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# Derivation of New Degrees for Best COCUNP Weighted Approximation: II

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**Abstract** Approximation Theory is a branch of analysis and applied mathematics requiring that the approximation process preserves certain $f$-shaped properties defined at a finite interval $[a,b]$, such as convexity in all or parts of the interval. The (Co)convex and Unconstrained Polynomial (COCUNP) approximation is one of the key estimations of the approximation theory that Kopotun has recently raised for ten years. Numerous studies have been conducted on modern methods of weighted approximation to construct the best degree of approximation. In developing COCUNP a novel technique, the Lebesgue Stieltjes integral-i technique is used to resolve certain disadvantages, such as Riemann's integrable functions, which do not have the degree of the best approximation in norm space. In order to achieve the main goal, Derivation of New Degree (DOND) of the best COCUNP approximation was constructions. The theoretical results revealed that, in general, the new degrees of best approximation were able to smaller errors compared to the existing literature in the same estimating. In conclusion, this study has successfully developed DOND for the best (Co)convex Polynomial (COCP) weighted approximation.

**Keywords** Constrained and Unconstrained Approximation, (Co)convex Polynomial Approximation, Lebesgue Stieltjes Integral

**2020 Mathematics Subject Classification** 41A10, 41A25, 26A42

## 1 Introduction

The $L_p$ space is one of the most important and interesting concepts in approximation theory. In other words, a large number of theorems of best approximation can be regarded depend on $L_p$ space, so that a normed space is probably the most important kind of spaces in functional analysis, at least from the field of approximation.

**Definition 1.1.** *[1] The space $L_p(\mathbb{D}, d\mathscr{L})$, $0<p<\infty$, denotes the space of all measurable functions $f$ on $\mathbb{D}=[-1,1]$, such that $\|f\|_p<\infty$, where $\|f\|_p=(\int_{-1}^{1}|f(x)|^p d\mathscr{L})^{\frac{1}{p}}$ is norm for $p\geq 1$, and quasi norm for $0<p<1$. Also, $\|f\|_\infty$ , which is called the essential supremum of $f$, is defined by*

$$\|f\|_\infty = ess\sup_{x\in\mathbb{D}}|f(x)|.$$

The following notation was introduced by [2]. For $\alpha,\beta\in J_p$, let us denote $J_p=(\frac{-1}{p},0)$ if $0<p<\infty$, and $J_p=[0,\infty)$ if $p=\infty$. The following is definition of $\mathbb{L}_{p,r}^{\alpha,\beta}$ and $\mathbb{L}_p^{\alpha,\beta}$ spaces.

**Definition 1.2.** *[2] Let $w_{\alpha,\beta}(x)=(1+x)^\alpha(1-x)^\beta$ be the (classical) Jacobi weight. Define*

$$\mathbb{L}_{p,r}^{\alpha,\beta}=\{f:\mathbb{D}\longrightarrow\mathbb{R}: f^{(r-1)}\in AC_{loc}(-1,1),\ \|f\|_{w_{\alpha,\beta},p}=$$

$$\|w_{\alpha,\beta}f^{(r)}\|_p<\infty\ ,\ and\ 1\leq p\leq\infty\},$$

$$\mathbb{L}_{p}^{\alpha,\beta}=\{f:\mathbb{D}\longrightarrow\mathbb{R}: \|f\|_{\alpha,\beta,p}=\|w_{\alpha,\beta}f\|_p<\infty,\ and\ 0<p\leq\infty\},$$

*and for convenience denote $\mathbb{L}_{p,0}^{\alpha,\beta}=\mathbb{L}_p^{\alpha,\beta}$ .*

The following is the definition of the knots of Chebyshev partition.

**Remark 1.3.** *[3] The partition $\hat{T}_\eta = \{t_j\}_{j=0}^{\eta}$, where*

$$t_j = t_{j,\eta} = \begin{cases} -\cos(\frac{j\Pi}{\eta})\,; & \text{if } 0 \le j \le \eta, \\ -1\,; & \text{if } j < 0, \\ 1\,; & \text{if } j > \eta, \end{cases}$$

*and $t_j$'s as the nots of Chebyshev partition and note that $t_j$, $1 \le j \le \eta - 1$ are the extremum points of the Chebyshev polynomial of the first kind of degree $\le n$. If $\eta > 1$, the set $t_j = t_{j,\eta} = -\cos(\frac{j\Pi}{\eta})$, $j = 0, \cdots, \eta$ is called the Chebyshev partition of $[-1,1]$.*

We will use the following notations below which was defined by [4]. Let $\theta_{\mathscr{N}}$ be a partition of $[-1,1]$ which have at least $k$ intervals, that is,

$$\theta_{\mathscr{N}} = \theta_{\mathscr{N}}[-1,1] = \{x_i\}_{i=0}^{\mathscr{N}} = \{-1 = x_\circ \le \cdots \le x_{\mathscr{N}-1} \le x_{\mathscr{N}} = 1\}.$$

Let $i$, $0 \le i \le \mathscr{N}$ be a fixed, and denote $j(i) = \max\{0, i-k+1\}$, then we write

$$x^{\#} = x_{j(i)+1},\ x_* = x_{j(i)+2}, \text{ and } \quad x_i \in [x_i - \frac{x^{\#}}{2}, x_i + \frac{x_*}{2}] \subseteq [x_\circ, x_{\mathscr{N}}] \tag{1}$$

and

$$\|\theta_{\mathscr{N}}\| = \max_{0 \le i \le \mathscr{N}-1} \{x_{i+1} - x_i\}$$

the length of the largest interval in that partition.

Let $\mathbb{D} = [-1,1]$ be measurable subset of $\mathbb{R}$ and $\mathbf{P} = \{\mathbb{D}_j\}_{j\in\mathbb{N}}$ be a family of finite subsets of $\mathbb{D}$. We have Lebesgue partition $\mathbf{P}$ of $\mathbb{D}$, if $\mathbb{D}_j$ are measurable sets, $\cup_{j\in\mathbb{N}}\mathbb{D}_j = \mathbb{D}$ and $\mathbb{D}_j \cap \mathbb{D}_\iota = \emptyset$, for $j \ne \iota$. Now, the following definition is referred to as Lebesgue Stieltjes integral-i, a term that will be used extensively throughout this paper.

**Definition 1.4.** *[5] Let $\mathbb{D}$ be measurable set, $f : \mathbb{D} \longrightarrow \mathbb{R}$ be a bounded function, and $\mathscr{L}_i : \mathbb{D} \longrightarrow \mathbb{R}$ be nondecreasing function for $i \in \mathbb{N}$. For a Lebesgue partition $\mathbf{P}$ of $\mathbb{D}$, put $\underline{\mathrm{LS}}(f, \mathbf{P}, \underline{\mathscr{L}}) = \sum_{j=1}^{n} \prod_{i\in\mathbb{N}} m_j\ \mathscr{L}_i(\mu(\mathbb{D}_j))$ and $\overline{\mathrm{LS}}(f, \mathbf{P}, \underline{\mathscr{L}}) = \sum_{j=1}^{n} \prod_{i\in\mathbb{N}} M_j\ \mathscr{L}_i(\mu(\mathbb{D}_j))$ where $\mu$ is a measure function of $\mathbb{D}$, $m_j = \inf\{f(x) : x \in \mathbb{D}_j\}$, $M_j = \sup\{f(x) : x \in \mathbb{D}_j\}$, and $\underline{\mathscr{L}} = \mathscr{L}_1, \mathscr{L}_2, \cdots$. Also, $\mathscr{L}_i(x_j) - \mathscr{L}_i(x_{j-1}) > 0$, $\underline{\mathrm{LS}}(f, \mathbf{P}, \underline{\mathscr{L}}) \le \overline{\mathrm{LS}}(f, \mathbf{P}, \underline{\mathscr{L}})$, $\prod_{i\in\mathbb{N}} \underline{\int}_i^{\mathbb{D}} f\, \mathrm{d}\underline{\mathscr{L}} = \sup\{\underline{\mathrm{LS}}(f, \underline{\mathscr{L}})\}$ and $\prod_{i\in\mathbb{N}} \overline{\int}_i^{\mathbb{D}} f\, \mathrm{d}\underline{\mathscr{L}} = \inf\{\overline{\mathrm{LS}}(f, \underline{\mathscr{L}})\}$ where $\underline{\mathrm{LS}}(f, \underline{\mathscr{L}}) = \{\underline{\mathrm{LS}}(f, \mathbf{P}, \underline{\mathscr{L}}) : \mathbf{P}$ part of set $\mathbb{D}\}$ and $\overline{\mathrm{LS}}(f, \underline{\mathscr{L}}) = \{\overline{\mathrm{LS}}(f, \mathbf{P}, \underline{\mathscr{L}}) : \mathbf{P}$ part of set $\mathbb{D}\}$. If $\prod_{i\in\mathbb{N}} \underline{\int}_i^{\mathbb{D}} f\, \mathrm{d}\underline{\mathscr{L}} = \prod_{i\in\mathbb{N}} \overline{\int}_i^{\mathbb{D}} f\, \mathrm{d}\underline{\mathscr{L}}$ where $\mathrm{d}\underline{\mathscr{L}} = d\mathscr{L}_1 \times d\mathscr{L}_2 \times \cdots$. Then $f$ is integral $\int_i$ according to $\mathscr{L}_i$ for $i \in \mathbb{N}$.*

The class of all functions of Lebesgue Stieltjes integral-i is defined as follows.

Let $I_f$ be the class of all functions of Lebesgue Stieltjes integral-i of $f$ that satisfying Definition 1.4, i.e.,

$$I_f = \{f : f \text{ is integrable function according to } \mathscr{L}_i,\ i \in \mathbb{N}\}$$

$$= \{f : \prod_{i\in\mathbb{N}} \underline{\int}_i^{\mathbb{D}} f\, \mathrm{d}\underline{\mathscr{L}} = \prod_{i\in\mathbb{N}} \overline{\int}_i^{\mathbb{D}} f\, \mathrm{d}\underline{\mathscr{L}}\}.$$

## 2 COCUNP and Auxiliary Results

In this section, we will mention the concepts COCP and Unconstrained Polynomial (UNP). The definition of convex function was given as follows.

**Definition 2.1.** *[6] A function $f : [a,b] \to \mathbb{R}$ is called convex (or convex downward) in the interval $[a,b]$ if*

$$f((1-\lambda)x + \lambda y) \le (1-\lambda)f(x) + \lambda f(y),$$

*for all $x, y \in [a,b]$ and all $\lambda \in [0,1]$.*

Note that $f : [a,b] \to \mathbb{R}$ is said to be concave if $-f$ is convex. Definition 2.1 is given in general abstract space. Since the real line (or an interval) is a convex set the definition of CP simply can be formulated as follows.

**Definition 2.2.** *A polynomial $p_n$ is called convex in the interval $[a,b]$ if*

$$p_n(\lambda x + (1-\lambda)y) \le \lambda p_n(x) + (1-\lambda)p_n(y),$$

*for all $x, y \in [a,b]$ and $\lambda \in (0,1)$.*

Next, we recall the definition of coconvex functions.

**Definition 2.3.** *[7] Let $\mathbb{D} = [-1,1]$ and $Y_s = \{y_i\}_{i=1}^{s}$, $s \in \mathbb{N}$ be a partition of $\mathbb{D}$, that is, a collection of $s$ fixed points $y_i$ such that*

$$y_{s+1} = -1 < y_s < \cdots < y_1 < 1 = y_\circ$$

*and let $\Delta^{(2)}(Y_s)$ be the set of continuous functions on $\mathbb{D}$ that are convex downwards on the segment $[y_{i+1}, y_i]$ if $i$ is even and convex upwards on the same segment if $i$ is odd. The functions from $\Delta^{(2)}(Y_s)$ are called coconvex.*

If the polynomial $p_n$ preserves the shape (or curve) of the function $f$, it is then known as a constrained polynomial. Otherwise, it is an unconstrained polynomial.

**Definition 2.4.** *[8] Let $f \in \Delta^{(2)}(Y_s)$, for $i \in \mathbb{N}$, the symmetric difference of $f$ is denoted by*

$$\mho_{h\phi}^{i}(f, x) = \begin{cases} \prod_{i\in\mathbb{N}} \int_i^{\mathbb{D}} f\, \mathrm{d}\underline{\mathscr{L}_\phi}\,; & \text{if } f \in I_f, \\ 0\,; & \text{otherwise.} \end{cases} \tag{2}$$

**Definition 2.5.** *[8] For $\alpha, \beta \in J_p$, $r \in \mathbb{N}_\circ$ and $0 < p \le \infty$, $\Phi^{p,r}(w_{\alpha,\beta})$ space defines as*

$$\Phi^{p,r}(w_{\alpha,\beta}) = \{f : f \in \mathbb{L}_{p,r}^{\alpha,\beta} \cap I_f \text{ and } \mho_{h\phi}^{i}(f,x) < \infty\},$$

*and $\Phi^{p,0}(w_{\alpha,\beta}) = \Phi^{p}(w_{\alpha,\beta})$.*

**Definition 2.6.** *[8] A weighted DTMS in $\Phi^{p,r}(w_{\alpha,\beta})$ is defined by*

$$\varpi_{i,r}^{\phi}(f^{(r)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p} = \sup\{\|w_{\alpha,\beta}\, \phi^r \mho_{h\phi}^{i}(f^{(r)}, x)\|_p,\ 0 < h \le \|\theta_{\mathscr{N}}\|\},$$

*where $\|\theta_{\mathscr{N}}\| < 2(i^{-1})$, $\mathscr{N} \ge 2$.*

**Theorem 2.7.** *[8] Let $s, r \in \mathbb{N}_\circ$, $0 < p \le \infty$ and $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$. Let $\mathbb{D}$ be defined in Definition 1.4, such that $|\mathbb{D}| \le \delta_\circ$ for some $\delta_\circ \in \mathbb{R}^+$. Then,*

$$\varpi^{\phi}_{i+1,r}\,(f^{(r)},\ \|\theta_{\mathscr{N}}\|,\mathbb{D})_{w_{\alpha,\beta},p} \le c(\delta_\circ)\varpi^{\phi}_{i,r+1}\,(f^{(r+1)},\ \|\theta_{\mathscr{N}}\|,\ \mathbb{D})_{w_{\alpha,\beta},p} \qquad (3)$$

*where the constant c depend on $\delta_\circ$.*

**Theorem 2.8.** *[8] Let $s, r \in \mathbb{N}_\circ$, $\alpha, \beta \in J_p$ and $0 < p \le \infty$. Let $\mathbf{P}$ be a Lebesgue partition of $\mathbb{D}$, and $\hat{T}_\eta$ be a Chebyshev partition with $\mathbf{P} \cap \hat{T}_\eta \ne \emptyset$, $1 \le \eta \le r$. If $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, then there is a constant c depend on $\eta$ and $J_{j,\eta}$ such that*

$$\varpi^{\phi}_{i+\eta}\,(f,\ \|\theta_{\mathscr{N}}\|,\mathbb{D})_{w_{\alpha,\beta},p} \le c\|\theta_{\mathscr{N}}\|^{-\eta}\varpi^{\phi}_{i,2\eta}\,(f^{(2\eta)},\ \|\theta_{\mathscr{N}}\|,\ \mathbb{D})_{w_{\alpha+\eta,\beta+\eta},p}\,.$$

This study aims to complement the DOND of best COCUNP approximation in our previous works to estimate the function, but based on the Chebyshev partition and the Lebesgue partition. In addition, we aim to develop the degree of best coconvex and unconstrained polynomial approximation of $f$, that has one inflection point.

# 3 Main Results

This section introduces a new definition called the degrees of best COCP and UNP weighted approximations of $f$ of $\Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})$.

**Definition 3.1.** *Let $f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})$, for $\alpha, \beta \in J_p$ respectively, the degrees of best (co)convex and unconstrained polynomial weighted approximations of f are denoted by*

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p = \inf\{\|f - p_n\|_{\alpha,\beta,p}\,,\ \ p_n \in \pi_n \cap \Delta^{(2)}(Y_s) \cap I_f\,,\ f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})\}$$

*and*

$$\mathbb{E}_n(f, w_{\alpha,\beta})_{\alpha,\beta,p} = \mathbb{E}_n(f)_{\alpha,\beta,p} = \inf\{\|f - p_n\|_{\alpha,\beta,p}\,,\ \ p_n \in \pi_n \cap I_f\,,\ f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})\}.$$

We are starting with the following main result.

**Theorem 3.2.** *Let $s \in \mathbb{N}_\circ$, $Y_s \in \mathbb{Y}_s$ and $f \in \Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})$. Then, for $1 \le \eta \le r$, $\hat{T}_n$ Chebyshev partition and $\mathbf{P}$ Lebesgue partition of $\mathbb{D}$, we have*

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le c\varpi^{\phi}_{i,\eta}(f^{(\eta)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p} \qquad (4)$$

*where c is a constant depend on $\delta_\circ$, $\eta$ and $\phi$.*
*In particular, if $\mathbf{P} \cap \hat{T}_n \ne \emptyset$, then,*

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le c\|\theta_{\mathscr{N}}\|^{-\frac{\eta}{2}}\,\varpi^{\phi}_{i,\eta}(f^{(\eta)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha+\frac{\eta}{2},\beta+\frac{\eta}{2}},p}. \qquad (5)$$

*Proof.* We first prove our theorem by virtue of Lebesgue Stieltjes integral-i properties in Definition 1.4.

Note that Definition 3.1, implies that

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le \|w_{\alpha,\beta}(f - p_n)\|_p,$$

such that $p_n \in \pi_n \cap \Delta^{(2)}(Y_s) \cap I_f$ and the function $f : \mathbb{D} \to \mathbb{R}$ belong to $\Delta^{(2)}(Y_s) \cap \Phi^p(w_{\alpha,\beta})$, then, there exist $\mathscr{L}_i : \mathbb{D} \to \mathbb{R}$ are nondecreasing functions and $i \in \mathbb{N}$. Thus, we have Lebesgue partition $\mathbf{P}$ of $\mathbb{D}$. Now, suppose that $\hat{T}_\eta$ is Chebyshev partition such that $\mathbb{D} \cap \hat{T}_\eta \ne \emptyset$ and $1 \le \eta \le r$. Then,

$$\begin{aligned}
\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p &\le \|w_{\alpha,\beta}(\prod_{i\in\mathbb{N}}\int_{i+\eta}^{\mathbb{D}}(f^{(\eta)} - p_n^{(\eta)})\underline{\mathrm{d}\mathscr{L}_\phi}\|_p,\\
&\le c(\phi)^{-\eta}\|w_{\alpha,\beta}\phi^\eta(\prod_{i\in\mathbb{N}}\int_{i+\eta}^{\mathbb{D}}(f^{(\eta)} - p_n^{(\eta)})\underline{\mathrm{d}\mathscr{L}_\phi}\|_p,\\
&\le c(\phi)^{-\eta}(\|w_{\alpha,\beta}\phi^\eta(\prod_{i\in\mathbb{N}}\int_{i+\eta}^{\mathbb{D}}f^{(\eta)}\underline{\mathrm{d}\mathscr{L}_\phi})\|_p +\\
&\quad \|w_{\alpha,\beta}\phi^\eta(\prod_{i\in\mathbb{N}}\int_{i+\eta}^{\mathbb{D}}p_n^{(\eta)}\underline{\mathrm{d}\mathscr{L}_\phi})\|_p).
\end{aligned}$$

Now, by virtue of Theorems 2.7 and 2.8, we will put $|\mathbb{D}| \le \delta_\circ$ and $|\phi| \le \delta_\circ$, then

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le c(\delta_\circ, \eta, \phi)\sup\|w_{\alpha,\beta}\phi^\eta\mho^i_{h\phi}(f^{(\eta)}, x)\|_p.$$

Therefore, (4) is proved.
If $\mathbf{P} \cap \hat{T}_n \ne \emptyset$, by use proof of Theorem 2.8, then, we lead to (5). ☐

The ensuing complements for Theorems 2.7 and 2.8 are given follows.

**Lemma 3.3.** *For any $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, then*

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le c(\delta_\circ)\varpi^{\phi}_{i-1,\eta+1}(f^{(\eta+1)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}\,. \qquad (6)$$

*In particular, if $\hat{T}_\eta$ is Chebyshev partition and $\mathbf{P}$ is Lebesgue partition of $\mathbb{D}$,*

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le c(\delta_\circ)\|\theta_{\mathscr{N}}\|^{\frac{-\eta-1}{2}}\,\varpi^{\phi}_{i-1,\eta+1}(f^{(\eta+1)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\ddot{\alpha},\ddot{\beta}},p}$$

*where $\ddot{\alpha} = \alpha + \frac{\eta+1}{2}$, $\ddot{\beta} = \beta + \frac{\eta+1}{2}$ and $\mathbf{P} \cap \hat{T}_n \ne \emptyset$.*

*Proof.* Let $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, then by Theorem 2.7, we have

$$\begin{aligned}
\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p &\le c\varpi^{\phi}_{i,\eta}(f^{(\eta)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}\\
&\le c(\delta_\circ)\varpi^{\phi}_{i-1,\eta+1}(f^{(\eta+1)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}\,.
\end{aligned}$$

Thus, (6) is proven.

Directly from (4), if $\frac{\eta}{2}$ replaced the $\frac{\eta+1}{2}$, therefore,

$$\mathscr{E}^{(2)}_n(f, w_{\alpha,\beta}, Y_s)_p \le c(\delta_\circ)(\|\theta_{\mathscr{N}}\|^{-(\frac{\eta+1}{2})}) \times \varpi^{\phi}_{i-1,\eta+1}(f^{(\eta+1)}, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\ddot{\alpha},\ddot{\beta}},p}\,,$$

if $\ddot{\alpha} = \alpha + \frac{\eta+1}{2}$ and $\ddot{\beta} = \beta + \frac{\eta+1}{2}$. ☐

Note that our main theorem and lemma can be extended for the COCUNP as we discussed earlier in our previous papers [8, 9, 10, 11].

## 4 Conclusions

In this work, the DOND for the best COCUNP weighted approximation was considered with the effect of the Chebyshev and Lebesgue partitions. Cocovex functions space have been used to convert Jackson type approximations into weighted approximations. The resultant approximations were converted by employing the Lebesgue Stieltjes integral-i technique to prove it. Also, we obtained the following: Suppose that $Y_s \in \mathbb{Y}_s$ , $\sigma, s, n \in \mathbb{N}$ and $\sigma \neq 4$. If $f \in \Delta^{(2)}(Y_s) \cap \Phi^{p,r}(w_{\alpha,\beta})$, then,

$$\mathscr{E}_n^{(2)}(f, w_{\alpha,\beta}, Y_s)_p \leq c(\delta_\circ) \times n^{-\sigma} \varpi^{\phi}_{i+1,r}\,(f^{(r)},\, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha+\frac{1}{2},\beta+\frac{1}{2}},p}$$

and

$$\mathscr{E}_n^{(2)}(f, w_{\alpha,\beta}, Y_s)_p \leq c(\eta, J_{j,\eta}) \times n^{-\sigma} \varpi^{\phi}_{i+2\eta,i+\eta}\,(f^{(i+\eta)},\, \|\theta_{\mathscr{N}}\|, \mathbb{D})_{w_{\alpha,\beta},p}\,.$$

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