \mathbf{d}_i(s,t)}{\partial s}=\frac{\partial Q(s,t)}{\partial s}\mathbf{e}_i=\mathbf{u}\times\overbrace{Q(s,t)\mathbf{e}_i}^{\mathbf{d}_i(s,t)}$$

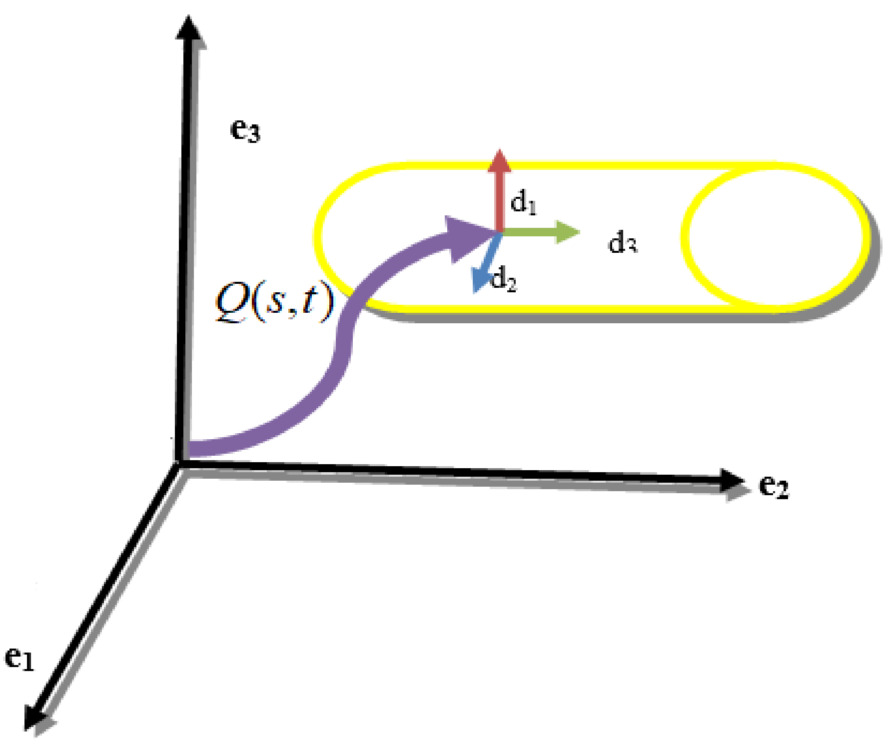

Figure 4. An orthogonal rotation matrix $\mathbf{Q}(s,t)$ transforms a vector $\mathbf{r}(s,t)$ from the global canonical basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ to the material frame of reference $\{\mathbf{d}_1,\mathbf{d}_2,\mathbf{d}_3\}$

Give $\{\mathbf{d}_i(s,t)\}_{i=1,2,3}$ we find $\mathbf{r}(s,t)=\mathbf{r}(s_1,t)+\int_{s_1}^{s} v_i \mathbf{d}_i(\xi,t)d\xi$. Let's consider an inextensible and unshearable rod, the strains v_i in any configuration equal the strains in the reference configuration. We choose the reference framing to be adapted to the centerline, that is $\frac{\partial \mathbf{r}(s,t)}{\partial s}=\mathbf{d}_3(s,t)$ and $v_1(s,t)=v_2(s,t)=0$, $v_3(s,t)=1$. Now, we may write $\mathbf{r}(s,t)=\mathbf{r}(s_1,t)+\int_{s_1}^{s}\mathbf{d}_3(\xi,t)d\xi$

To solve the problem, we proceed as is follow:

1. First, we obtain $(n_1,n_2,n_3)^T$ and $(k_1,k_2,k_3)^T$ solving equations (14)
2. Then we find $Q(s,t)$ solving equation (17) and using $\mathbf{u}$, that is.

$$\begin{pmatrix} \frac{\partial q_{11}}{\partial s} & \frac{\partial q_{12}}{\partial s} & \frac{\partial q_{13}}{\partial s} \\ \frac{\partial q_{21}}{\partial s} & \frac{\partial q_{22}}{\partial s} & \frac{\partial q_{23}}{\partial s} \\ \frac{\partial q_{31}}{\partial s} & \frac{\partial q_{32}}{\partial s} & \frac{\partial q_{33}}{\partial s} \end{pmatrix} = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix}\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

The above equation can be written as a system of first order differential equations,

$$\frac{\partial q_{11}}{\partial s}=-k_3q_{21}+k_2q_{31},\ldots,\frac{\partial q_{33}}{\partial s}=-k_2q_{13}+k_1q_{23}$$

3. After that using equation (15) we find $\mathbf{d}(s,t)$ then we use $\mathbf{r}(s,t)=\mathbf{r}(s_1,t)+\int_{s_1}^{s}\mathbf{d}_3(\xi,t)d\xi$ to generate the curve (in other words to generate the centerline we solve the following differential equations

$$\frac{\partial \mathbf{r}(s,t)}{\partial s}=\mathbf{d}_3(s,t)\leftrightarrow\frac{\partial \mathbf{r}(s,t)}{\partial s}=\begin{pmatrix} q_{31}(s) \\ q_{31}(s) \\ q_{31}(s) \end{pmatrix}$$

Writing all our equations in a general form

$$\frac{\partial \mathbf{\Theta}(s,t)}{\partial s}=\mathbf{\Sigma}(s,t)$$

If we consider the gravity force $\mathbf{f}(s,t)=-\rho A(s)ge_3$ as an external force, then the external moment is zero. To insert $\mathbf{f}(s,t)$ in equation (14), we write $\mathbf{f}(s,t)$ in terms of local basis as follows

$$\mathbf{f}(s,t)=\overbrace{-\rho A(s)g\left(q_{22}q_{31}-q_{21}q_{32}\right)\mathbf{d}_1(s,t)}^{f_1(s,t)}\underbrace{-\rho A(s)g\left(q_{32}q_{11}-q_{12}q_{31}\right)\mathbf{d}_2(s,t)}_{f_2(s,t)}\underbrace{-\rho A(s)g\left(q_{12}q_{21}-q_{22}q_{11}\right)\mathbf{d}_3(s,t)}_{f_3(s,t)}$$

Below we present some numerical results to illustrate some solutions. As can be seen from Figure 5, the deformation of the centerline predicted by the Euler's elastica theory and Kirchhoff rod theory and discrete elastic rod formulation have an excellent agreement

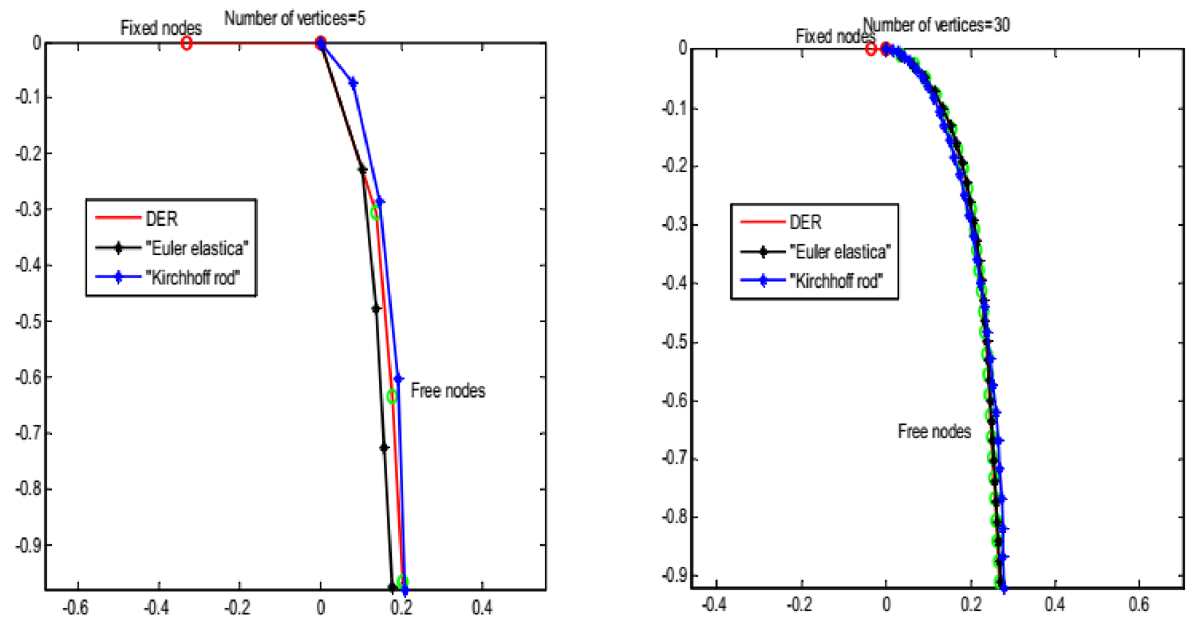

Figure 5. The deformation of the centerline for a cantilevered rod rod (with 5 and 30 vertices) using Euler's elastica theory and the Kirchhoff rod theory and the discrete elastic rod formulation under its own weight. The first is assumed fixed: x_0 and x_1 are constant. The parameter values for this rod: l=0.25m, r=10^{-3}m, E=10^{6}Pa, ρ =10^{3}kg/m^3

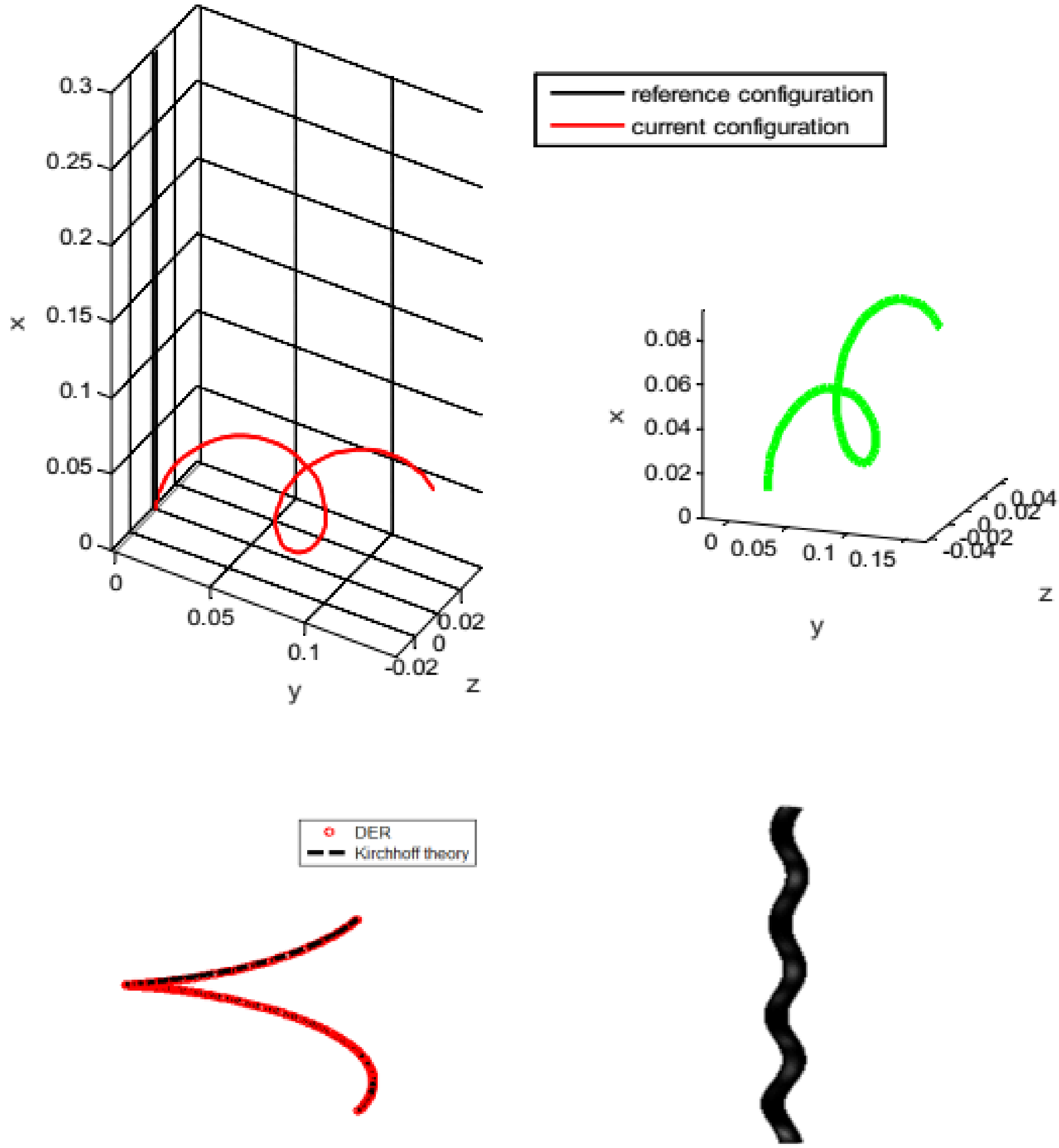

Figure 6. The deformation of the centerline for a rod under the action of terminal moments $\mathbf{m} = -25EI_1\mathbf{e}_1 + 10EI_2\mathbf{e}_2 + -25D\mathbf{e}_3$. The centerline has a constant curvature and torsion. The parameter values for this rod: $\ell = 0.3m$, $r = 0.5mm$ $EI_1 = EI_2 = 4\times10^{-7}Nm^2$ $D = 2.6\times10^{-7}Nm^2$

Figure 6 shows a numerical simulation of the rod under the action of terminal moment. In Figure 7, we consider two helices with different orientations under the action of gravity, the first one is clamped vertically, while the other one is a horizontal helix.

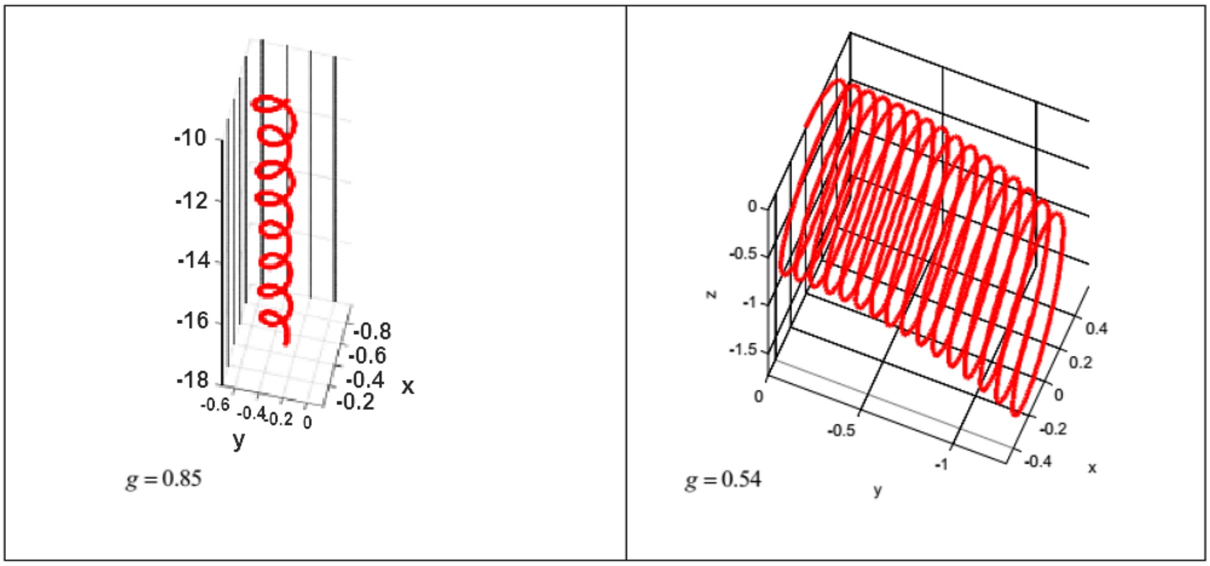

Figure 7. The deformation of the centerline for a rod under the action of gravity, stretching the filament vertically (left) and horizontal helix (right). The parameter values for this rod: $E=1, D=0.5$ $k_1=0, k_2=2$ and $k_3=0.1$, $n_1=0, n_2=0$, $n_3=0$, $I_1=3, I_2=2$, $A=1$ and $r=1$.

4. Conclusion

We concluded this work with some numerical experiments. In this paper, a static solution for Kirchhoff's equations is obtained. We have integrated the static equations as an initial value problem, to integrate our equations the ODE45 scheme is used, we compare our solution with classical Euler theory of the Elastica and Discreet Elastic Rod model and see an excellent agreement. Using the proposed technique, solutions under the action of external forces can be easily obtained. In the present paper, we have only considered the gravity force as an external force, but we believe the technique presented here can be used in other classes of static solutions for Kirchhoff's equations, we can mention here, filament in a rotating system, where centrifugal acceleration plays the role of the external force and the electric force for a charged filament as well.

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