Mathematics and Statistics 9(6): 1011-1018, 2021 http://www.hrpub.org
DOI: 10.13189/ms.2021.090617

Properties of Sakaguchi Kind Functions Associated with Bessel Function

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Received August 22, 2021; Revised October 19, 2021; Accepted November 11, 2021

Cite This Paper in the following Citation Styles

***(a)**: [1] H.Priya, B. Srutha Keerthi, "Properties of Sakaguchi Kind Functions Associated with Bessel Function," Mathematics and Statistics, Vol.9, No.6, pp. 1011-1018, 2021. DOI: 10.13189/ms.2021.090617*

***(b)**: H.Priya, B. Srutha Keerthi, (2021). Properties of Sakaguchi Kind Functions Associated with Bessel Function. Mathematics and Statistics, 9(6), 1011-1018. DOI: 10.13189/ms.2021.090617*

Abstract The aim of the paper is to obtain the First Hankel Determinant and the Second Hankel determinant. We shall make use of few lemmas which are based on Caratheodory's class of analytic functions. We establish a new Sakaguchi class

$$\frac{(1-t)[\rho\xi^2(N_{\nu,q}^{\lambda}f(\xi))''+\xi(N_{\nu,q}^{\lambda}f(\xi))']}{\rho\xi[(N_{\nu,q}^{\lambda}f(\xi))'-t(N_{\nu,q}^{\lambda}f(t\xi))']+(1-\rho)[N_{\nu,q}^{\lambda}f(\xi)-N_{\nu,q}^{\lambda}f(t\xi)]} \prec \Psi(\xi)$$

of univalent function, further we estimate the sharp bound for initial coefficients a_2 and a_3 using the Bessel function expansion. We have discussed about the coefficient a_4 as well for the Second Hankel Determinant. The results are obtained for Sakaguchi kind. Our results travel along exploring the stages of Hankel Determinants. Various types of technologies like wire, optical or other electromagnetic systems are used for the transmission of data in one device to another. Filters play an important role in the process that can remove disorted signals. By using different parameter values for the function belongs to Sakaguchi kind of functions the Low pass filter and High pass filter can be designed and that can be done by the coefficient estimates.

Keywords Coefficient Inequality, Subordination Techniques, Two Kinds of Bessel Function, Second Hankel Determinant

AMS Subject Classification rimary: 30C45, 30C80

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_\kappa \xi^\kappa, \quad \xi \in \mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\} \tag{1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$ which are univalent in $\mathbb{U}$. If $\kappa \in \mathcal{A}$ is represented as:

$$\kappa(\xi) = \xi + \sum_{\kappa=2}^{\infty} b_\kappa \xi^\kappa, \quad \xi \in \mathbb{U} \tag{2}$$

then, the Hadamard product of f and κ is constructed as follows:

$$(f \times \kappa)(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa \xi^\kappa, \quad \xi \in \mathbb{U} \tag{3}$$

If the function F is univalent in $\mathbb{U}$, then the following holds (see [1] and [2]):

$$f(\xi) \prec F(\xi) \Leftrightarrow f(0) = F(0) \ \ and \ \ f(\mathbb{U}) \subset F(\mathbb{U}).$$

The infinite series is given by:

$$J_\nu(\xi) = \sum_{\kappa \geq 0} \frac{(-1)^\kappa (\frac{\xi}{2})^{2\kappa+\nu}}{\kappa! \Gamma(\kappa+\nu+1)}, \ \ \xi \in \mathbb{C}, (\nu \in \mathbb{R}),$$

where Γ denotes the Gamma function.[3] the normalized Bessel function of the first kind $g_\nu : \mathbb{U} \to \mathbb{C}$ defined by (see also [4] - [6])

$$\begin{aligned} g_\nu(\xi) &= 2^\nu \Gamma(\nu+1) \xi^{1-\frac{\nu}{2}} J_\nu(\xi^{\frac{1}{2}}) \\ &= \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}\Gamma(\nu+1)}{4^{\kappa-1}(\kappa-1)!\Gamma(\kappa+\nu)} \xi^\kappa, \ \ \xi \in \mathbb{U}, (\nu \in \mathbb{R}) \end{aligned}$$

For the strict inquality $0 < q < 1$, g_ν is defined by;

$$\begin{aligned} \partial_q g_\nu(\xi) &= \partial_q \left\{ \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}\Gamma(\nu+1)}{4^{\kappa-1}(\kappa-1)!\Gamma(\kappa+\nu)} \xi^\kappa \right\} \\ &= \frac{g_\nu(q\xi) - g_\nu(\xi)}{\xi(q-1)} \\ &= 1 + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}\Gamma(\nu+1)}{4^{\kappa-1}(\kappa-1)!\Gamma(\kappa+\nu)} [\kappa, q] \xi^{\kappa-1}, \ \ \xi \in \mathbb{U} \end{aligned}$$

where

$$[\kappa, q] = \frac{1-q^\kappa}{1-q} = 1 + \sum_{j=1}^{\kappa-1} q^j, \ \ [0, q] = 0 \tag{4}$$

using (4), the next two products are obtained:
(i) The q-shifted fractional for a positive integer k is given by;

$$[\kappa, q]! = \begin{cases} 1, & \text{if } \kappa = 0 \\ [1,q][2,q][3,q]...[\kappa,q], & \text{if } \kappa \in \mathbb{N} \end{cases}$$

(ii) The q-generalised Pochhammer symbol for a positive number r is defined by;

$$[r, q]_\kappa = \begin{cases} 1, & \text{if } \kappa = 0 \\ [r,q][r+1,q]...[r+\kappa-1,q], & \text{if } \kappa \in \mathbb{N} \end{cases}$$

For the conditions $\nu > 0, \lambda > -1$, and $0 < q < 1$, we can define the function $I^\lambda_{\nu,q} : \mathbb{U} \to \mathbb{C}$ by;

$$I^\lambda_{\nu,q}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{(-1)^{\kappa-1}\Gamma(\nu+1)}{4^{\kappa-1}(\kappa-1)!\Gamma(\kappa+\nu)} \frac{[\kappa,q]!}{[\lambda+1,q]_{\kappa-1}} \xi^\kappa, \ \ \xi \in \mathbb{U}.$$

The Hankel determinants $H_\xi(1) = a_3 - a_2^2$ and $H_\xi(2) = a_2 a_4 - a_3^2$ are discussed.

Remark 1. *A simple reckoning shows that:*
$I^\lambda_{\nu,q}(\xi) \times M_{q\lambda+1}(\xi) = \xi \partial_q g_\nu \ \ \xi \in \mathbb{U}$, *where* $M_{q\lambda+1}$ *is given by:*

$$M_{q\lambda+1}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{[\lambda+1,q]_{\kappa-1}}{[\kappa-1,q]!} \xi^\kappa \ \ \xi \in \mathbb{U}.$$

By making using of idealogy of q-derivative, we instigate the linear operator $N^{\lambda}_{\nu,q}: \mathcal{A} \to \mathcal{A}$ defined by:

$$N^{\lambda}_{\nu,q} f(\xi) = I^{\lambda}_{\nu,q}(\xi) \times f(\xi) = \xi + \sum_{\kappa=2}^{\infty} \Psi_{\kappa} a_{\kappa} \xi^{\kappa}, \ \ \xi \in \mathbb{U}. \tag{5}$$

with the conditions $(\nu > 0, \lambda > -1, 0 < q < 1)$,
where

$$\Psi_{\kappa} = \frac{(-1)^{\kappa-1}\Gamma(\nu+1)}{4^{\kappa-1}(\kappa-1)!\Gamma(\kappa+\nu)} \frac{[\kappa,q]!}{[\lambda+1,q]_{\kappa-1}} \tag{6}$$

Remark 2. *From (5),*
(i) $[\lambda+1,q]N^{\lambda}_{\nu,q}f(\xi) = [\lambda,q]N^{\lambda+1}_{\nu,q}f(\xi) + q^{\lambda}\xi\partial_q(N^{\lambda+1}_{\nu,q}f(\xi)), \quad \xi \in \mathbb{U}$
(ii) $\lim_{q\to 1^-} N^{\lambda}_{\nu,q}f(\xi) = I^{\lambda}_{\nu,1} \times f(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{\kappa!}{[\lambda+1]_{\kappa-1}} \frac{(-1)^{\kappa-1}\Gamma(\nu+1)}{4^{\kappa-1}(\kappa-1)!\Gamma(\kappa+\nu)} a_{\kappa}\xi^{\kappa}$,
$\xi \in \mathbb{U}$

Proof. (i)To prove we consider RHS and arrive at the LHS

$$[\lambda,q]N^{\lambda+1}_{\nu,q}f(\xi) + q^{\lambda}\xi\partial_q(N^{\lambda+1}_{\nu,q}f(\xi)) = [\lambda,q]\left[\xi + \sum_{\kappa=2}^{\infty} \Upsilon \frac{[\kappa,q]!}{[\lambda+2,q]_{\kappa-1}} a_{\kappa}\xi^{\kappa}\right] + q^{\lambda}\left[\xi + \sum_{\kappa=2}^{\infty} \Upsilon \frac{[\kappa,q]![\kappa,q]}{[\lambda+2,q]_{\kappa-1}} a_{\kappa}\xi^{\kappa}\right]$$

$$= \xi\{[\lambda,q] + q^{\lambda}\} + \sum_{\kappa=2}^{\infty}\left\{\Upsilon \frac{[\kappa,q]!}{[\lambda+2,q]_{\kappa-1}} a_{\kappa}\xi^{\kappa}\right\}\{[\lambda,q] + q^{\lambda}[\kappa,q]\}$$

$$= \xi\left\{\frac{1-q^{\lambda}}{1-q} + q^{\lambda}\right\} + \frac{(-1)^{2-1}\Gamma(\nu+1)}{4^{2-1}(2-1)!\Gamma(2+\nu)} \frac{[2,q]!}{[\lambda+2,q]_{2-1}} a_2\xi^2\{[\lambda,q] + q^{\lambda}[2,q]\} + \frac{(-1)^{3-1}\Gamma(\nu+1)}{4^{3-1}(3-1)!\Gamma(3+\nu)} \frac{[3,q]!}{[\lambda+2,q]_{3-1}} a_3\xi^3\{[\lambda,q] + q^{\lambda}[3,q]\} + \cdots$$

$$= [\lambda+1,q]\xi + \sum_{\kappa=2}^{\infty} \Upsilon \frac{[\kappa,q]![\lambda+1,q]}{[\lambda+1,q]_{\kappa-1}} a_{\kappa}\xi^{\kappa}$$

$$= [\lambda+1,q]\left[\xi + \sum_{\kappa=2}^{\infty} \Upsilon \frac{[\kappa,q]!}{[\lambda+1,q]_{\kappa-1}} a_{\kappa}\xi^{\kappa}\right]$$

$$= [\lambda+1,q]N^{\lambda}_{\nu,q}f(\xi)$$

(ii)W.K.T,

$$N^{\lambda}_{\nu,q}f(\xi) = \xi + \sum_{\kappa=2}^{\infty} \Upsilon \frac{[\kappa,q]!}{[\lambda+1,q]_{\kappa-1}} a_{\kappa}\xi^{\kappa}$$

Applying limit, we get

$$\lim_{q\to 1^-} N^{\lambda}_{\nu,q}f(\xi) = \xi + \sum_{\kappa=2}^{\infty} \Upsilon a_{\kappa}\xi^{\kappa} \frac{(1,1)(2,1)(3,1)(4,1)...(\kappa,1)}{[\lambda+1,1][\lambda+2,1]...[\lambda+(\kappa-1),1]}$$

$$= \xi + \sum_{\kappa=2}^{\infty} \Upsilon \frac{\kappa!}{[\lambda+1]_{\kappa-1}} a_{\kappa}\xi^{\kappa}$$

where $\Upsilon = \frac{(-1)^{\kappa-1}\Gamma(\nu+1)}{4^{\kappa-1}(\kappa-1)!\Gamma(\kappa+\nu)}$ □

Now, we bring in the class of functions $M^{\lambda}_{\nu,q}(\rho,t,\Psi)$ as follows

Definition 3. Let $\Psi(\xi)=1+B_1\xi+B_2\xi^2+...,\ \ \xi\in\mathbb{U}$, with the condition $B_1>0$, then the function $f\in\mathcal{A}$ is said to be in the class $M^{\lambda}_{\nu,q}(\rho,t,\Psi)$ if the function

$$\frac{(1-t)[\rho\xi^2(N^{\lambda}_{\nu,q}f(\xi))''+\xi(N^{\lambda}_{\nu,q}f(\xi))']}{\rho\xi[(N^{\lambda}_{\nu,q}f(\xi))'-t(N^{\lambda}_{\nu,q}f(t\xi))']+(1-\rho)[N^{\lambda}_{\nu,q}f(\xi)-N^{\lambda}_{\nu,q}f(t\xi)]}$$

is analytic in $\mathbb{U}$ and satisfies:

$$\frac{(1-t)[\rho\xi^2(N^{\lambda}_{\nu,q}f(\xi))''+\xi(N^{\lambda}_{\nu,q}f(\xi))']}{\rho\xi[(N^{\lambda}_{\nu,q}f(\xi))'-t(N^{\lambda}_{\nu,q}f(t\xi))']+(1-\rho)[N^{\lambda}_{\nu,q}f(\xi)-N^{\lambda}_{\nu,q}f(t\xi)]}\prec\Psi(\xi)$$

$(\nu>0,\ \lambda>-1, 0<q<1, 0\le\rho\le1, |t|\le1$ but $t\neq1)$
In this paper, we obtain the Fekete-Szego inequalities and Second Hankel Determinant for the function of the class $M^{\lambda}_{\nu,q}(\rho,t,\Psi)$.

2 First Hankel Dterminant Problem

Lemma 4. *([8], Lemma 3) If $p(\xi)=1+C_1\xi+C_2\xi^2+....\in\mathcal{P}$ and α is said to be a complex number, then*

$$max|C_2-\alpha C_1^2|=2max\{1;|2\alpha-1|\}.$$

Lemma 5. *([9], $Lemma1$) If $p(\xi)=1+C_1\xi+C_2\xi^2+....\in\mathcal{P}$, then*

$$|C_2-\alpha C_1^2|\le\begin{cases}-4\alpha+2, & if\,\alpha\le0\\ 2, & if\,0\le\alpha\le1\\ 4\alpha-2, & if\,\alpha\ge1.\end{cases}$$

When $\alpha<0$ or $\alpha>1$, the equality holds iff

$$p(\xi)=\tfrac{1+\xi}{1-\xi}$$

or one of its rotations.
If the strict inequality $0<\alpha<1$ is considered, then the equality holds if and only if

$$p(\xi)=\tfrac{1+\xi^2}{1-\xi^2}$$

or one of its rotations.
If $\alpha=0$, the equality holds if and only if:

$$p(\xi)=(\tfrac{1}{2}+\tfrac{\lambda}{2})\tfrac{1+\xi}{1-\xi}+(\tfrac{1}{2}-\tfrac{\lambda}{2})\tfrac{1-\xi}{1+\xi},$$

with $0\le\lambda\le1$, or one of its rotations.
If $\alpha=1$, the equality holds iff:

$$\tfrac{1}{p(\xi)}=(\tfrac{1}{2}+\tfrac{\lambda}{2})\tfrac{1+\xi}{1-\xi}+(\tfrac{1}{2}-\tfrac{\lambda}{2})\tfrac{1-\xi}{1+\xi},$$

with $0\le\lambda\le1$. From the reference $([9],pages162-163)$, it can be improved in the following way when $0<\alpha<1$:

$$|C_2-\alpha C_1^2|+\alpha|C_1^2|\le2,\ \ if\ \ 0<\alpha\le\frac{1}{2} \tag{7}$$

and

$$|C_2-\alpha C_1^2|+(1+\alpha)|C_1^2|\le2,\ \ if\ \ \frac{1}{2}\le\alpha<1 \tag{8}$$

Theorem 6. *When $f(\xi)$ given by (1) belongs to the class $M^{\lambda}_{\nu,q}(\rho,t,\Psi)$, with $\Psi(\xi)$ which satisfies Definition (1), and μ is said to be a complex number, then:*

$$|a_3-\mu a_2^2|\le\frac{B_1}{\Psi_3(1+2\rho)(3-u_3)}max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1u_2}{(2-u_2)}-\frac{\mu B_1\Psi_3(3-u_3)(1+2\rho)}{\Psi_2^2(1+\rho)^2(2-u_2)^2}\right|\right\}$$

where $\Psi_\kappa,\ \kappa\in\{2,3\}$, are given by (6).

Proof. When $f(\xi) \in M^{\lambda}_{\nu,q}(\rho, t, \Psi)$,then we observe that there exists a Schwarz function W, which is analytic in $\mathbb{U}$, with $W(0) = 0$ and $|W(\xi)| < 1$, $z \in \mathbb{D}$, such that:

$$\frac{(1-t)[\rho\xi^2(N^{\lambda}_{\nu,q}f(\xi))'' + \xi(N^{\lambda}_{\nu,q}f(\xi))']}{\rho\xi[(N^{\lambda}_{\nu,q}f(\xi))' - t(N^{\lambda}_{\nu,q}f(t\xi))'] + (1-\rho)[N^{\lambda}_{\nu,q}f(\xi) - N^{\lambda}_{\nu,q}f(t\xi)]} = \Psi(W(\xi)), \ \ \xi \in \mathbb{U} \tag{9}$$

Since W is a Schwarz function, it follows that the function p_1 defined by;

$$p_1(\xi) = \frac{1 + W(\xi)}{1 - W(\xi)} = 1 + C_1\xi + C_2\xi^2 + ..., \ \ \xi \in \mathbb{U} \tag{10}$$

belongs to $\mathcal{P}$. By defining the function p by:

$$p(\xi) = \frac{(1-t)[\rho\xi^2(N^{\lambda}_{\nu,q}f(\xi))'' + \xi(N^{\lambda}_{\nu,q}f(\xi))']}{\rho\xi[(N^{\lambda}_{\nu,q}f(\xi))' - t(N^{\lambda}_{\nu,q}f(t\xi))'] + (1-\rho)[N^{\lambda}_{\nu,q}f(\xi) - N^{\lambda}_{\nu,q}f(t\xi)]} \tag{11}$$

$$= 1 + d_1\xi + d_2\xi^2 + ... \tag{12}$$

In view of (9) and (10), we have:

$$p(\xi) = \Psi\left(\frac{p_1(\xi) - 1}{p_1(\xi) + 1}\right), \ \xi \in \mathbb{U} \tag{13}$$

By making use of (10), we easily obtain:

$$\tfrac{p_1(\xi)-1}{p_1(\xi)+1} = \tfrac{1}{2}[C_1\xi + (C_1 - \tfrac{C_1^2}{2})\xi^2 + (C_3 + \tfrac{C_1^3}{4} - C_1C_2)\xi^3 + ...], \ \xi \in \mathbb{U}$$

therefore

$$\Psi\left(\tfrac{p_1(\xi)-1}{p_1(\xi)+1}\right) = 1 + \tfrac{1}{2}B_1C_1\xi + \left(\tfrac{1}{2}B_1\left(C_2 - \tfrac{C_1^2}{2}\right) + \tfrac{1}{4}B_2C_1^2\right)\xi^2 + ..., \ \xi \in \mathbb{U}$$

and from (12), we obtain

$$d_1 = \tfrac{1}{2}B_1C_1$$
$$d_2 = \tfrac{1}{2}B_1\left(C_2 - \tfrac{C_1^2}{2}\right) + \tfrac{1}{4}B_2C_1^2$$
$$d_3 = \tfrac{1}{2}B_1\left(C_3 + \tfrac{C_1^3}{4} - C_1C_2\right) + \tfrac{1}{2}B_2C_1\left(C_2 - \tfrac{C_1^2}{2}\right)$$

On the other hand, from (11), according to (5), it follows that

$$d_1 = (1+\rho)(2-u_2)\Psi_2a_2$$
$$d_2 = (1+2\rho)\Psi_3a_3(3-u_3) - (1+\rho)^2\Psi_2^2a_2^2(2-u_2)u_2$$
$$d_3 = (1+3\rho)\Psi_4a_4(4-u_4) - \Psi_2a_2\Psi_3a_3[(1+\rho)(2-u_2)u_3 + (1+\rho)(1+2\rho)(3-u_3)u_2] + (1+\rho)^3\Psi_2^3a_2^3u_2^2(2-u_2)$$

and combning d_1, d_2 and d_3 values, we have

$$a_2 = \frac{B_1C_1}{2(1+\rho)(2-u_2)\Psi_2} \tag{14}$$

$$a_3 = \frac{B_1}{2(1+2\rho)(3-u_3)\Psi_3}\left[C_2 - \frac{C_1^2}{2} + \frac{1}{2}\frac{B_2}{B_1}C_1^2 + \frac{B_1C_1^2u_2}{2(2-u_2)}\right] \tag{15}$$

$$a_4 = \frac{1}{2(1+3\rho)(4-u_4)\Psi_4}\Bigg[B_1\left(C_3 + \frac{C_1^3}{4} - C_1C_2\right) + B_2C_1\left(C_2 - \frac{C_1^2}{2}\right) + ((2-u_2)u_3 + (1+2\rho)(3-u_3)u_2)\left(\frac{B_1^2C_1}{2(1+2\rho)(3-u_3)(2-u_2)}\right) \left(C_2 - \frac{C_1^2}{2} + \frac{B_2C_1^2}{2B_1} + \frac{B_1C_1^2u_2}{2(2-u_2)}\right) - \frac{B_1^3C_1^3u_2^2}{4(2-u_2)^2}\Bigg] \tag{16}$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{B_1}{2\Psi_3(1+2\rho)(3-u_3)}\{C_2 - \alpha C_1^2\} \tag{17}$$

where

$$\alpha = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1u_2}{(2-u_2)} + \frac{\mu B_1\Psi_3(3-u_3)(1+2\rho)}{\Psi_2^2(1+\rho)^2(2-u_2)^2}\right] \tag{18}$$

and by making use of the Lemma 1, the upcomming results are discussed. □

Theorem 7. *If $f(\xi)$ given by (1), with $\Psi(\xi)$ satisfyies Definition (1) and μ, $B_2 \in \mathbb{R}$ and $0 \leq \rho \leq 1$ and $|t| \leq 1$ but $t \neq 1$, belongs to the class $M^{\lambda}_{\nu,q}(\rho, t, \Psi)$ then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{\Psi_3(1+2\rho)(3-u_3)}\left[\frac{B_2}{B_1} + \frac{B_1 u_2}{(2-u_2)} - \frac{\mu B_1 \Psi_3(3-u_3)(1+2\rho)}{\Psi_2^2(1+\rho)^2(2-u_2)^2}\right], & \text{if } \mu \leq \sigma_1 \\ \frac{B_1}{\Psi_3(1+2\rho)(3-u_3)}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{-B_1}{\Psi_3(1+2\rho)(3-u_3)}\left[\frac{B_2}{B_1} + \frac{B_1 u_2}{(2-u_2)} - \frac{\mu B_1 \Psi_3(3-u_3)(1+2\rho)}{\Psi_2^2(1+\rho)^2(2-u_2)^2}\right], & \text{if } \mu \geq \sigma_2. \end{cases}$$

with

$$\sigma_1 = \frac{\Psi_2^2(1+\rho)^2(2-u_2)^2}{B_1\Psi_3(3-u_3)(1+2\rho)}\left[\frac{B_2}{B_1} + \frac{B_1 u_2}{(2-u_2)} - 1\right] \tag{19}$$

$$\sigma_2 = \frac{\Psi_2^2(1+\rho)^2(2-u_2)^2}{B_1\Psi_3(3-u_3)(1+2\rho)}\left[\frac{B_2}{B_1} + \frac{B_1 u_2}{(2-u_2)} + 1\right] \tag{20}$$

where $\Psi_\kappa, \kappa \in \{2, 3\}$, are given by (6)

Proof. The proof of the theorem follows from Lemma (4).

□

Theorem 8. *If the function f given by (1) belongs to the class $M^{\lambda}_{\nu.q}(\rho, t, \Psi)$, with $\Psi(\xi)$ satisfies Definition (1) and μ, $B_2 \in \mathbb{R}, 0 \leq \rho \leq 1$ and $|t| \leq 1$ but $t \neq 1$ then*
(i)When $\sigma_1 < \mu \leq \sigma_3$, we have

$$|a_3 - \mu a_2^2| + P\left[1 - \frac{B_2}{B_1} - \frac{B_1 u_2}{(2-u_2)} + \frac{\mu B_1 \Psi_3(3-u_3)(1+2\rho)}{\Psi_2^2(1+\rho)^2(2-u_2)^2}\right]|a_2^2| \leq Q \tag{21}$$

(ii)When $\sigma_3 < \mu \leq \sigma_2$, we have

$$|a_3 - \mu a_2^2| + P\left[1 + \frac{B_2}{B_1} + \frac{B_1 u_2}{(2-u_2)} - \frac{\mu B_1 \Psi_3(3-u_3)(1+2\rho)}{\Psi_2^2(1+\rho)^2(2-u_2)^2}\right]|a_2^2| \leq Q \tag{22}$$

where

$$P = \frac{(1+\rho)^2(2-u_2)^2\Psi_2^2}{\Psi_3 B_1(1+2\rho)(3-u_3)},$$

$$Q = \frac{B_1}{\Psi_3(1+2\rho)(3-u_3)}$$

where the values of σ_1 and σ_2 are defined by (19) and (20) accordingly.

$$\sigma_3 = \frac{\Psi_2^2(1+\rho)^2(2-u_2)^2}{B_1\Psi_3(3-u_3)(1+2\rho)}\left[\frac{B_2}{B_1} + \frac{B_1 u_2}{(2-u_2)}\right]$$

and $\Psi_\kappa, \kappa \in \{2, 3\}$ are given by (6).

Proof. Using Theorem 1, we can obtain (15) and (16) from (14) it is obvious that:

$$C_1 = \frac{2a_2(1+\rho)(2-u_2)\Psi_2}{B_1} \tag{23}$$

The proof follows from Lemma(5). □

3 Second Hankel Determinant

Theorem 9. *If $f(\xi) \in M^{\lambda}_{\nu,q}(\rho, t, \Psi)$ then*

$$|a_2 a_4 - a_3^2| \leq \left|\frac{B_1}{4}\left\{A\left[B_1\left(C_3 + \frac{C_1^3}{4} - C_1 C_2\right) + B_2 C_1\left(C_2 - \frac{C_1^2}{2}\right)\right] + (E)\left[A((2-u_2)u_3 + (1-2\rho)(3-u_3)u_2)\left(\frac{B_1^2 C_1}{2(1+\rho)(3-u_3)(2-u_2)}\right) - \frac{B_1}{(1+2\rho)^2(3-u_3)^2\Psi_3^2}(E)\right]\right\}\right| \tag{24}$$

where

$$A=\frac{C_1}{(1+\rho)(2-u_2)(1+3\rho)(4-u_4)\Psi_2\Psi_4}$$

and

$$E=C_2-\frac{C_1^2}{2}+\frac{1}{2}\frac{B_2}{B_1}C_1^2+\frac{B_1C_1^2u_2}{2(2-u_2)}$$

Proof. From (14), (15) and (16), we have

$$|a_2a_4-a_3^2|\leq\left|\frac{B_1}{4}\left\{A\left[B_1\left(C_3+\frac{C_1^3}{4}-C_1C_2\right)+B_2C_1\left(C_2-\frac{C_1^2}{2}\right)\right]+(E)\left[A((2-u_2)u_3+(1-2\rho)(3-u_3)u_2)\left(\frac{B_1^2C_1}{2(1+\rho)(3-u_3)(2-u_2)}\right)-\frac{B_1}{(1+2\rho)^2(3-u_3)^2\Psi_3^2}(E)\right]\right\}\right| \quad (25)$$

Where,

$$A=\frac{C_1}{(1+\rho)(2-u_2)(1+3\rho)(4-u_4)\Psi_2\Psi_4}$$

and

$$E=C_2-\frac{C_1^2}{2}+\frac{1}{2}\frac{B_2}{B_1}C_1^2+\frac{B_1C_1^2u_2}{2(2-u_2)}$$

Which gives the desired inequality (24) □

Corollary 10. *If* $\rho=0; t=0; u_n=1; C_n=\frac{(-1)^{n+1}}{(2n)!}$ *then*

$$|a_2a_4-a_3^2|\leq\left|\frac{B_1}{4}\left\{\frac{1}{6\Psi_2\Psi_4}\left(\frac{77B_1}{1440}-\frac{B_2}{12}\right)+\left(\frac{-1}{6}+\frac{B_2}{8B_1}+\frac{B_1}{8}\right)\left[\frac{B_1^2}{16\Psi_2\Psi_4}-\frac{B_1}{4\Psi_3^2}\left(\frac{-1}{6}+\frac{B_2}{8B_1}+\frac{B_1}{8}\right)\right]-\frac{B_1^3}{192\Psi_2\Psi_4}\right\}\right|$$

4 Conclusion

By setting the values of $\rho, t\ and\ u_n$ we bring out the interesting coefficient inequality and subordination techniques, for the subclasses of $M_{\nu,q}^{\lambda}(\rho,t,\Psi)$. The investigation of initial coefficient bounds Fekete-Szego inequality and Second Hankel determinant for various subclasses can be a scope of future research for disorted signals.

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