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On Recent Advances in Divisor Cordial Labeling of Graphs

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Abstract An assignment of intergers to the vertices of a graph $\bar{G}$ subject to certain constraints is called a vertex labeling of $\bar{G}$. Different types of graph labeling techniques are used in the field of coding theory, cryptography, radar, missile guidance, x-ray crystallography etc. A DCL of $\bar{G}$ is a bijective function $\bar{f}$ from node set $\bar{V}$ of $\bar{G}$ to $\{1, 2, 3, ..., |\bar{V}|\}$ such that for each edge rs, we allot 1 if $\bar{f}(r)$ divides $\bar{f}(s)$ or $\bar{f}(s)$ divides $\bar{f}(r)$ & 0 otherwise, then the absolute difference between the number of edges having 1 & the number of edges having 0 do not exceed 1, i.e., $|e_{\bar{f}}(0) - e_{\bar{f}}(1)| \leq 1$. If $\bar{G}$ permits a DCL, then it is called a DCG. A complete graph K_n, is a graph on n nodes in which any 2 nodes are adjacent and lilly graph I_n is formed by $2K_{1,n}$ joining $2P_n, n \geq 2$ sharing a common node. i.e., $I_n = 2K_{1,n} + 2P_n$, where $K_{1,n}$ is a complete bipartite graph & P_n is a path on n nodes. In this paper, we propose an interesting conjecture concerning DCL for a given $\bar{G}$, besides, discussing certain general results concerning DCL of complete graph K_n-related graphs. We also prove that I_n admits a DCL for all $n \geq 2$. Further, we establish the DCL of some I_n-related graphs in the context of some graph operations such as duplication of a node by an edge, node by a node, extension of a node by a node, switching of a node, degree splitting graph, & barycentric subdivision of the given $\bar{G}$.

Keywords Graph Labeling, DCL, Lilly Graph

1 Introduction

By $\bar{G}$, we denote a simple, finite, & undirected graph with node set $\bar{V}$ & edge set $\bar{E}$. An allocation of labels to nodes or edges or sometimes both, under some constraints is called as graph labeling. Graph labeling is a close association of graph theory & number theory. Being interdisciplinary, graph labeling is attracting the attention of numerous researchers and software developers. For number theory and graph theory related terms, we refer to [1] and [4], respectively. For further study on various graph labeling problems, see [3]. We use DCL and DCG to denote divisor cordial labeling and divisor cordial graph, respectively.

Cahit [2] introduced the idea of cordial labeling. Sundaram et al. [9] coined the notion of prime cordial labeling. The concept of DCL was given by Vartharajan et al. [10]. Vartharajan et al. [11] proved some general results especially the DCL of full binary tree.

Definition 1. [10] A DCL of $\bar{G}$ having $\bar{V}$ is a bijection $\bar{f}$ from $\bar{V}$ to $\{1, 2, 3, ..., |\bar{V}|\}$ such that each edge rs is alloted 1 if $\bar{f}(r)/\bar{f}(s)$ or $\bar{f}(s)/\bar{f}(r)$ & 0 otherwise, then $|e_{\bar{f}}(0) - e_{\bar{f}}(1)| \leq 1$. If $\bar{G}$ admits a DCL, then it is said to be a DCG.

For further results on DCL, refer to [3, 6, 10, 11].

2 Main Results

This section is devoted to derive some general results on DCG. Also, DCL of lilly graph in the context of different graph operations has been explored.

2.1 DCL of K_n Related Graphs

Let $N(u)$ and $N[u]$ represent the open and closed neighbourhood of u, respectively. In this section, we deal with K_n

related graphs in establishing DCL.

Definition 2. [6] Let $\bar{G}$ with $V(\bar{G}) = \bar{S}_1 \cup \bar{S}_2, ... \cup \bar{S}_t \cup \bar{T}$, where each $\bar{S}_i$ is a set of nodes having at least two nodes of same degree & $\bar{T} = V - \bigcup \bar{S}_i$. $DS(\bar{G})$ is the degree splitting graph of $\bar{G}$, which is created from $\bar{G}$ by adding nodes $\bar{w}_1, \bar{w}_2, ..., \bar{w}_t$ & connecting $\bar{w}_i$ to each node of $\bar{S}_i$; $1 \leq i \leq t$.

Lemma 1. *[7] $DS(K_n)$ yields K_{n+1}.*

Theorem 1. *[10] K_n does not admit a DCL for $n \geq 7$.*

Theorem 2. *$DS(K_n)$ does not permit a DCL for $n \geq 6$.*

Proof. The proof follows clearly from Theorem 1 and Lemma 1. □

Definition 3. [5] Extension of a node u_i^{α}, is achieved by adding a new node u_i' such that $N(u_i') = N[u_i^{\alpha}]$.

Lemma 2. *Extension of any arbitrary node of K_n yields K_{n+1}.*

The proof follows from the fact that the newly added node is joined with all the nodes of K_n including the node itself as every pair of nodes in K_n are adjacent, which eventually gives rise to K_{n+1}.

Theorem 3. *The graph G obtained by performing extension of any arbitrary node in K_n does not admit a DCL for $n \geq 6$.*

Proof. It follows from Theorem 1 and Lemma 2. □

Definition 4. [6] Switching of a node v_1 in H_1 is obtained by eliminating all edges incident to v_1 in H_1 & adding new edges joining v_1 to every other node which were not adjacent to v_1 in H_1.

Lemma 3. *The graph formed by switching any arbitrary node in K_n admits a DCL for $n \leq 8$.*

Proof. Switching of any arbitary node in K_n results in a disconnected graph whose components are K_{n-1} and K_1. The result clearly follows for switching of node in K_n for $n = 3, 4, 6, 7$ (see Figure 1). Now we discuss DCL of switching of a node in K_5 and K_8.
Case(i). When $n = 5$. Label the isolated node with 4 and assign the remaining labels to the nodes of K_4. Clearly, $e(0) = e(1) = 3$.
Case(ii). When $n = 8$. Label the isolated node with 7 and assign the remaining labels to the nodes of K_7. Here $e(0) = 10$ and $e(1) = 11$. □

Theorem 4. *Switching of an arbitrary node in K_n for $n \geq 9$ does not admit a DCL.*

Proof. Switching of an arbitrary node in K_n; $n \geq 9$ results in a disconnected graph G whose components are K_{n-1} and K_1. We take $n = 9$ for the sake of discussion. We obtain a disconnected G obtained by switching of a node in K_9 whose

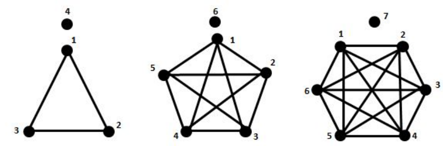

Figure 1. DCL of switching of a node in K_4, K_6 and K_7

componenets are K_8 and K_1. We prove by a method of contradiction. We assume that G admits a DCL. Without loss of generality, label the isolated node with the largest prime p where $p \leq 9$ (i.e., 7) in order to get more edges having label 1, and assign the remaining labels to nodes of K_8 in any order. Here $e(0) = 15$, $e(1) = 13$, and therefore $|e(0) - e(1)| > 1$, a contradiction. The other possibilities of assigning different labels to the isolated node can be dealt in the similar lines. The similar argument holds good for $n \geq 10$. Hence the theorem. □

Considering the fact that characterization of DCGs is challenging in general, we propose the following conjecture.

Conjecture 1. *For a given finite graph $\bar{G}$, establishing a DCL of $\bar{G}$ is NP-hard.*

Remark 1. *We believe that the conjecture is true as there are no algorithm available in the literature and devising a particular pattern of DCG is also the hardest.*

2.2 DCL of Lilly Related Graphs

Here, we explore the DCL of lilly graph and its related graphs in the context of various graph operations.

Definition 5. [8] Lilly graph I_n is defined by $I_n = 2K_{1,n} + 2P_n$ (see Figure 2).

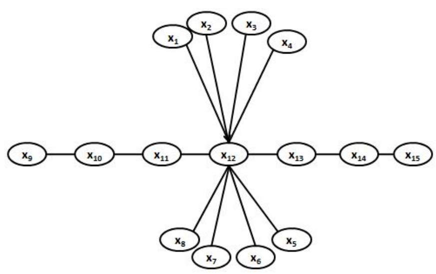

Figure 2. I_4

Theorem 5. *I_n admits a DCL.*

Proof. Let $x_1, x_2, ..., x_{4n-1}$ represent the nodes of I_n. Clearly, the number of elements in node & edge sets are equal to $4n - 1$ and $4n - 2$, respectively. Define a function (bijective) $T : V(I_n) \to \{1, 2, ..., 4n - 1\}$ by letting $T(x_{3n}) = 2$, $T(x_1) = 4$, $T(x_i) = T(x_{i-1}) + 2$; $2 \leq i \leq 2n - 2$, $T(x_{2n-1}) = 1$, $T(x_{2n-1+i}) = T(x_{2n-1+i-1}) + 2$; $1 \leq i < n + 1$, $T(x_{3n+1}) = T(x_{3n-1}) + 2$, $T(x_{3n+1+i}) = T(x_{3n+1+(i-1)}) + 2$; $1 \leq i \leq n - 2$. See that $e_T(0) =$

$e_T(1) = 2n - 1$ which establishes that I_n is a DCG (see Figure 3). □

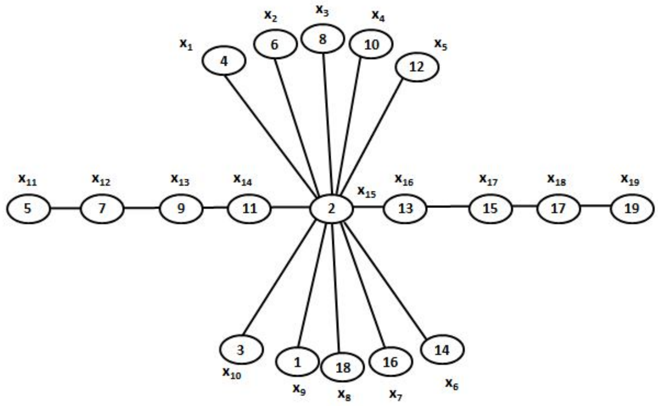

Figure 3. DCL of I_5

Theorem 6. *Switching of any pendant node in I_n admits a DCL.*

Proof. Let $x_1, x_2, ..., x_{4n-1}$ represent the nodes of I_n. Here, $x_1, x_2, ..., x_n, x_{n+1}, ..., x_{2n}, x_{2n+1}, x_{4n-1}$ represent the pendant nodes. Let $\bar{G}$ be the newly constructed graph by switching any pendant node of I_n, say x_k. Clearly, the number of elements in node & edge sets of $\bar{G}$) are equal to $4n - 1$ & $8n - 6$, respectively. Define a function (bijective) $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 4n-1\}$ by assigning $T(x_k) = 1$ & $T(x_{3n}) = p$ where p is the largest prime $\leq 4n - 1$. Label the rest of the nodes by using unutilized labels out of $\{1, 2, ..., 4n - 1\}$. One can see that $4n - 3$ edges recieve the label 1 as $T(x_k) = 1$ & there are exactly $4n - 3$ edges whose one end node is x_k. Consequently, the remaining edges get the label 0. So $|e_T(0) - e_T(1)| \leq 1$ which shows that $\bar{G}$ is a DCG. □

Corollary 7. *Switching of any node of degree 2 in I_n admits a DCL.*

Proof. Switching a node of degree 2 in I_n results in a graph having $4n - 1$ nodes and $8n - 8$ edges. The labeling is done on the similar lines as in Theorem 6. □

Theorem 8. *Switching of the apex node in I_n admits a DCL.*

Proof. Let $\bar{G}$ be obtained by switching the apex node of I_n, namely, x_{3n}. Clearly, the number of elements in node & edge sets of $\bar{G}$ are equal to $4n - 1$ & $4n - 8$, respectively. Define a function $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 4n-1\}$ by assigning $T(x_{3n}) = 1$, $T(x_{2n+1}) = 4$, $T(x_i) = T(x_{i-1})+2$; $2n+2 \leq i \leq 3n-1$, $T(x_{3n+1}) = T(x_{3n-1}) + 2$, $T(x_i) = T(x_{i-1}) + 2$; $3n + 2 \leq i \leq 4n - 1$. Now assigning the unutilized labels to unlabeled nodes in any order yields that $|e_T(0) - e_T(1)| \leq 1$ (see Figure 4). □

Definition 6. [6] Duplication of a node z_i with a node z_i' results in $\bar{G}$ having the property that $N(z_i) = N(z_i')$.

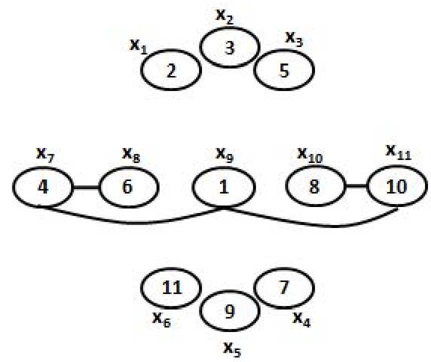

Figure 4. DCL of switching of x_{3n} in I_3

Theorem 9. *Duplication of the apex node in I_n admits a DCL.*

Proof. Let $\bar{G}$ be formed by duplicating the apex node x_{3n} of I_n by the newly added node s. Clearly, the number of elements in node & edge sets of $\bar{G}$ are given by $4n$ and $6n$, respectively. Define a function $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 4n\}$ by assigning $T(x_{3n}) = 2$, $T(s) = 4$, $T(x_1) = 1$, $T(x_2) = 6$, $T(x_i) = T(x_{i-1}) + 2$; $3 \leq i \leq 2n - 1$, $T(x_{2n}) = 5$, $T(x_{2n+1}) = 3$ and $T(x_{2n+2}) = 9$. Now allocate the unutilized labels from $\{1, 2, ..., 4n\}$ simultaneously to the unlabeled nodes x_j ; $2n + 3 \leq j \leq 4n - 1$ and $j \neq 3n$. It can be easily seen that $|e_T(0) - e_T(1)| \leq 1$ which shows that $\bar{G}$ is a DCG. □

Theorem 10. *The duplication of an arbitrary node of degree 1 or 2 in I_n permits a DCL.*

Proof. Let $\bar{G}$ be formed by duplicating any arbitrary node x_k of I_n by the the newly inserted node s. Then the cardinality of node set of $\bar{G}$ is $4n$. Now arise two cases.
Case(i). Duplication of a node of degree 1.
In this case, the cardinality of edge set of $\bar{G}$ is $4n - 1$. Define a function (bijective) $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 4n\}$ by fixing $T(x_{3n}) = 2$, $T(x_1) = 1$ and $T(s)$ be the largest prime $p \leq 4n$. Assign the unutilized even labels to x_i; $2 \leq i \leq 2n$ and odd labels simultaneously to x_j ; $2n + 1 \leq j \leq 4n - 1, j \neq 3n$.
Case(ii). Duplication of a node of degree 2.
In this case, the cardinality of edge set of $\bar{G}$ is $4n$. Labeling is done by using the pattern of case(i) (see Figure 5).
In both the cases, we observe that the difference of edges having labels 1 and 0 is not more than 1 which verifies that $\bar{G}$ is a DCG. □

Theorem 11. *$DS(I_n)$ permits a DCL.*

Proof. Let $\bar{G}$ denote the $DS(I_n)$ with $\bar{V}(\bar{G}) = V(I_n) \cup \{v, w\}$ and $\bar{E}(\bar{G}) = E(I_n) \cup \{x_i v : 1 \leq i \leq 2n\} \cup \{x_{2n+1}v, x_{4n-1}v\} \cup \{x_i w : 2n + 2 \leq i \leq 4n - 2, i \neq 3n\}$. The number of elements in node & edge sets of $\bar{G}$ are respectively equal to $4n + 1$ & $8n - 4$. Consider a bijective map $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 4n + 1\}$ determined by choosing $T(x_{3n}) = 1$, $T(v) = 4$, $T(w) = 2$, $T(x_{4n-1}) = 4n - 2$, $T(x_1) = 3$, $T(x_i) = T(x_{i-1}) + 2$; $2 \leq i \leq 2n$ and

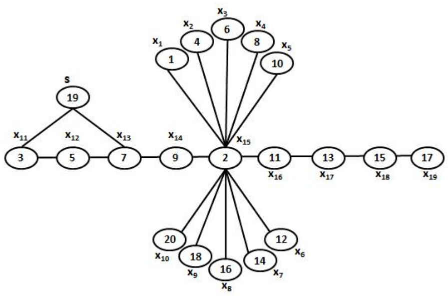

Figure 5. DCL of duplication of node x_{12} of degree 2 in I_5

$T(x_{2n+1}) = 6$. Assign the unutilized labels to the remaining nodes in any order. It follows that $\bar{G}$ is a DCG (see Figure 6). □

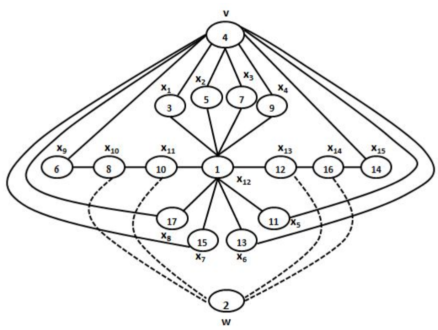

Figure 6. DCL of degree splitting graph of I_4

Definition 7. [6] Let $e = rs \in \bar{G}$. Then e is subdivided once it is substituted by edges $e' = rw$ & $e'' = ws$. If all edges in $\bar{G}$ are subdivided, we obtain a graph which we call as barycentric subdivision of $\bar{G}$.

Theorem 12. *Barycentric subdivision of I_n permits a DCL.*

Proof. Let $\bar{G}$ represent the barycentric subdivision of I_n. Here, $\bar{V}(\bar{G}) = V(I_n) \cup \{v_1, v_2, ..., v_{2n}, v_{2n+1}, ..., v_{4n-2}\}$ & $\bar{E}(\bar{G}) = \{x_{3n}v_i : 1 \le i \le 2n\} \cup \{v_i x_i : 1 \le i \le 2n\} \cup \{x_i v_i : 2n+1 \le i \le 4n-2\} \cup \{v_i x_{i+1} : 2n+1 \le i \le 4n-2\}$. Clearly, the number of elements in node & edge sets of $\bar{G}$ are respectively equal to $8n-3$ & $8n-4$. Consider a function (bijective) $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 8n-3\}$ by choosing $T(x_{3n}) = 2$, $T(x_1) = 1$, $T(v_1) = 4$, $T(v_2) = 6$,
$T(v_i) = T(v_{i-1}) + 4; 3 \le i \le 2n-1$,
$T(x_i) = \frac{T(v_i)}{2}; 2 \le i \le 2n-1$,
$T(v_{2n}) = T(x_{2n-1}) + 2$ & $T(x_{2n}) = T(v_{2n}) + 2$.
Next, fix $T(x_{2n+1}) = 8, T(v_{2n+1}) = 12$,
$T(x_i) = T(x_{i-1}) + 8; 2n+2 \le i \le 3n-1$,
$T(v_i) = T(v_{i-1}) + 8; 2n+2 \le i \le 3n-2$,
$T(x_{4n-1}) = T(x_{3n-1}) + 4, T(v_{3n-1}) = T(x_{2n}) + 2$,
$T(v_{3n}) = T(v_{3n-1}) + 2, T(x_{3n+1}) = T(v_{3n}) + 2$,
$T(v_i) = T(v_{i-1}) + 4; 3n+1 \le i \le 4n-2$
$T(x_i) = T(x_{i-1}) + 4; 3n+2 \le i \le 4n-2$.
Evidently, $e_T(1) = 4n-2$ & $e_T(0) = 4n-2$ which proves that $\bar{G}$ is DCG (see Figure 7). □

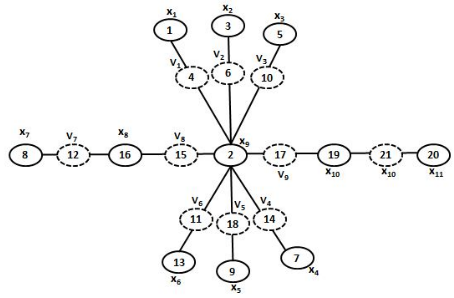

Figure 7. DCL of barycentric subdivision of I_3

Theorem 13. *Extension of all pendant nodes in I_n permits a DCL.*

Proof. Let $\bar{G}$ be formed by performing the extension of all pendant nodes of I_n with $\bar{V}(\bar{G}) = V(I_n) \cup \{v_1, v_2, ..., v_{2n}, \; v_{2n+1}, v_{4n-1}\}$ and $\bar{E}(\bar{G}) = E(I_n) \cup \{x_i v_i; 1 \le i \le 2n\} \cup \{v_i x_{3n}; 1 \le i \le 2n\} \cup \{x_{2n+1}v_{2n+1}, v_{2n+1}x_{2n+2}, x_{4n-1}v_{4n-1}, v_{4n-1}x_{4n-2}\}$. Clearly, the number of elements in node & edge sets of $\bar{G}$ are given by $6n+1$ & $8n+2$ respectively. Consider a function $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 6n+1\}$ defined by letting $T(x_{3n}) = 2$, $T(x_1) = 1, T(x_2) = 6, T(v_1) = 4$. Now two cases arise.
Case 1. when n is even.
$T(x_i) = T(x_{i-1}) + 4; 3 \le i \le n + \frac{n}{2}$,
$T(v_i) = \frac{T(x_i)}{2}; 2 \le i \le n + \frac{n}{2}$,
$T(x_{n+\frac{n}{2}+1}) = 8, T(x_i) = T(x_{i-1}) + 8; n + \frac{n}{2} + 2 \le i \le 2n$,
$T(v_i) = T(x_i) + 4; \; n + \frac{n}{2} + 1 \le i \le 2n$,
$T(v_{2n+1}) = T(v_{n+\frac{n}{2}}) + 2$ and $T(v_{4n-1}) = 6n+1$.
Assign all the unutilized even labels and then odd labels to $x_{2n+1}, x_{2n+2}, ..., x_{4n-2}$ (excluding x_{3n}) simultaneously from $\{1, 2, ..., 6n+1\}$.
Case 2. When n is odd.
$T(x_i) = T(x_{i-1}) + 4; 3 \le i \le n + \lceil \frac{n}{2} \rceil$,
$T(v_i) = \frac{T(x_i)}{2}; 2 \le i \le n + \lceil \frac{n}{2} \rceil$,
$T(x_{n+\lceil \frac{n}{2} \rceil+1}) = 8, T(x_i) = T(x_{i-1}) + 8; n + \lceil \frac{n}{2} \rceil + 2 \le i \le 2n$,
$T(v_i) = T(x_i) + 4; \; n + \lceil \frac{n}{2} \rceil + 1 \le i \le 2n$,
$T(v_{2n+1}) = T(v_{n+\lceil \frac{n}{2} \rceil}) + 2$ and $T(v_{4n-1}) = 6n+1$.
Allocate all unutilized even labels & then odd labels to $x_{2n+1}, x_{2n+2}, ..., x_{4n-2}$ (excluding x_{3n}) simultaneously from $\{1, 2, ..., 6n+1\}$.
In both the cases T induces DCL for $\bar{G}$.

□

Theorem 14. *Extension of the apex node in I_n permits a DCL.*

Proof. Let $\bar{G}$ be constructed by taking the extension of the apex node of I_n. Clearly, $\bar{V}(\bar{G}) = V(I_n) \cup \{w\}$ & $\bar{E}(\bar{G}) = E(I_n) \cup \{x_i w : 1 \le i \le 2n\} \cup \{x_{3n}w, x_{3n-1}w, x_{3n+1}w\}$. Note here, $|\bar{V}(\bar{G})| = 4n$ & $|\bar{E}(\bar{G})| = 6n+1$. Define a function (bijective) $T : \bar{V}(\bar{G}) \to \{1, 2, ..., 4n\}$ by choosing $T(x_{3n}) = 2, T(w) = 4, T(x_1) = 6, T(x_i) = T(x_{i-1}) + 2; 2 \le i \le 2n-2, T(x_{2n-1}) = 1$ and $T(x_{2n}) = 3$. Assign the unutilized odd labels simultaneously to $x_{2n+1}, x_{2n+2}, ..., x_{4n-1}$. It follows that $\bar{G}$ is a DCG. □

Conclusion: This paper has dealt with certain interesting general results on DCL for K_n related graphs besides, formulating an impressive conjecture on DCL. Further, we have proved that lilly graph admits a DCL and discussed DCL for some lilly related graphs under various graph operations.

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